HYPERBOLICITY OF DELAY EQUATIONS VIA FOURIER MULTIPLIERS

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Abstract. We apply recently obtained results on the hyperbolicity of semigroups to delay equations. The main tool is the theory of operator valued Fourier multipliers.

Dedicated to Rainer Nagel on the occasion of his 60th birthday

1. Introduction

Partial differential equations with delay have been studied for many years and by many different methods. In an abstract way and using the standard notation (see [20]), they can be written as

$$\text{(DE)} \qquad \begin{cases} u'(t) = Bu(t) + \Phi u_t, \quad t \geq 0, \\ u(0) = x, \\ u_0 = f, \end{cases}$$

in a Banach space X, where $(B, D(B))$ is a (unbounded) linear operator on X, $u_t(\cdot) = u(t + \cdot)$ on $[-1, 0]$, and the delay operator Φ is supposed to belong to, e.g., $\mathcal{L}(W^{1,p}([-1, 0], X), X)$ for some $1 \leq p < \infty$.

J. Hale [7] and G. Webb [19] were among the first to apply semigroup theory to delay equations, and we refer to [20] for more recent references on partial differential equations with delay. There are deep results on stability and hyperbolicity using compactness properties of the semigroup by J. Milota [14] (see also in [20]), or the positivity [16].

As a first step one has to choose an appropriate state space. One of the possibilities is to work in the space of continuous X-valued functions. In this case, the relationship between solutions of (DE) and a corresponding semigroup has been studied intensively (see for example [8], [20] or [6, Section VI.6]) and is well understood. On the other hand, the state space $\mathcal{E} := X \times L^p([-1, 0], X)$ turns out to be a better choice

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Date: December 13, 2008.

1991 *Mathematics Subject Classification.* 34K05, 34K20, 47D06.

Key words and phrases. C_0-semigroups, hyperbolicity, delay equations, exponential stability, Fourier-multipliers.

with regards to certain applications (e.g., to control theory, see [17], to numerical methods, see [15]) and will be used in this paper.

The aim of this work is to give robust hyperbolicity and stability results for linear partial differential equations with delay, especially for the cases where no spectral mapping theorems are available and we cannot use the powerful technics of characteristic equations. Our approach is based on a recently obtained characterization of hyperbolicity resp. stability in terms of the resolvent being a Fourier multiplier (see [12] resp. [10]).

In the next section we collect some results on the semigroup approach for delay equations in the L^p history space, mainly from [2, 3]. A similar method have been used by H. Petzeltová [18] with different spaces; they allow less space regularity of the delay operator, but require analyticity of the semigroup generated by B, and also more time regularity in the delay term. Our approach is especially useful in the Hilbert space case and allows stability results in the case where the semigroup generated by $(B, D(B))$ is not compact, see [4] for applications.

In Section 3 we collect the results on the hyperbolicity and uniform exponential stability of semigroups which we will apply to the delay semigroup in Section 4.

The main problem is that in applications we have quite often a good characterization of the generator but no explicit knowledge on the semigroup itself, i.e., the solutions of the differential equation. Thus, it would be useful if we could find conditions on the generator or its resolvent implying the hyperbolicity or uniform exponential stability of the semigroup. In Hilbert spaces the celebrated theorem of Gearhart allows us to make the connection. Recently, a generalization of this stability theorem has been formulated in Banach spaces using the notion of Fourier multipliers [5, 10], and hyperbolicity of a semigroup has been also characterized in this way [12, 11].

The contributions of this paper are formulated and proved in Section 4. We present robust hyperbolicity results of the following kind: assume that $(B, D(B))$ generates a hyperbolic semigroup and that the delay operator Φ is "small" in some sense, then the delay semigroup remains hyperbolic. As a special case we consider uniform exponential stability.

We make a small remark about another approach to the problem (DE). Formally, taking the Laplace transform of the solution u (respectively Fourier transform when considered on $\mathbb{R}$ extended by zero to the negative axis) we get

$$\widehat{u}(\lambda) = (\lambda - B - \Phi_\lambda)^{-1} \left(x + \Phi \left(e^{\lambda \cdot} \int_{\cdot}^{0} e^{-\lambda s} f(s) ds \right) \right),$$

where $\Phi_\lambda x := \Phi(e^{\lambda\cdot}x)$. Arranging the terms at $\lambda = \alpha + i\varrho$, after a straightforward computation, using the Neumann series representation of the resolvent, we obtain

$$\begin{aligned}&\widehat{u}(\alpha+i\varrho)\\ &= (\alpha+i\varrho-B)^{-1}\sum_{n=0}^{\infty}\left(\Phi_{\alpha+i\varrho}(\alpha+i\varrho-B)^{-1}\right)^{n}\\ &* \left[x+\mathcal{F}_z(\Phi(f(z+.)\chi_{(.,0)}(z+.)e^{-\alpha z})(\varrho)\right],\end{aligned}$$

where χ_I is the characteristic function of the interval I. Applying the inverse Fourier transform with respect to ϱ we get for $t > 0$

$$\begin{aligned}u(t) &= e^{t\alpha}\left[e^{(-\alpha+B)s}H(s)\star_s \mathcal{F}_\varrho^{-1}(\textstyle\sum)(s)\right](t)x\\ &+ e^{t\alpha}\left[e^{(-\alpha+B)s}H(s)\star_s \mathcal{F}_\varrho^{-1}(\textstyle\sum)(s)\star_s \Phi(f(s+.)\chi_{(.,0)}(s+.)e^{-\alpha s})\right](t),\end{aligned}$$

If one could show that the middle term

$$\mathcal{F}_\varrho^{-1}\left(\sum_{n=0}^{\infty}\left(\Phi_{\alpha+i\varrho}(\alpha+i\varrho-B)^{-1}\right)^{n}\right) \in L^1(\mathbb{R},\mathcal{L}_s(X)),$$

then for the solution one would get that $e^{-\alpha\cdot}u(.)$ is in L^p resp. C_0, as a convolution with an L^1-function. Here, we denoted by $\mathcal{L}_s(X)$ the set of continuous linear operators on X endowed with the strong operator topology. A weaker condition is that the expression in the brackets is a Fourier multiplier. Such estimates can yield stability results. To make the computations correct, and to be able to cover also hyperbolicity results, we have chosen the semigroup approach on a product space to the problem (DE).

2. THE SEMIGROUP APPROACH TO DELAY EQUATIONS

Let us summarize here some results from [2, 3] on the semigroup approach to linear partial differential equations with delay.

Consider the equation

$$\text{(DE)}\qquad \begin{cases} u'(t) = Bu(t) + \Phi u_t, & t \geq 0,\\ u(0) = x, \\ u_0 = f,\end{cases}$$

where

- $x \in X$, X is a Banach space,
- $B : D(B) \subseteq X \longrightarrow X$ is a linear, closed, and densely defined operator,
- $f \in L^p([-1,0],X)$, $p \geq 1$,

- $\Phi : W^{1,p}([-1,0],X) \longrightarrow X$ is a linear, bounded operator,
- $u : [-1,\infty) \longrightarrow X$ and $u_t : [-1,0] \longrightarrow X$ is defined by $u_t(\sigma) := u(t+\sigma)$.

Definition 2.1. *We say that a function $u : [-1,\infty) \longrightarrow X$ is a (classical) solution of (DE) if*

(i) $u \in C([-1,\infty),X) \cap C^1([0,\infty),X)$,
(ii) $u(t) \in D(B)$ *and* $u_t \in W^{1,p}([-1,0],X)$ *for all* $t \geq 0$, *and*
(iii) u *satisfies (DE) for all* $t \geq 0$.

To be able to solve (DE) by semigroup methods, we introduce the Banach space

$$\mathcal{E} := X \times L^p([-1,0],X)$$

and the operator

$$\mathcal{A} := \begin{pmatrix} B & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \tag{1}$$

with domain

$$D(\mathcal{A}) := \left\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(B) \times W^{1,p}([-1,0],X) \ : \ f(0) = x \right\}. \tag{2}$$

Consider now the abstract Cauchy problem

$$\text{(ACP)} \qquad \begin{cases} v'(t) = \mathcal{A}\, v(t), & t \geq 0, \\ v(0) = v_0 \end{cases}$$

associated to the operator matrix $(\mathcal{A}, D(\mathcal{A}))$ on the Banach space $\mathcal{E}$ with initial value $v_0 := \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)$. There is a natural correspondence between the solutions of the two problems (see [3, Proposition 2.3 and 2.4]).

Lemma 2.2. *(i) If u is a solution of (DE), then $t \mapsto \left(\begin{smallmatrix} u(t) \\ u_t \end{smallmatrix}\right)$ is a solution of the equation (ACP).*
(ii) If $t \mapsto \left(\begin{smallmatrix} u(t) \\ v(t) \end{smallmatrix}\right)$ is a solution of (ACP), then $v(t) = u_t$ for all $t \geq 0$ and u is a solution of (DE).

We can then give the following definition for well-posedness.

Definition 2.3. *We say that (DE) is well-posed if*

(i) for every $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A})$ there is a unique solution $u(x,f,\cdot)$, and
(ii) the solutions depend continuously on the initial values, i.e., if a sequence $\left(\begin{smallmatrix} x_n \\ f_n \end{smallmatrix}\right)$ in $D(\mathcal{A})$ converges to $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A})$, then $u(x_n,f_n,t)$ converges to $u(x,f,t)$ uniformly for t in compact intervals.

There is also a correspondence between the well-posedness of equation (DE) and of the abstract Cauchy problem (ACP), see also [3, Theorem 2.8].

Proposition 2.4. *Let $(\mathcal{A}, D(\mathcal{A}))$ be the operator matrix defined by (1) and (2). Then the following assertions are equivalent.*

(i) Equation (DE) is well-posed.

(ii) $(\mathcal{A}, D(\mathcal{A}))$ is the generator of a strongly continuous semigroup on $\mathcal{E}$.

As an easy consequence of Lemma 2.2 and Proposition 2.4, we have that if $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$, then the solutions of equation (DE) are given by the first component of the function $t \mapsto \mathcal{T}(t)\binom{x}{f}$ for $\binom{x}{f} \in D(\mathcal{A})$.

By means of the perturbation theorem of Miyadera-Voigt (see [6, Corollary III.3.16]) one can formulate the following sufficient condition on the well-posedness, see [3, Theorem 3.2]. Here $(B, D(B))$ is assumed to be the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X, $(T_0(t))_{t\geq 0}$ is the nilpotent left shift semigroup on $L^p([-1,0], X)$ and $S_t : X \to L^p([-1,0], X)$ is defined by

$$(S_t\, x)(\tau) := \begin{cases} S(t+\tau)x, & -t < \tau \leq 0, \\ 0, & -1 \leq \tau \leq -t. \end{cases}$$

Theorem 2.5. *Let $(B, D(B))$ be the generator of a strongly continuous semigroup on X and assume that there exist $t_0 > 0$ and $0 \leq q < 1$ such that*

$$\text{(M)} \qquad \int_0^{t_0} \|\Phi(S_r\, x + T_0(r)f)\|\, dr \leq q\, \left\|\binom{x}{f}\right\|$$

for all $\binom{x}{f} \in D(\mathcal{A})$. Then the operator $(\mathcal{A}, D(\mathcal{A}))$ is the generator of a strongly continuous semigroup on $\mathcal{E}$. Thus, (DE) is well-posed.

One can formulate the following special case of the above result, see [3, Theorem 3.3, Examples 3.4], [13].

Corollary 2.6. *Assume that $(B, D(B))$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X, $\infty > p \geq 1$, and that there exists a function $\eta : [-1,0] \to \mathcal{L}(X)$ of bounded variation such that $\Phi : C([-1,0], X) \to X$ is given by the Riemann-Stieltjes integral*

$$\text{(3)} \qquad \Phi(f) := \int_{-1}^{0} d\eta f.$$

Then $(\mathcal{A}, D(\mathcal{A}))$ is a generator on $\mathcal{E}$.

Important special cases are operators Φ defined by

$$\Phi(f) := \sum_{k=0}^{n} B_k f(h_k), \quad f \in W^{1,p}([-1,0], X),$$

where $B_k \in \mathcal{L}(X)$ and $h_k \in [-1,0]$ for $k = 0, \dots, n$.

We now characterize the resolvent set and the resolvent operator of $\mathcal{A}$ (see [3, Lemma 4.1]). Here $\epsilon_\lambda(t) := e^{\lambda t}$ and $\Phi_\lambda \in \mathcal{L}(X)$ is defined by $\Phi_\lambda x := \Phi(\epsilon_\lambda \otimes Id)x = \Phi(e^{\lambda \cdot} x)$ for $x \in X$. The operator $(A_0, D(A_0))$ is the generator of the nilpotent left shift semigroup $(T_0(t))_{t\geq 0}$ in $L^p([-1,0],X)$.

Lemma 2.7. *Let X be a Banach space, $(B, D(B))$ be linear, closed and densely defined, and $\Phi : W^{1,p}([-1,0],X) \longrightarrow X$ be linear and bounded. Let $(\mathcal{A}, D(\mathcal{A}))$ be the operator matrix defined in (1) and (2). Then $\lambda \in \rho(\mathcal{A})$ if and only if $\lambda \in \rho(B + \Phi_\lambda)$. Moreover, for $\lambda \in \rho(\mathcal{A})$ the resolvent $R(\lambda, \mathcal{A})$ is given by*

$$\begin{pmatrix} R(\lambda, B+\Phi_\lambda) & R(\lambda, B+\Phi_\lambda)\Phi R(\lambda, A_0) \\ \epsilon_\lambda \otimes R(\lambda, B+\Phi_\lambda) & [\epsilon_\lambda \otimes R(\lambda, B+\Phi_\lambda)\Phi + Id]R(\lambda, A_0) \end{pmatrix}. \tag{4}$$

3. Fourier multiplier theorems and stability

It was shown recently by Clark, Latushkin, Montgomery-Smith, Randolph [5] and Hieber [10] that Fourier multipliers also play an important role in the stability theory of semigroups. Hyperbolicity is characterized in [12, 11]. For the theory of operator valued Fourier multipliers and their applications in semigroup theory see also [1, 9].

Definition 3.1. *We say that the function $m \in L^\infty(\mathbb{R}, \mathcal{L}(X,Y))$ is an L^p Fourier multiplier from $L^p(\mathbb{R},X)$ to $L^p(\mathbb{R},Y)$ if*

$$f \mapsto \mathcal{F}^{-1}(m\mathcal{F}f) = \mathcal{F}^{-1}m * f,$$

$f \in \mathcal{S}(\mathbb{R},X)$ extends to an element of $\mathcal{L}(L^p(\mathbb{R},X), L^p(\mathbb{R},Y))$. The set of L^p Fourier multipliers from $L^p(\mathbb{R},X)$ into itself will be denoted by $\mathcal{M}_p^{\mathcal{L}(X)}$ and normed with

$$\|m\|_{\mathcal{M}_p^{\mathcal{L}(X)}} := \|\mathcal{F}^{-1}m\mathcal{F}\|_{\mathcal{L}(L^p(\mathbb{R},X))}.$$

The following results play an important role in our investigations.

Theorem 3.2. [12] *Let $(G, D(G))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ in the Banach space X. Then the semigroup is hyperbolic if and only if*

$$R(i\cdot, G) \in \mathcal{M}_p^{\mathcal{L}(X)}.$$

Theorem 3.3. [10] *Assume that $(G, D(G))$ is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$. Then*

$$\omega_0(G) = \inf\left\{\mu > s(G) \,:\, \sup_{\alpha \geq \mu} \|R(\alpha + i\cdot, G)\|_{\mathcal{M}_p^{\mathcal{L}(X)}} < \infty\right\}.$$

Here $\omega_0(G)$ is the growth bound and $s(G)$ is the spectral bound of the generator G, see [6] for the notations. We make the remark that the infimum in the previous theorem in never attained.

4. HYPERBOLICITY AND STABILITY OF DELAY EQUATIONS

Definition 4.1. *We call the delay operator* $\Phi \in \mathcal{L}(W^{1,p}([-1,0],X),X)$ *Fourier-admissible if*

(a) *the operator* $(\mathcal{A}, D(\mathcal{A}))$ *is a generator for each generator* $(B, D(B))$ *and*
(b) *the function* $\Phi R(\alpha + i\cdot, A_0)$ *is an* L^p *Fourier multiplier from* $L^p(\mathbb{R}, L^p([-1,0],X))$ *to* $L^p(\mathbb{R}, X)$ *for all* $\alpha \in \mathbb{R}$.

Proposition 4.2. *Assume that* Φ *is given by (3) as in Corollary 2.6. Then* Φ *is Fourier-admissible.*

Proof. Condition (a) of Definition 4.1 is satisfied by Corollary 2.6. So we only have to prove that condition (b) is satisfied. Let us fix an arbitrary $\alpha \in \mathbb{R}$. Denote by M the operator acting from $L^p(\mathbb{R}, L^p([-1,0],X)) =: Z$ into $L^p(\mathbb{R}, X)$ by the rule

$$(Mf)(t) := \int_{\mathbb{R}} \Phi R(\alpha + is, A_0)\hat{f}(s)e^{its}ds.$$

We need to prove that for all f from a dense subspace $D \subset Z$ and for some constant $c_{\alpha,\eta} > 0$, the following inequality holds:

$$\int_{\mathbb{R}} \|(Mf)(t)\|_X^p \, dt \leq c_{\alpha,\eta} \|f\|_Z^p. \tag{5}$$

We take D to be the space of all Schwartz functions from $\mathbb{R}$ to $L^p([-1,0],X)$ whose Fourier transform has compact support. This will be enough to justify the use of the theorem of Fubini in our further computations.

Before we prove (5), let us simplify the expression for $(Mf)(t)$ using (3) and the definition of A_0.

$$\begin{aligned}
(Mf)(t) &= \int_{\mathbb{R}} \int_{-1}^{0} [R(\alpha + is, A_0)\hat{f}(s)](\tau) d\eta(\tau) e^{ist} ds \\
&= \int_{\mathbb{R}} \int_{-1}^{0} \int_{\tau}^{0} e^{(\alpha+is)(\tau-r)} \hat{f}(s)(r) dr d\eta(\tau) e^{ist} ds \\
&= \int_{-1}^{0} \int_{\tau}^{0} \int_{\mathbb{R}} e^{(\alpha+is)(\tau-r)+ist} \hat{f}(s)(r) ds dr d\eta(\tau) \\
&= \int_{-1}^{0} \int_{\tau}^{0} e^{\alpha(\tau-r)} \int_{\mathbb{R}} e^{is(\tau-r+t)} \hat{f}(s)(r) ds dr d\eta(\tau) \\
&= \int_{-1}^{0} \int_{\tau}^{0} e^{\alpha(\tau-r)} f(\tau - r + t)(r) dr d\eta(\tau).
\end{aligned}$$

Now observe that $e^{\alpha(\tau-r)}$ is bounded by some constant C for all τ and r from the domain of integration. Therefore, using this and the Hölder inequality, we obtain the following estimates:

$$\begin{aligned}
\int_{\mathbb{R}} \|(Mf)(t)\|_X^p \, dt &\le \int_{\mathbb{R}} \int_{-1}^{0} \int_{\tau}^{0} \|f(\tau - r + t)(r)\|_X^p dr d|\eta|(\tau) dt \\
&= C \int_{-1}^{0} \int_{\tau}^{0} \int_{\mathbb{R}} \|f(\tau - r + t)(r)\|_X^p dt dr d|\eta|(\tau) \\
&= C \int_{-1}^{0} \int_{\tau}^{0} \int_{\mathbb{R}} \|f(t)(r)\|_X^p dt dr d|\eta|(\tau) \\
&\le C \int_{-1}^{0} \int_{-1}^{0} \int_{\mathbb{R}} \|f(t)(r)\|_X^p dt dr d|\eta|(\tau) \\
&= C|\eta|([-1, 0]) \int_{\mathbb{R}} \int_{-1}^{0} \|f(t)(r)\|_X^p dr dt = C_{\alpha,\eta} \|f\|_Z^p,
\end{aligned}$$

which is the desired result. Here $|\eta|$ is the positive valued Borel measure defined from η by its total variation. ☐

Theorem 4.3. *Assume that* Φ *is admissible,* $(B, D(B))$ *generates a hyperbolic semigroup and let*

$$a_n := \|(\Phi_i . R(i\cdot, B))^n\|_{\mathcal{M}_p^{\mathcal{L}(X)}} < \infty. \tag{6}$$

If

$$a := \sum_{n=0}^{\infty} a_n < \infty, \tag{7}$$

then $(\mathcal{A}, D(\mathcal{A}))$ *generates a hyperbolic semigroup.*

Proof. By Theorem 3.2, we have to show that the resolvent operator given by (4) is a Fourier multiplier on the line $\{\lambda \in i\mathbb{R}\}$. Since Φ is admissible, this follows if the function $R(i\cdot, B+\Phi_{i\cdot})$ is bounded as a Fourier multiplier on the line $i\mathbb{R}$. We make here the remark that if m is a Fourier-multiplier, then $\epsilon_{\alpha+i\cdot} \otimes m(\cdot)$ is a Fourier multiplier between the Banach spaces $L^p(\mathbb{R}, L^p([-1,0]), X))$ and $L^p(\mathbb{R}, X)$, and that $R(i., A_0)$ is an L^p Fourier multiplier from $L^p(\mathbb{R}, L^p([-1,0]), X))$ into itself.

Defining

$$M := \|R(i\cdot, B)\|_{\mathcal{M}_p^{\mathcal{L}(X)}},$$

which is finite by Theorem 3.2, we obtain that

$$R(i\omega, B) \sum_{n=0}^{\infty} (\Phi_{i\omega} R(i\omega, B))^n \in \mathcal{L}(X)$$

for all $\omega \in \mathbb{R}$ and that

$$\begin{aligned} \left\| R(i\cdot, B) \sum_{n=0}^{\infty} (\Phi_{i\cdot} R(i\cdot, B))^n \right\|_{\mathcal{M}_p^{\mathcal{L}(X)}} &\le M \sum_{n=0}^{\infty} \| (\Phi_{i\cdot} R(i\cdot, B))^n \|_{\mathcal{M}_p^{\mathcal{L}(X)}} \\ &\le M \sum_{n=0}^{\infty} a_n = Ma. \end{aligned}$$

Easy calculations show that this operator defines an inverse for $(i\omega - B - \Phi_{i\omega})$. ☐

Theorem 4.4. *Assume that Φ is admissible, $\omega_0(B) < 0$ and let $\alpha \in (\omega_0(B), 0]$ such that*

$$a_{\alpha,n} := \sup_{\beta \ge \alpha} \|(\Phi_{\beta+i\cdot} R(\beta + i\cdot, B))^n\|_{\mathcal{M}_p^{\mathcal{L}(X)}} < \infty. \tag{8}$$

If

$$a_\alpha := \sum_{n=0}^{\infty} a_{\alpha,n} < \infty, \tag{9}$$

then $\omega_0(\mathcal{A}) < \alpha \le 0$.

Proof. By our assumptions and using Theorem 4.3, we have that $(\mathcal{A} - \beta)$ generates a hyperbolic semigroup for all $\beta \ge \alpha$. Thus, $\omega_0(\mathcal{A}) < \alpha \le 0$. ☐

We now formulate an immediate consequence of the theorem.

Corollary 4.5. *Assume that* Φ *is admissible,* $\omega_0(B) < 0$ *and let* $\alpha \in (\omega_0(B), 0]$*. If*

$$\sup_{\beta\geq\alpha} \|\Phi_{\beta+i\cdot}R(\beta+i\cdot, B)\|_{\mathcal{M}_p^{\mathcal{L}(X)}} < 1, \tag{10}$$

then $\omega_0(\mathcal{A}) < \alpha \leq 0$*.*

Corollary 4.6. *Consider the case* $\Phi = C\delta_{-\tau}$*, where* $C \in \mathcal{L}(X)$ *and* $\tau \in [0,1]$*. Assume that* $(B, D(B))$ *generates an exponentially stable semigroup* $(S(t))_{t\geq 0}$ *, let* $\alpha \in (\omega_0(B), 0]$ *and take* $\omega \in [\omega_0(B), \alpha)$ *and* $M \geq 1$ *such that* $\|S(t)\| \leq Me^{\omega t}$*. If*

$$\frac{Me^{-\alpha\tau}}{\alpha-\omega}\|C\| < 1,$$

then $\omega_0(\mathcal{A}) < \alpha \leq 0$*.*

Proof. It follows from the elementary properties of the Fourier transform and from the connection of the semigroup and its resolvent that for $t \in \mathbb{R}$, $\beta \geq \alpha$ and for $x \in X$

(11)

$$\begin{aligned}\mathcal{F}^{-1}(\Phi_{\beta+i\cdot}R(\beta+i\cdot,B)x)(t) &= Ce^{-\beta\tau}\mathcal{F}^{-1}(e^{-i\tau\cdot}R(i\cdot,B-\beta)x)(t)\\ &= Ce^{-\beta\tau}\tilde{S}_\beta(t+\tau)x = \begin{cases} Ce^{-\beta\tau}e^{-\beta(\tau+t)}S(\tau+t)x, & \text{if } t+\tau>0,\\ 0, & \text{if } t+\tau<0.\end{cases}\end{aligned}$$

Here we have used the notations of [10] and denoted with $S_\beta(t) := e^{-\beta t}S(t)$ the rescaled, exponentially stable semigroup and with

$$\tilde{S}_\beta(t) := \begin{cases} S_\beta(t), & \text{if } t \geq 0,\\ 0, & \text{if } t < 0.\end{cases}$$

It follows that $\tilde{S}_\beta \in L^1(\mathbb{R}, \mathcal{L}_s(X))$. Taking $\omega \in [\omega_0(B), \alpha)$, $M := M_\omega \geq 1$ such that $\|S(t)\| \leq Me^{\omega t}$, we obtain from (11) using the Hausdorff-Young inequality

$$\begin{aligned}\|\Phi_{\beta+i\cdot}R(\beta+i\cdot,B)\|_{\mathcal{M}_p^{\mathcal{L}(X)}} &\leq \left\|\mathcal{F}^{-1}(\Phi_{\beta+i\cdot}R(\beta+i\cdot,B))\right\|_{L^1} = \int_0^\infty \left\|Ce^{-\beta(s+\tau)}S(s)\right\| ds\\ &\leq \|C\| \int_0^\infty e^{-\beta(s+\tau)}e^{\omega s}ds = M\|C\|e^{-\beta\tau}\int_0^\infty e^{(\omega-\beta)s}ds\\ &= \frac{Me^{-\beta\tau}}{\beta-\omega}\|C\| \leq \frac{Me^{-\alpha\tau}}{\alpha-\omega}\|C\|.\end{aligned}$$

We can finish the proof using Corollary 4.5. □

Corollary 4.7. *Consider the case* $\Phi = C\delta_{-\tau}$*, where* $C \in \mathcal{L}(X)$ *commuting with* $(B, D(B))$ *and* $\tau \in [0,1]$*. Assume that* $(B, D(B))$ *generates an exponentially stable semigroup, let* $\alpha \in (\omega_0(B), 0]$ *and take* $\omega \in [\omega_0(B), \alpha)$ *and* $M \geq 1$ *such that* $\|S(t)\| \leq Me^{\omega t}$*. If*

$$\frac{Me^{-\alpha\tau}}{\alpha - \omega} r(C) < 1,$$

then $\omega_0(\mathcal{A}) < \alpha \leq 0$*.*

Proof. Taking $\frac{Me^{-\alpha\tau}}{\alpha-\omega} r(C) < q < 1$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$\left(\frac{Me^{-\alpha\tau}}{\alpha - \omega}\right)^n \|C^n\| < q^n.$$

Then it follows for $n \geq N$ that

$$\begin{aligned} a_{\alpha,n} &= \sup_{\beta \geq \alpha} \|(\Phi_{\beta+i\cdot} R(\beta + i\cdot, B))^n\|_{\mathcal{M}_p^{\mathcal{L}(X)}} = \left\|\Phi^n_{\beta+i\cdot} R(\beta + i\cdot, B))^n\right\|_{\mathcal{M}_p^{\mathcal{L}(X)}} \\ &\leq \left\|\left(e^{-\alpha\tau} C\right)^n\right\| \left\|e^{-i\cdot\tau} R(\beta + i\cdot, B))^n\right\|_{\mathcal{M}_p^{\mathcal{L}(X)}} \leq \|C^n\| \, e^{-\alpha\tau n} \left\|e^{-i\cdot\tau} R(\beta + i\cdot, B))\right\|^n_{\mathcal{M}_p^{\mathcal{L}(X)}} \\ &\leq \|C^n\| \left(\frac{Me^{-\alpha\tau}}{\alpha - \omega}\right)^n < q^n, \end{aligned}$$

and the statement follows applying Theorem 4.4. ☐

Corollary 4.8. *Assume that* Φ *is of the form (3), that* $(B, D(B))$ *generates an exponentially stable semigroup* $(S(t))_{t \geq 0}$ *, let* $\alpha \in (\omega_0(B), 0]$ *and take* $\omega \in [\omega_0(B), \alpha)$ *and* $M \geq 1$ *such that* $\|S(t)\| \leq Me^{\omega t}$*. If*

$$|\eta|([-1, 0]) \frac{M}{\alpha - \omega} < 1,$$

then $\omega_0(\mathcal{A}) < \alpha \leq 0$*.*

Proof. Analogously to the proof of Corollary 4.6, we have for $t \in \mathbb{R}$ and $\beta \geq \alpha$

(12)

$$\begin{aligned}
\mathcal{F}^{-1}(\Phi_{\beta+i\cdot}R(\beta+i\cdot,B)x)(t) &= \mathcal{F}^{-1}\left(\int_{-1}^{0} e^{(\beta+i\cdot)s}d\eta(s)R(\beta+i\cdot,B)x\right)(t)\\
&= \int_{\mathbb{R}}\int_{-1}^{0} e^{(\beta+iw)s}e^{iwt}d\eta(s)R(\beta+iw,B)x\\
&= \int_{-1}^{0} e^{\beta s}d\eta(s)\int_{\mathbb{R}} e^{iwt}e^{iws}R(\beta+iw,B)xdw\\
&= \int_{-1}^{0} e^{\beta s}d\eta(s)\mathcal{F}^{-1}\left(e^{is\cdot}R(\beta+i\cdot,B)x\right)(t)\\
&= \int_{-1}^{0} e^{\beta s}d\eta(s)\tilde{S}_{\beta}(t-s)x = \Phi\left(S_{\beta,t}x\right).
\end{aligned}$$

Here $S_{\beta,t}$ is defined from the semigroup S_β analogously to the expression S_t. Taking now ω and M as in the text of the Corollary, we obtain

$$\begin{aligned}
\|\Phi_{\beta+i\cdot}R(\beta+i\cdot,B)\|_{\mathcal{M}_p^{\mathcal{L}(X)}} &\leq \left\|\mathcal{F}^{-1}(\Phi_{\beta+i\cdot}R(\beta+i\cdot,B))\right\|_{L^1} = \int_0^\infty \|\Phi\left(S_{\beta,t}\right)\|\,dt\\
&= \int_0^\infty \left\|\int_{-1}^{0} d\eta(s)S_{\beta,t}(s)\right\|dt \leq \int_0^\infty\int_{-1}^{0} \|S_{\beta,t}(s)\|\,d|\eta|(s)dt\\
&= \int_{-1}^{0}\int_0^\infty \|e^{-\beta(t-s)}S(t-s)\|dtd|\eta|(s) \leq |\eta|([-1,0])\frac{M}{\beta-\omega}\\
&\leq |\eta|([-1,0])\frac{M}{\alpha-\omega}.
\end{aligned}$$

The proof can be finished by using Corollary 4.5. ☐

ACKNOWLEDGMENTS

A. B. was supported by the OTKA grant Nr. F034840, E. F. was partially supported by the grants No. MSM113200007, GAUK 166/1999 and R. S. was partially supported by the student part of the NSF grant DMS-9800027. All the authors thank the organizers of the "TULKA-Conference on Evolution Equations and Semigroups" held in Blaubeuren on the occasion of the 60th birthdays of J. Goldstein and R. Nagel, to enable the finishing of this work.

REFERENCES

[1] Amann, H., *Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications*, Math. Nachr. **186** (1997), 5–56.

[2] Bátkai, A., *Hyperbolicity of linear partial differential equations with delay*, to appear in Int. Eq. Oper. Th.

[3] Bátkai, A., Piazzera, S., *Semigroups and linear partial differential equations with delay*, to appear in J. Math. Anal. Appl.

[4] Bátkai, A., Piazzera, S., *Damped wave equations with delay*, Fields Institute Communications **29** (2001), 51–61.

[5] Clark, S., Latushkin, Y., Montgomery-Smith, S, Randolph, T., *Stability radius and internal versus external stability in Banach spaces: an evolution semigroup approach*, SIAM J. Control. Optim., **38** (2000), 1757–1793 (electronic).

[6] Engel, K.-J., Nagel R., "One-parameter Semigroups for Linear Evolution Equations", Springer-Verlag, Graduate Texts in Mathematics **194**, 1999.

[7] Hale, J. K., "Functional Differential Equations", Appl. Math. Sci., vol. 3, Springer-Verlag, 1971.

[8] Hale, J. K., Verduyn Lunel, S. M., "Introduction to Functional Differential Equations", Appl. Math. Sci. 99, Springer-Verlag, 1993.

[9] Hieber, M., *Operator valued Fourier multipliers*, "Topics in Nonlinear Analysis, The Herbert Amann Anniversary Volume" (J. Escher, G. Simonett, eds.), Birkhäuser, 1999, pp. 363–380.

[10] Hieber, M., *A characterization of the growth bound of a semigroup via Fourier multipliers*, "Evolution Equations and Their Applications in Physical and Life Sciences" (G. Lumer, L. Weis, eds.), Lecture Notes in Pure and Applied Mathematics Series **215**, Marcel Dekker, 2001, pp. 121–124.

[11] Latushkin, Y., Räbiger, F., *Fourier multipliers in stability and control theory*, Preprint, 2000.

[12] Latushkin, Y., Shvidkoy, R., *Hyperbolicity of semigroups and Fourier multipliers*, Preprint, 2000.

[13] Maniar, L., Voigt, J., *Linear delay equations in the L^p context*, Preprint, 2000.

[14] Milota, J., *Stability and saddle-point property for a linear autonomous functional parabolic equation*, Comment. Math. Univ. Carolinae, **27** (1986), 87–101.

[15] Kappel, F., *Semigroups and delay equations*, "Semigroups, Theory and Applications", Vol. II., (Brezis, H., Crandall, M. G., Kappel, F. eds.), Pitman Research Notes in Mathematics **152**, Longman, 1986, pp. 136–176.

[16] Nagel, R. (ed.), "One-parameter Semigroups of Positive Operators", Springer-Verlag, Lecture Notes Math. **1184**, 1986.

[17] Nakagiri, S., *Optimal control of linear retarded systems in Banach spaces*, J. Math. Anal. Appl. **120** (1986), 169–210.

[18] Petzeltová, H., *Solution semigroup and invariant manifolds for functional equations with infinite delay*, Mathematica Bohemica, **118** (1993), 175–193.

[19] Webb, G., *Functional differential equations and nonlinear semigroups in L^p-spaces*, J. Diff. Eq. **29** (1976), 71–89.

[20] Wu, J., "Theory and Applications of Partial Functional Differential Equations", Springer-Verlag, Appl. Math. Sci. **119**, 1996.

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