# ON RECENT DEVELOPMENTS IN THE SPECTRAL PROBLEM FOR THE LINEARIZED EULER EQUATION

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## 1. Introduction

The purpose of this article is to survey results concerning the unstable spectrum of the Euler equation linearized about a steady state. The Euler equations of the motion of an inviscid, incompressible fluid are the basic equations of fluid mechanics and they have been the object of much study by mathematicians over the centuries since Euler "unveiled" them in 1755. However, many significant problems concerning the Euler equation remain unsolved. Some of these problems arise from the particularly challenging nature of the nonlinearity of the equations. But even the linearized equations give rise to complex issues and open questions. Among the most fundamental of these is the structure of the spectrum of the linear Euler operator. Because this operator is, generally, degenerate, non-self adjoint and non-elliptic the standard well known theorems concerning the spectra of elliptic operators do not apply. It has been necessary to develop specialized tools using several branches of mathematics to investigate the spectrum of the Euler operator. Techniques used include those from PDE, functional analysis, operator theory and ODE. In this paper we will discuss our understanding of results achieved and the problems that remain open.

A prime motivation for the study of the spectrum of the Euler equation linearized about a steady flow $u(x)$ is the question of (linear) stability or instability of this flow. The "classical" works on the subject in the $19^{\text{th}}$ and earlier $20^{\text{th}}$ centuries concerned an examination of the discrete unstable spectrum (i.e. eigenvalues for a linear PDE with positive real part) for a very special class of flows $u(x)$ [1]. We discuss

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[1] See, for example, Chandrasekhar [7], Drazin and Reid [12], Kelvin [29], Lin [33], Rayleigh [42].

these results and recent developments concerning the discrete unstable spectrum in Section 3.

Unstable eigenvalues are important from the point of view of instabilities because they indicate a "strong" instability that exists independent of the norm in which the growth of the perturbation is measured. However for most flows $u(x)$ we do not know of the existence of such eigenvalues. For a generic steady flow we have much more information about the more delicate issue of the unstable essential spectrum. This is associated with highly localized short wave perturbations. In this case the existence of the spectrum is strongly dependent on the function space for the perturbation. Methods that combine geometric optics to generate an ODE system that characterizes the essential spectrum and arguments from semigroup theory produce results that give a rather complete description of the unstable essential spectrum for very general flows $u(x)$. We discuss these results in Section 4.

## 2. The linearized Euler equation

The Euler equations for the motion of an inviscid, incompressible fluid are

$$\frac{\partial q}{\partial t} + (q, \nabla)q + \nabla p = 0 \tag{1}$$

$$\nabla \cdot q = 0. \tag{2}$$

Here $q(x,t)$ is the velocity vector of the fluid and $p(x,t)$ is the scalar pressure field. We consider the system in a domain $M \subset \mathbb{R}^n$ or in torus $\mathbb{T}^n$ with $n = 2, 3$. When $M$ is bounded we impose the boundary condition

$$(\hat{n}, q)|_{\partial M} = 0 \tag{3}$$

In the case when $M$ is unbounded (3) is replaced the condition

$$|q| \to 0 \text{ as } |x| \to \infty \tag{4}$$

The spectral problem we will study is sensitive to the nature of the boundary. The analysis is less difficult in the case of $\mathbb{T}^n$ and we will tend to concentrate on the results in this case.

Let $u(x)$ be a steady solution to (1)-(2) satisfying the appropriate boundary condition. We will assume sufficient smoothness for $u(x)$ where necessary in our discussions. The equation governing a steady state can be written as

$$u \times \operatorname{curl} u + \nabla H = 0, \quad \nabla \cdot u = 0. \tag{5}$$

The problem of finding vector fields $u(x)$ that satisfy (5) plus appropriate boundary conditions is notoriously difficult except for an equilibrium depending only on one spatial variable (e.g. plane parallel shear flow or differentially rotating cylinders). There are a couple of known classes of equilibria depending on two spatial variables. Fully three dimensional equilibria are exceedingly difficult to describe (for some details about equilibria see, for example, Friedlander and Lifschitz [21] and references therein).

There are two interesting observations about 3-dimensional Euler equilibria that follow directly from (5). The first is that so called Beltrami flows, for which the velocity and the vorticity $\operatorname{curl} u$ are proportional. These are exact solutions to (5) with $\nabla H \equiv 0$. A specific example of a Beltrami flow in $\mathbb{T}^3$ is an ABC flow which in coordinate form is given by

$$u = (A \sin x_3 + C \cos x_2, B \sin x_1 + A \cos x_3, C \sin x_2 + B \cos x_1). \tag{6}$$

This flow is non-integrable. It exhibits a phenomenon referred to as Lagrangian chaos (cf. [11, 37]). The second observation was made by Arnold [3] and concerns integrable flows where $\nabla H \neq 0$ and the level surfaces $H = H_0$ are compact. For such flows the stream lines (i.e. integral curves of the field $u$) and the vortex lines (i.e. integral curves of the field $\operatorname{curl} u$) cover the surfaces $H = H_0$ which are diffeomorphic to the surface of a torus.

Consider a perturbation $v(x, t)$ of a given steady Euler flow $u(x)$. Neglecting the terms that are nonlinear in $v(x, t)$, we obtain from (1), (2) the linearized Euler equation

$$\frac{\partial v}{\partial t} = -(u, \nabla)v - \partial u \cdot v - \nabla p \equiv Lv \tag{7}$$

$$\nabla \cdot v = 0 \tag{8}$$

where $\partial u$ denotes the gradient matrix with entries $\frac{\partial u_i}{\partial x_j}$ with $i, j = 1, ..., n$. We refer to $L$ as the linear Euler operator associated with the equilibrium $u(x)$. The results we will present concern both the spectrum of $L$ and the spectrum of the evolution operator $e^{tL}$ for $t > 0$ on function spaces of divergence free vector fields. We denote by $\omega(L)$ its exponential type and by $s(L)$ the spectral bound of the generator $L$ (see, for example, [16]). For much of our discussion the semigroup acts on $(L^2(\mathbb{T}^n))^n_{\text{divfree}}$ with $n = 2, 3$.

For general steady flows $u(x)$ the spectra are unions of discrete and essential parts. We define the essential spectrum as follows (c.f. Browder [6]). For any Banach space $X$ and a bounded, or closed unbounded, operator $A$ acting on $X$ we use the following classification of spectral points. A point $z \in \sigma(A)$ is called a point of the *discrete* spectrum if it satisfies the following conditions:

(1) $z$ is an isolated point in $\sigma(A)$.
(2) $z$ has finite multiplicity, i.e. $\cup_{r=1}^{\infty} \mathrm{Ker}(z-A)^r = N$ is finite dimensional in $X$.
(3) The range of $z - A$ is closed, which implies that there is a complementary subspace $Q \subset X$ such that $X = N \oplus Q$, $A(Q) \subset Q$ and $z - A$ is invertible on $Q$.

If, on the contrary, $z$ does not satisfy conditions $(1) - (3)$, then it is called a point of the *essential spectrum*. Thus,

$$\sigma(A) = \sigma_{\mathrm{ess}}(A) \cup \sigma_{\mathrm{disc}}(A). \tag{9}$$

In addition to the formulation of the linearized Euler equation as the evolution equation for the perturbation velocity $v(x,t)$ given by (7)-(8), it is often useful to consider the evolution equation for the perturbation vorticity $w(x,t) = \mathrm{curl}\, v$. The linear vorticity equation is obtained by taking the curl of (7) to give

$$\frac{\partial w}{\partial t} = \{u, w\} + \{\mathrm{curl}^{-1} w, \Omega\} \equiv L_{\mathrm{vor}} w, \tag{10}$$

where $\Omega = \mathrm{curl}\, u$ and $\{,\}$ denotes the Poisson bracket of two vector fields, i.e.

$$\{a, b\} = (b, \nabla)a - (a, \nabla)b.$$

In 3 dimensions, generically, the second Poisson bracket in (10) is very hard to analyze. However in 2 dimensions the problem greatly simplifies and the vorticity equation reduces to

$$\frac{\partial w}{\partial t} = -(u, \nabla)w - (\mathrm{curl}^{-1} w, \nabla)\Omega. \tag{11}$$

In 2 dimensions the role of the vorticity is that of the scalar field

$$w = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}. \tag{12}$$

Furthermore, the second term on the right hand side of (11), i.e. $(\mathrm{curl}^{-1} w, \nabla)\Omega$, defines a compact operator on $w$ in any Sobolev space of mean zero functions,

$$H_0^m(\mathbb{T}^2) = \left\{ w : \|w\|_m^2 = \sum_{k \in \mathbb{Z}^2} |\hat{w}(k)|^2 |k|^{2m} < \infty, \hat{w}(0) = 0 \right\}, \tag{13}$$

where $\hat{w}$ denotes the Fourier transform of $w$, and $m \in \mathbb{Z}$.

In 2 dimensions we can also define a scalar stream function to replace a divergence free vector field. We write

$$u = \nabla^{\perp}\Psi(x), \quad v = \nabla^{\perp}\phi(x,t) \tag{14}$$

where $\nabla^{\perp}$ denotes the 2-D perpendicular gradient operator. The steady state equations in 2-D will be satisfied when $\Psi$ satisfies an elliptic equation of the form

$$\Omega = \Delta\Psi = -F(\Psi), \tag{15}$$

for some function $F$. The 2-D vorticity equation (11) becomes

$$\frac{\partial}{\partial t}\Delta\phi = \left(\frac{\partial\Psi}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial\Psi}{\partial x_1}\frac{\partial}{\partial x_2}\right)(\Delta\phi + F'(\Psi)\phi). \tag{16}$$

It is interesting to consider the relationship between the spectra of the operator $L$ defined in (7) and the operators acting on the vorticity or the stream function (whenever the last is available). Let us note that the curl-operator defines an isomorphism between the Sobolev spaces of divergence-free vector fields of zero mean on $\mathbb{T}^n$. Namely,

$$\mathrm{curl} : H_0^m(\mathbb{T}^n) \mapsto H_0^{m-1}(\mathbb{T}^n).$$

It is an easy exercise to prove this using the Fourier series definition (13) adapted to the vector case, and the fact that since $w$ is a divergence-free field $w$, then $(\hat{w}(k), k) = 0$ for all $k \in \mathbb{Z}^n$. In particular, the inverse to curl is well defined on the corresponding spaces, and is an isomorphism too.

The way we obtained the vorticity form of (7) readily implies the following relationship between $L$ and $L_{\mathrm{vor}}$,

$$L_{\mathrm{vor}} = \mathrm{curl}\, L\, \mathrm{curl}^{-1}. \tag{17}$$

This tells us that the two operators are similar on the respective spaces of the Sobolev tower. As a consequence of (17) we obtain equality between their spectra. Moreover, any isomorphism preserves all conditions (1)-(3) listed in the definition of the discrete spectrum. Thus, we have

$$\sigma_{\mathrm{ess}}(L_{\mathrm{vor}}) = \sigma_{\mathrm{ess}}(L), \quad \sigma_{\mathrm{disc}}(L_{\mathrm{vor}}) = \sigma_{\mathrm{disc}}(L), \tag{18}$$

where $L_{\mathrm{vor}}$ acts on the space of vorticities $H_0^{m-1}(\mathbb{T}^n)$, and $L$ acts on the space of velocities from $H_0^m(\mathbb{T}^n)$, for any $m \in \mathbb{Z}$. Since the corresponding semigroups satisfy the same similarity relation (17), their respective spectra are equivalent too.

In dimension 2, the perpendicular gradient defines an isomorphism between the scalar-$H_0^{m+1}(\mathbb{T}^2)$ and vector-$H_0^m(\mathbb{T}^2)$. So, we analogously obtain the same result for the spectra of $L$ and its stream function formulation defined from (16) by application of the inverse Laplacian to both side of the equation.

## 3. The discrete unstable spectra

At present only very limited information exists concerning discrete eigenvalues of the linearized Euler operator $L$ given by (7). The very extensive "classical" literature concerning linear instability of fluid motion has concentrated almost exclusively on the relatively simple examples of plane parallel shear flow or rotating shear flow (see, for example the books of Drazin and Reid [12] or Chandrasekhar [7]). In the case of plane parallel shear flow, i.e. $u = U(x_2)\hat{x}_1$ the eigenvalue problem is given by the well known Rayleigh equation

$$\left(\frac{\lambda}{ik} + U(x_2)\right)\left(\frac{d^2}{dx_2^2} - k^2\right)\Phi(x_2) - U''(x_2)\Phi(x_2) = 0. \tag{19}$$

This equation is obtained by substituting

$$\phi(x_1, x_2, t) = \Phi(x_2)e^{ikx_1}e^{\lambda t} \tag{20}$$

into (16) and writing $\frac{d\Psi}{dx_2} = U(x_2)$.

We consider (19) with $2\pi$-periodic boundary conditions on $\Phi(x_2)$. The celebrated Rayleigh stability criterion [42] says that a sufficient condition for stability is an absence of an inflection point in the profile $U(x_2)$. This criterion follows from a simple application of the so called "energy method" for stability in which an energy integral is constructed by integrating (19) multiplied by the complex conjugate of $\Phi(x_2)$ to give

$$\mathrm{Re}\lambda \int_0^{2\pi} \frac{U''|\Phi|^2}{|U - i\lambda/k|^2} dx_2 = 0. \tag{21}$$

We note that (21) implies that if $U'' \neq 0$ for $0 < x_2 < 2\pi$, then not only is the unstable discrete spectrum empty but also the discrete spectrum in the left half of complex plane. This is a part of a more general result concerning the symmetry of the spectrum of any Hamiltonian system with real-valued linear Hamiltonian vector field. The linearized Euler equation is just a particular example of such a system defined on an infinite dimensional manifold (see [38]).

When the techniques studied in Section 4 for analyzing the essential spectrum are applied to the special case of plane parallel shear flow, it is shown that the unstable essential spectrum of (19) is empty for any profile $U(x_2)$. In fact, the only contribution to the continuous spectrum of (19) with periodicity conditions comes when

$$i\lambda \in \mathrm{Range}(kU(x_2)), \tag{22}$$

and the term in the denominator of the integrand in (21) can become zero.

The classical treatment of the discrete unstable spectrum of the Rayleigh equation was based on formal asymptotics around a so called neutral mode (see Tollmien [49], Lin [33]). However the formal treatment does not give complete information about the asymptotic behavior and does not exclude the possibility that the neutral mode is isolated (see Drazin and Reid [12]). Howard [28] proved that for a special class of continuous profiles with inflection points there were no more unstable eigenvalues than the number of inflection points. Rosenbluth and Simon [44] used an alternative perturbation approach to establish a sufficient condition for instability, however no explicit profiles were exhibited for which this condition is satisfied. The first rigorous proof of the existence of unstable eigenvalues was given by Faddeev [17] for monotonic profiles with inflection points. In the case of monotonic profiles, Faddeev observed that the spectral operator could be written in terms of an operator $A$ belonging to a class known as the Friedrich's model. He remarked that the problem for non-monotonic profiles is more difficult since the spectrum of $A$ becomes multiple.

Meshalkin and Sinai [39], followed by Yudovich [53], Frenkel [18], and Zhang and Frenkel [56] investigated the instability of a viscous shear flow $U(x_2) = \sin(mx_2)$ using techniques of continued fractions. More recently Friedlander *et al.* [5, 20, 23] showed that these techniques could be used for the inviscid equation (19) with $U(x_2) = \sin(mx_2)$ (so called Kolmogorov flows). Eigenfunctions are constructed in terms of Fourier series that converge to $C^\infty$-smooth functions for eigenvalues $\lambda$ that satisfy the characteristic equation. In [5] this characteristic equation is analyzed using continued fractions and a complete description is given for the unstable spectrum when $U(x_2) = \sin(mx_2)$. It is shown that for all $m > 1$ the discrete unstable spectrum is nonempty and for appropriate values of the wave number $k$ there are both eigenvalues $\lambda$ that are purely real and occurring in complex conjugate pairs.

Results concerning the discrete spectrum for the Rayleigh equation with more general shears $U(x_2)$ are also given in [5] where the existence of an unstable eigenvalue is proved for any rapidly oscillating profile. In this case the spectral asymptotics are determined using the method of averaging. Recently Lin [34] has applied operator theory to the Rayleigh equation to prove for a certain class of shear flows with inflection points that the discrete unstable spectrum is nonempty.

In contrast with the results for periodic profiles, we note that if the profile is close to linear, i.e. $u(x_2) = x_2 + \varepsilon f(x_2)$, for any smooth function $f$ and sufficiently small $\varepsilon > 0$, then for a fixed wave number

$k$ there are no unstable eigenvalues even though $u(x_2)$ may have an arbitrary number of inflection points. This follows from results in the paper of Faddeev [17].

Once we go beyond shear flows there are very few concrete results concerning the discrete spectrum of the linearized Euler equation. Construction of an energy integral from the eigenvalue version of (16) gives, using the skew-symmetry of the operator on the right hand side,

$$\text{Re}\lambda \int_{\mathbb{T}^2} \left[ |\nabla\phi|^2 - \frac{|\nabla\phi|^2}{F'(\Psi)} \right] dx = 0 \tag{23}$$

(see, for example, [19]). Hence $F'(\Psi) < 0$ (or $F'(\Psi) > 0$ and sufficiently large) is a sufficient condition to ensure that the discrete unstable spectrum of the 2-D vorticity equation is empty. A variant of this sufficient condition for stability is given recently in [35]. We note that earlier Arnold [2], exploiting the Hamiltonian nature of the full nonlinear Euler equations, proved that $F'(\Psi) < 0$ is a sufficient condition for nonlinear stability with respect to the enstrophy norm (i.e. $L^2$ norm of the vorticity).

In [24], Friedlander *et al.* study the discrete unstable spectrum of (16) in an example where $F'(\Psi) > 0$, namely a flow with a "cats-eye" type structure where the stream function is

$$\Psi(x_1, x_2) = \cos(x_1 + mx_2) + a\cos(x_1 - mx_2). \tag{24}$$

The method of averaging is used to obtain a formal asymptotic expansion in powers of $1/m$ for large $m$. This produces an "averaged" equation which is an ODE for the leading order terms of the eigenfunctions and eigenvalues of (16). It is proved that for all constants $a$ (except the degenerate 4-fold symmetric case of $|a| = 1$), there exists an unstable eigenvalue. In [35] Lin takes a more abstract approach and proves that there is a purely growing mode for equation (16) if a related dispersion operator has an odd number of negative eigenvalues.

As we noted, in general in 3-D the operator $L_{\text{vor}}$ defined in (10) is very hard to analyze and there are almost no specific 3-D examples where the existence of the discrete spectrum of $L_{\text{vor}}$ has been demonstrated. However the case of motion in 3-D that is invariant with respect to one coordinate is more tractable and the eigenvalue problem reduces to the form of (16) (see [24]). Let the steady velocity be written as

$$u = \nabla^{\perp}\Psi(x_1, x_2) + \hat{k}W(x_1, x_2) \tag{25}$$

where $\hat{k}$ is that unit vector perpendicular to the $(x_1, x_2)$ plane and $\Psi(x_1, x_2)$ is any periodic function such that $\Delta\Psi = -F(\Psi)$. The spectral problem for $x_3$-independent perturbations decouples the vertical and planar motion and results in a pair of 2-D spectral problem for eigenfunctions $\phi(x_1, x_2)$, $\alpha(x_1, x_2)$:

$$\lambda \Delta \phi = D(\Delta \phi + F'(\Psi)\phi) \tag{26a}$$

and

$$(\lambda - D)\alpha = -\nabla W \cdot \nabla^{\perp}\phi \tag{26b}$$

where $D$ is the skew symmetric operator

$$D \equiv \Psi_{x_2}\frac{\partial}{\partial x_1} - \Psi_{x_1}\frac{\partial}{\partial x_2}.$$

For $\mathrm{Re}\lambda > 0$ the operator $(\lambda - D)$ is invertible, hence $\alpha$ is obtained from (26b) for any eigenfunction to the 2-D vorticity equation (26a). In principle, an algebraic instability may also occur as a result of the Jordan cell connected with the purely imaginary spectrum. However, in this problem the existence of integrals of (26b) may preclude such an algebraic instability.

## 4. The structure of the essential spectrum

In general the spectrum of the linearized Euler operator is the union of discrete eigenvalues and the essential spectrum. In contrast to the "strong" stability associated with unstable eigenvalues, instabilities in the essential spectrum are associated with localized short wave length perturbations and are amenable to investigation using techniques of microlocal analysis. Asymptotic methods employing high frequency perturbations have classical roots in fluid dynamics (e.g. Kelvin [29]) and were revitalized in the context of hyperbolic systems by Eckhoff [13, 14] and Ludwig [36] and in the context of the Euler equations in the 1980's by Eckhoff and Storesletten [15]. The next decade produced a number of papers on localized instabilities known under the name of broadband instabilities (c.f. [19] and references therein). This approach was refined in the work of Friedlander and Vishik [25, 26, 52], and Lifschitz and Hameri [32, 31] who developed geometric optics techniques to produce effective criteria for detecting instabilities in the essential spectrum of the Euler equation. Such instabilities are associated with stretching along the streamlines of the flow $u(x)$. They are characterized by a quantity that we call a fluid Lyapunov exponent in analogy with the classical Lyapunov exponent of dynamical systems. In 2D the classical Lyapunov exponent and the fluid Lyapunov exponent are equal. In 3D a positive classical exponent implies a positive fluid exponent but it is also possible in 3D that stretching, weaker than exponential, can produce a positive fluid exponent (c.f. [25, 26, 32]). Vishik

[50], as we will discuss below, proved that the fluid Lyapunov exponent determines the essential spectrum radius of $e^{tL}$ for the Euler equation in any dimension.

We now summarize results and indicate lines of development concerning the essential spectra related to Lyapunov exponents.

4.1. **2-D case.** In the case of two space variables, as we noted earlier, the Euler operator in vorticity form reduces to

$$L_{\text{vor}} = -(u, \nabla)w - (\operatorname{curl}^{-1} w, \nabla)\Omega,$$

for scalar functions $w$ defined by (12). On any Sobolev space $H_0^m(\mathbb{T}^2)$, it is a compact perturbation of the main convection term given by

$$Aw = -(u, \nabla)w.$$

The spectral problem for $A$ itself has its own history of study (see, for example, the book of Antonevich [1]). The classical results – although proved in a different setting – suggest considering the upper Lyapunov exponent of the underlying integral flow $\{\varphi_t\}$, namely

$$\lambda_{\max} = \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbb{T}^2} \|\partial\varphi_t(x)\|,$$

as a determining quantity for the spectral radius of the corresponding transport semigroup defined by the formula

$$e^{tA}w = w \circ \varphi_{-t}.$$

In particular, for the energy norm $v \in (L^2(\mathbb{T}^2))^2_{\text{divfree}}$ or equivalently $w \in H_0^{-1}(\mathbb{T}^2)$, the formula for the spectral radius reads

$$r(e^{tA}) = e^{t\lambda_{\max}}.$$

The use of Nussbaum's formula [40] for essential spectral radius yields

$$r_{\text{ess}}(e^{tL_{\text{vor}}}) = r_{\text{ess}}(e^{tA}) \le e^{t\lambda_{\max}},$$

and by the symmetry,

$$r_{\text{ess}}(e^{tL_{\text{vor}}}) \ge e^{-t\lambda_{\max}}.$$

This "soft" analysis shows that the essential spectrum of the semigroup in the energy norm for velocity is confined to the annulus

$$\{e^{-t\lambda_{\max}} \le |z| \le e^{t\lambda_{\max}}\}.$$

A complete description of the spectrum obtained in papers [46, 47] reveals the opposite inclusion also. Thus, the following theorem holds.

**Theorem 4.1.** *Suppose $u(x)$ is any continuously differentiable steady solution of (1),(2) on the torus $\mathbb{T}^2$. If $\lambda_{\max} > 0$, then the following identities for the essential spectra in $H_0^{-1}(\mathbb{T}^2)$ hold,*

$$\sigma_{\rm ess}(e^{tL_{\rm vor}}) = \{e^{-t\lambda_{\max}} \le |z| \le e^{t\lambda_{\max}}\} \tag{27}$$

$$\sigma_{\rm ess}(L_{\rm vor}) = \{-\lambda_{\max} \le {\rm Re} z \le \lambda_{\max}\}. \tag{28}$$

*The same identities hold for the essential spectra of $L$ and $e^{tL}$ in the energy space $(L^2(\mathbb{T}^2))^2_{\rm divfree}$.*

*More generally, in any Sobolev space $H_0^m(\mathbb{T}^2)$, $m \in \mathbb{Z}$, the spectra are*

$$\sigma_{\rm ess}(e^{tL_{\rm vor}}) = \{e^{-t|m|\lambda_{\max}} \le |z| \le e^{t|m|\lambda_{\max}}\} \tag{29}$$

$$\sigma_{\rm ess}(L_{\rm vor}) = \{-|m|\lambda_{\max} \le {\rm Re} z \le |m|\lambda_{\max}\}. \tag{30}$$

*The same formulas hold for the spectrum of $L$ and $e^{tL}$ on $(H_0^{m+1}(\mathbb{T}^2))^2_{\rm divfree}$.*

In practice, procedure of finding $\lambda_{\max}$ is especially simple. Since in 2 dimensions $\lambda_{\max}$ can only be attained at a hyperbolic point of the flow $\{\varphi_t\}$, one has to find all zeros of the field $u(x)$ and pick the largest real part of an eigenvalue of the corresponding gradient matrix $\partial u(x)$.

Theorem 4.1 is proved under the extra assumption $\lambda_{\max} > 0$. However, the identities (27)-(30) remain valid even if $\lambda_{\max} = 0$ provided the underlying fluid flow has arbitrarily long trajectories, the condition usually needed for translational and rotational invariance of spectra. In this case we simply have

$$\sigma_{\rm ess}(e^{tL_{\rm vor}}) = \mathbb{T}, \qquad \sigma_{\rm ess}(L_{\rm vor}) = i\mathbb{R}. \tag{31}$$

A topological argument shows that the condition on orbits is satisfied if $u \not\equiv 0$ and $u$ has at least two distinct stagnation points [46]. If, on the contrary, all trajectories have uniformly bounded periods, the spectra are still included into the unit circle and imaginary axis respectively, and hence, they do not intersect the unstable region.

We consider Theorem 4.1 in the context of two types of 2D flows. It follows from this theorem that no parallel shear flow has essential spectrum in the unstable region, because in this case the matrix $\partial u(x)$ has only zero eigenvalues. So, the corresponding evolution semigroup may become exponentially unstable only in the presence of the discrete part of the spectrum. For the cellular flow with a hyperbolic point given by

$$u(x) = (-\sin x \cos y, \cos x \sin y),$$

the maximal Lyapunov exponent is realized at the origin, and it is equal to 1. So, in this case there are exponentially growing solutions to (7) and to (11) in respective Sobolev norms, and in particular, in the energy norm.

The proof of Theorem 4.1 presented in [47] contains an explicit construction of a sequence of approximate eigenfunctions for $L_{\mathrm{vor}}$ corresponding to every point on the strip. These functions are localized near the stable or unstable orbit associated with a hyperbolic point where $\lambda_{\max}$ is attained. Their supports are shrinking towards the orbit. Since such a sequence is weakly null in every Sobolev space, this implies that the strip and the annulus are, in fact, in the "weakly-null approximate point spectrum" of $L_{\mathrm{vor}}$ and $e^{tL_{\mathrm{vor}}}$ respectively.

We note that the formulas (27)-(30) automatically prove the Spectral Mapping Theorem for the Euler operator,

$$\sigma(e^{tL})\backslash\{0\} = e^{t\sigma(L)}. \tag{32}$$

Indeed, for the essential spectrum it follows from Theorem 4.1, and for the discrete spectrum it holds generally for all $C_0$-semigroups (see [16]). In particular, this implies the equality between the exponential growth type and the spectral bound, observed also in [30],

$$\omega(L) = s(L). \tag{33}$$

4.2. **3-D case.** In dimension three the structure of essential spectrum becomes increasingly difficult to understand. The major problem arises from the presence of a crucial vortex stretching term, $(w, \nabla)u$, which in vorticity formulation (10) adds a generically bounded perturbation to the convection term. To take into account its contribution an approach that originated in geometric optics was developed by in [25, 52, 32]. A crucial point of this approach was to study instabilities in the evolution operator as opposed to the generator.

The main idea behind the method employed in [25, 52, 32] is to restrict the Euler semigroup to a space of functions with highly oscillating profiles. Such a restriction will not change the semigroup considerably. In fact, it changes only by a finite dimensional operator which, say, for the purposes of finding the essential spectral radius is negligible [40]. On the other hand, it allows us to approximate the semigroup action by a finite dimensional ODE.

So, we hypothetically assume that the instabilities in (7) resulting from essential part of the spectrum are detectable by initial conditions of the form

$$v(x, t = 0) = b_0(x, \xi_0)e^{i\xi_0\cdot x/\delta}, \tag{34}$$

where $\xi_0$ is to be a designated direction of the wave front, $\delta > 0$ is small, and $b_0$ is an amplitude vector-function localized around some point

$x_0 \in \mathbb{T}^3$. The incompressibility condition imposes an extra restriction on $\xi_0$ and $b_0$, namely $\xi_0 \perp b_0$.

As $x_0$ evolves in time along its Lagrangian trajectory and $\xi_0$ evolves accordingly, we expect the amplitude $b_0$ to grow exponentially fast, so that the resulting solution of the linearized equation (7) is unstable. Moreover, we hope that the quantitative measure of the growth expressed by the limit

$$\lim_{t\to\infty} \frac{1}{t} \log |b(t)|$$

will determine certain structural features of the essential spectrum, in particular its radius.

Since the exact solution of the perturbation problem is unknown, we write $v(x,t)$ in the form of a formal WKB-type expansion in powers of $\delta$,

$$v(x,t) = b(x,\xi_0,t)e^{iS(x,\xi_0,t)/\delta} + ...$$

Substituting it into equation (7) we read off the evolution laws for $b$ and $\xi = \nabla S$. Details of these calculations can be found in [19]. Thus, for $b = b(t,\varphi_t(x_0),\xi_0,b_0)$ and $\xi = \xi(t,\varphi_t(x_0),\xi_0)$ written in Lagrangian coordinates, the equations are

$$\left\{ \begin{array}{lll} \dot{x} & = u(x), & x(0) = x_0 \in \mathbb{T}^3 \\ \dot{\xi} & = -\partial u^\top \xi, & \xi(0) = \xi_0 \in (\mathbb{R}^3)^* \backslash \{0\} \\ \dot{b} & = -\partial u \cdot b + 2(\partial u \cdot b, \frac{\xi}{|\xi|})\frac{\xi}{|\xi|}, & b(0) = b_0 \in \mathbb{R}^3,\ b_0 \perp \xi_0 \end{array} \right. , \tag{35}$$

Traditionally, they are referred to as a bicharacteristic-amplitude system of ODE's associated with the equilibrium $u(x)$. We also introduce the so-called fluid Lyapunov exponent of (35) by the formula

$$\mu_{\max} = \lim_{t\to\infty} \frac{1}{t} \log \sup |b(t,x_0,\xi_0,b_0)|, \tag{36}$$

where the supremum is taken over all triplets $(x_0,\xi_0,b_0)$ such that $|\xi_0| = |b_0| = 1$ and $\xi_0 \perp b_0$, and $b(t,x_0,\xi_0,b_0)$ is determined by the b-equation in (35).

In 1996 Vishik [50] gave a rigorous application of microlocal techniques, whose heuristics are described above, to prove the following important theorem.

**Theorem 4.2.** *Consider a smooth steady solution $u(x)$ of the Euler equation in three dimensions with either periodic or free boundary conditions. Suppose $\mu_{\max}$ is the fluid Lyapunov exponent* (36) *of the bicharacteristic-amplitude system* (35) *associated with the equilibrium $u(x)$. Then in the energy norm*

$$r_{\text{ess}}(e^{tL}) = e^{t\mu_{\max}}, \tag{37}$$

*for all* $t > 0$.

This theorem is proved by writing the evolution operator $e^{tL}$ as a product of a pseudodifferential operator and a shift operator along trajectories of the equilibrium flow $u(x)$. This allows the growth of $e^{tL}$ to be studied to precise exponential asymptotics.

In fact, the formula works in any dimension $n$. However, for $n = 2$ the values of $\mu_{\max}$ and $\lambda_{\max}$ are the same, as shown in [30]. Hence, (37) is in agreement with the results discussed earlier in the section.

Theorem 4.2 provides an effective tool for establishing instability in a variety of examples. We refer to [21, 22] and references therein for detailed accounts of the results. For instance, it follows that the ABC flows in certain range of parameters A, B and C are linearly unstable, as predicted by analytical and numerical studies. A sufficient condition for $\mu_{\max} > 0$ in the case of integrable flows $u \times \operatorname{curl} u = -\nabla H$ with $\nabla H \neq 0$ is given in [26].

Besides its applicability, Theorem 4.2 provides us with an important piece of information on the structure of the essential spectrum, namely

$$\sigma_{\text{ess}}(e^{tL}) \cap \{|z| = e^{t\mu_{\max}}\} \text{ is not empty.}$$

In paper [48] the same statement was proved for any Lyapunov exponent of the system (35). To put it in proper context, let us notice that the first two equations of (35) define a flow on $K = \mathbb{T}^3 \times \mathbb{PR}^2$ given by

$$(x, \xi) \to \left( \varphi_t(x), \frac{\partial \varphi_t^{-\top}(x)\xi}{|\partial \varphi_t^{-\top}(x)\xi|} \right),$$

where '$-\top$' stands for the 'inverse transpose', and $\mathbb{PR}^2$ denotes the two dimensional projective space. The corresponding mapping

$$(x_0, \xi_0, b_0) \to b(t, x_0, \xi_0, b_0)$$

generated by solutions of the third equation in (35) defines a linear cocycle

$$B_t(x_0, \xi_0)b_0 = b(t, x_0, \xi_0, b_0)$$

over this flow (see [8] for definition). Since the b-equation is homogenous on $\xi$ and smooth in both variables, so is the cocycle. Furthermore, since $K$ is compact, the classical Oseledets Multiplicative Ergodic Theorem [8, 41] applies. In our case it asserts that for almost every pair $(x_0, \xi_0)$ on $K$ and all $b_0 \perp \xi_0$ the following limit exists

$$\mu = \lim_{t \to \infty} \frac{1}{t} \log |B_t(x_0, \xi_0)b_0|. \tag{38}$$

The value of the limit is called a Lyapunov-Oseledets exponent of the cocycle $\{B_t\}$ or system (35). The largest of them can be determined from the formula introduced earlier by (36). This is a part of a more general result concerning the growth type of any continuous cocycle proved, for example, in [8].

Thus, we have following theorem [48].

**Theorem 4.3.** *Suppose $u(x)$ is a smooth equilibrium solution of the Euler equation on $\mathbb{T}^3$. Let $\mu$ be any Lyapunov-Oseledets exponent of the associated system* (35). *Then for all $t > 0$,*

$$\sigma_{\text{ess}}(e^{tL}) \cap \{|z| = e^{t\mu}\} \neq \emptyset. \tag{39}$$

4.3. **Conjectures.** By analogy with the two dimensional case, one may conjecture that

$$\sigma_{\text{ess}}(e^{tL}) = \{e^{-t\mu_{\max}} \leq |z| \leq e^{t\mu_{\max}}\}. \tag{40}$$

This amounts to proving two facts. First, that the essential spectrum has no circular gaps, and second, that it is rotationally invariant. The analysis carried out in [50] shows that the Euler semigroup behaves in a way very similar to that of the evolution Mather semigroup associated with the $b$-cocycle $B_t(x, \xi)$. It follows from de la Llave's Theorem [9], that a Mather semigroup has no circular spectral gaps when defined on a space of divergence-free vector fields. Even though it remains open whether such a property is inherited by the Euler semigroup, this at least suggests the validity of the first fact.

As to the second, unfortunately, all known mechanisms producing rotational invariance of the spectrum require presence of long periodic or non-periodic orbits in the underlying flow. In two dimensions, this is automatically guaranteed by existence of a hyperbolic point whenever $\lambda_{\max} > 0$. On the other hand, fully three dimensional equilibria may exhibit a much more complicated topology in which even periodic flows can produce positive Lyapunov exponents. Therefore, a more plausible, although wide open to date, statement would be the annular hull version of (40),

$$\bigcup_{0 \leq \theta \leq 2\pi} e^{i\theta} \sigma_{\text{ess}}(e^{tL}) = \{e^{-t\mu_{\max}} \leq |z| \leq e^{t\mu_{\max}}\}.$$

As we indicated above, the availability of long orbits is suggestive of rotational invariance. Indeed, in this special case we obtain a stronger result than in Theorem 4.3, which is stated in Theorem 4.4.

A point $x_0 \in \mathbb{T}^3$ is called *aperiodic* if its every neighborhood is crossed by an orbit of $\{\varphi_t\}$ with arbitrarily large (possibly infinite) period. A Lyapunov-Oseledets exponent $\mu$ is called *aperiodic* if the point $x_0$ in formula (38) can be chosen aperiodic.

**Theorem 4.4.** *Suppose $u(x)$ is a smooth equilibrium solution of the Euler equation on $\mathbb{T}^3$. Let $\mu$ be an aperiodic Lyapunov-Oseledets exponent of (35). Then in the energy norm,*

$$\{|z| = e^{t\mu}\} \subset \sigma_{\mathrm{ess}}(e^{tL}),\ t > 0.$$

4.4. **A new approach and further results.** The original philosophy of the method we have described manifests itself in a Fourier analysis treatment presented in [45]. The new idea proposed there is to use an exact representation of the semigroup $\{e^{tL}\}$ in the Calkin algebra in place of its approximation in the original space of operators as in [50]. By the Calkin algebra we mean the factor-space of the Banach algebra of all bounded linear operators on a given Banach space by the ideal of compact operators. So, two operators are in the same conjugacy class if and only if they differ by a compact operator.

This approach greatly simplifies the analysis. Directly from the equation (7) one deduces the following identity up to a compact perturbation,

$$e^{tL}w(x) = \sum_{q\in\mathbb{Z}^3\setminus\{0\}} B_t(\varphi_{-t}(x), q)\hat{w}(q)e^{iq\cdot\varphi_{-t}(x)}. \tag{41}$$

The right hand side is composed of the flow mapping and a pseudo-differential operator. Via a geometric argument, one can show that its norm in the Calkin algebra is comparable to $\sup_{x\in\mathbb{T}^3,|\xi|=1}\|B_t(x,\xi)\|$. With an application of Nussbaum's formula, this readily leads to a variant of the original proof of Vishik's Theorem 4.2.

The representation formula displayed in (41) takes a significantly simpler form when restricted to the space of frequencies supported in a cylinder of fixed radius and infinite in a designated direction. More precisely, for a fixed $\xi_0 \in \mathbb{R}^3$ with $|\xi_0| = 1$, and $R > 0$, let us consider the following space

$$X_{\xi_0,R} = \left\{w \in (L_2(\mathbb{T}^3))^3_{\mathrm{divfree}} : \hat{w}(k) = 0 \text{ if } k\cdot\xi_0^\perp > R\right\}.$$

The Euler semigroup restricted to this space

$$e^{tL} : X_{\xi_0,R} \mapsto (L_2(\mathbb{T}^3))^3_{\mathrm{divfree}}$$

reduces to the following up to a compact perturbation,

$$e^{tL}w(x) = B_t(\varphi_{-t}(x), \xi_0)w(\varphi_{-t}(x)). \tag{42}$$

So, $e^{tL}w(\varphi_t(x))$ is an exact solution of the $b$-equation in the system given by (35) modulo some vanishing term. This reveals the "true" asymptotic picture of the semigroup – it is nothing but a composition of a simple multiplication operator with the flow map.

Let us illustrate again how (42) can be used in proving Theorem 4.2. We choose $(x_0, \xi_0, b_0)$ as in formula (38) with $\mu = \mu_{\max}$. Let us fix a scalar function $h \in L^2(\mathbb{T}^3)$, $\|h\| = 1$, with Fourier transform supported in the ball of radius $R$, also fixed and large. Choose $h$ to be localized near $x_0$, and for small $\delta > 0$ define

$$\begin{aligned} w_\delta(x) &= \delta \nabla \times \left( i\xi_0 \times b_0 \cdot h(y) e^{i\xi_0 \cdot y/\delta} \right) \\ &= b_0 h(x) e^{i\xi_0 \cdot x/\delta} + O(\delta). \end{aligned}$$

So, $\|w\| = 1 + O(\delta)$. From the construction we see that $w_\delta \in X_{\xi_0,R}$ and as $\delta \to 0$ the sequence $w_\delta$ tends to zero weakly. Hence, for every compact operator $K$, $\|Kw_\delta\| \to 0$. Using formula (42) we deduce that

$$\|e^{tL} w_\delta\| - \|B_t(x, \xi_0) w_\delta(x)\| \to 0.$$

On the other hand,

$$\|B_t(x, \xi_0) w_\delta(x)\| = \|B_t(x, \xi_0) b_0 h(x)\| + O(\delta) \to \|B_t(x, \xi_0) b_0 h(x)\|$$

as $\delta \to 0$. Taking $h$ sufficiently concentrated at $x_0$ we can ensure the last norm to be as close to $|B_t(x_0, \xi_0) b_0|$ as we wish. This implies that the exponential growth type of the semigroup modulo any compact perturbation is at least as that of the $b$-cocycle, i.e. $\mu_{\max}$.

The representation (42) can be used to obtain the analogue of Theorems 4.2 – 4.4 in an arbitrary Sobolev space. We note that a generalization of the spectral radius formula was obtained earlier by Friedlander and Vishik [25]. They introduced the following definition of the maximal Lyapunov exponent adapted to $H_0^m(\mathbb{T}^n)$,

$$\mu_{\max}^{(m)} = \lim_{t \to \infty} \frac{1}{t} \log \sup |b(t, x_0, \xi_0, b_0)| \cdot |\xi(t, x_0, \xi_0)|^m \tag{43}$$

where, as before, the supremum is taken over all permissible initial data of (35). The formula for the essential spectral radius reads

$$r_{\text{ess}}(e^{tL}) = e^{t\mu_{\max}^{(m)}}. \tag{44}$$

In dimension $n = 2$, the product $|b||\xi|$ is conserved in time, as shown in [26]. Therefore, if the velocity smoothness $m$ is 1, then $\mu_{\max}^{(m)}$ must vanish, and according to new formula (44) the unstable essential spectrum is empty. This result is in agreement with Theorem 4.1 discussed in Section 4.1, because $L_{\text{vor}}$ defined on $L_2(\mathbb{T}^2)$ is merely a compact perturbation of a unitary operator.

We can now provide a heuristic proof of formula (44) based on our representation (42). To justify the use of (42) we note that the norm of $e^{tL}$ in the Calkin algebra is almost attained on one of the spaces $X_{\xi_0,R}$. So, let us fix a $\xi_0 \in \mathbb{R}^3$ and $R > 0$. Taking derivative of both sides of

(42) $m$ times and discarding lower order derivatives of $w$ (which only contribute compact summands), we obtain

$$\begin{aligned} \partial^{(m)} e^{tL} v(x) &= B_t(\varphi_{-t}(x), \xi_0) \otimes \\ &\otimes \underbrace{\partial \varphi_{-t}^{\top}(x) \otimes \ldots \otimes \partial \varphi_{-t}^{\top}(x)}_{m \text{ times}} \partial^{(m)} v(\varphi_{-t}(x)). \end{aligned} \tag{45}$$

Observe that $\partial\varphi_{-t}^{\top}(x) = \partial\varphi_t^{-\top}(\varphi_{-t}(x))$. So, the norm of the right hand side of (45) is equal to the product $\sup_{x\in\mathbb{T}^3} \|B_t(x,\xi_0)\| \cdot \|\partial\varphi_t^{-\top}(x)\|^m$. Taking supremum over all possible directions $\xi_0$, and recalling that $b(t, x_0, \xi_0, b_0) = B_t(x_0, \xi_0)b_0$ and $\xi(t, x_0, \xi_0) = \partial\varphi_t^{-\top}(x_0)\xi_0$ are solutions to the $b$- and $\xi$-equations respectively, we obtain (44).

In order to investigate further structural properties of the essential spectrum in the Sobolev settings, it is beneficial to interpret $\mu^{(m)}_{\max}$ as the maximal Lyapunov exponents of a suitable cocycle. This cocycle is to be defined as an $(m+1)$-fold tensor product of the $b$- and $\xi$-cocycles associated with the bicharacteristic-amplitude system (35). In mathematical terms, for two or more cocycles defined over the same base flow, their tensor product is the pointwise product of cocycle operators acting on the respective product of fibers. So, if we are to tensor multiply two cocycles $\Phi_t^{(1)}(x)$ and $\Phi_t^{(2)}(x)$ over a flow $\psi_t(x)$, we form the tensor product of the fibers on which these act, $X_x^{(1)} \otimes X_x^{(2)}$, and define new cocycle operators

$$\left(\Phi_t^{(1)} \otimes \Phi_t^{(2)}\right)(x) : X_x^{(1)} \otimes X_x^{(2)} \mapsto X_{\psi_t(x)}^{(1)} \otimes X_{\psi_t(x)}^{(2)}$$

by the rule

$$\left(\Phi_t^{(1)} \otimes \Phi_t^{(2)}\right)(x)\left(v^{(1)} \otimes v^{(2)}\right) = \left(\Phi_t^{(1)}(x)v^{(1)}\right) \otimes \left(\Phi_t^{(2)}(x)v^{(2)}\right).$$

If the tensor product of fibers is endowed with the projective or injective norm introduced by Grothendieck [27], then the corresponding operator norm is multiplicative,

$$\left\|\left(\Phi_t^{(1)} \otimes \Phi_t^{(2)}\right)(x)\right\| = \left\|\Phi_t^{(1)}(x)\right\| \cdot \left\|\Phi_t^{(2)}(x)\right\|.$$

Thus, the norm of the tensor product of the $b$-cocycle with $m$ copies of the $\xi$-cocycle (suitably extended to $K$) is exactly the expression inside the logarithm of (43). So, we see that $\mu^{(m)}_{\max}$ determines the maximal Lyapunov exponent of this cocycle.

This purely algebraic construction puts our spectral problem into a framework suitable for further investigations. In particular, we can extend Theorems 4.2 – 4.4 to any Sobolev space by taking the corresponding $(b \otimes \xi^m)$-cocycle. In this regard, it would be interesting to see if there is any connection between Lyapunov exponents of this new cocycle and its components. However, it is also interesting to pursue the subject of a product-cocycle on its own. The relevant theory of Banach spaces and operators developed in recent years might be a good source of tools for this direction (see [10]).

4.5. **In search for the spectrum of $L$.** Even though instability of the semigroup, i.e. the solutions of (7), is exactly what is needed in most applications, it is still interesting to understand the spectrum of the generator, partly because it is easier to examine numerically, and also because it produces a stronger type of instability. Indeed, if $Lv \sim \lambda v$, then $e^{tL}v \sim e^{t\lambda}v$, but not vice versa. Moreover, it is a general result valid for any $C_0$-semigroup, that

$$e^{t\sigma(L)} \subset \sigma(e^{tL})\backslash\{0\}.$$

In Section 4.1 we indicated that the opposite inclusion is valid for the 2 dimensional Euler semigroup. We say that such a semigroup has the Spectral Mapping Property, or satisfies the Spectral Mapping Theorem,

$$e^{t\sigma(L)} = \sigma(e^{tL})\backslash\{0\}. \tag{46}$$

Certain types of spectrum, such as exact eigenvalues or residual spectrum, are known to exponentiate in general, i.e. to satisfy the relation (46) automatically (see [16]). So, the only discrepancy between the sides of the identity (46) may come from the essential part of approximate point spectrum of $e^{tL}$. There are examples of semigroups, arising naturally in hyperbolic PDE's, that fail to have the Spectral Mapping Property (see [43]). And it is still completely open whether such a property is possessed by the 3 dimensional Euler. Moreover, we do not know if the identity

$$\omega(L) = s(L) \tag{47}$$

holds, which is a consequence of (46).

In the classical theory, to order to prove (46) or at least (47), one starts with a sequence $\{f_n\}$ of approximate eigenfunctions for the semigroup corresponding to an eigenvalue $e^{t\lambda}$, so that

$$\|e^{tL}f_n - e^{t\lambda}f_n\| \to 0 \text{ as } n \to \infty$$
$$\|f_n\| = 1,\ n = 1, 2, ...$$

Using the identity

$$\left(L - \lambda - \frac{2\pi k}{t}i\right)\int_0^t e^{s(L-\lambda-\frac{2\pi k}{t}i)}f ds = e^{t(L-\lambda)}f - f,$$

which is valid for all $k \in \mathbb{Z}$, one obtains

$$\left\| Lv_n - \left(\lambda + \frac{2\pi k}{t} i\right) v_n \right\| \to 0,$$

for the sequence $v_n$ defined by

$$v_n = \int_0^t e^{s(L-\lambda-\frac{2\pi k}{t}i)} f_n ds. \tag{48}$$

The problem is to show that the sequence of norms $\{\|v_n\|\}$ or its subsequence stays away from zero at least for one $k \in \mathbb{Z}$. Some assumptions on the semigroup such as uniform continuity are sufficient to ensure this condition. For those semigroups the Spectral Mapping Theorem is well-known [16].

Let us now go back to the Euler semigroup. All known approximate eigenfunctions for $e^{tL}$ have the form

$$f_\delta = h(x) e^{i\xi_0 \cdot x/\delta} + O(\delta),$$

where $h(x)$ may vary slowly with the $\delta$. So, we can roughly assume that the sequence $f_\delta$ is in one of the spaces $X_{\xi_0,R}$. According to the representation (42), we must have

$$e^{tL} f_\delta(x) = B_t(\varphi_{-t}(x), \xi_0) h(\varphi_{-t}(x)) e^{i\xi_0 \cdot \varphi_{-t}(x)/\delta} + O(\delta).$$

Substituting this into (48) yields

$$v_\delta(x) = \int_0^t e^{-s(\lambda+\frac{2\pi k}{t}i)} e^{i\xi_0\cdot\varphi_{-s}(x)/\delta} B_s(\varphi_{-s}(x), \xi_0) h(\varphi_{-s}(x)) ds.$$

We see that if the term $\xi_0 \cdot \varphi_{-s}(x)$ changes substantially with the variable of integration, then the corresponding exponent exhibits fast oscillations inside the integral. So, by the Lebesgue lemma, $v_\delta \to 0$ pointwise, and hence, in the energy norm for every $k$. This is why the WKB-method fails to work for $L$, at least in the way we just described: the highly oscillating components of the eigenfunctions essentially commute with the action of the semigroup, and make the truncated Laplace transform (48) vanish.

In the opposite case when $\xi_0 \cdot \varphi_{-s}(x)$ is almost constant we have $\xi_0 \cdot \varphi_{-s}(x) \sim \xi_0 \cdot x$. Expanding it into Taylor series as a function of $s$, we estimate

$$\xi_0 \cdot \varphi_{-s}(x) \sim \xi_0 \cdot x - s\, \xi_0 \cdot u(x).$$

So, to balance out this with the previous approximation the last term must vanish, and we arrive at the condition $\xi_0 \perp u(x)$. Under this special orthogonality assumption, the spectral problem for $L$ has been treated in a recent work of Latushkin and Vishik [30]. The authors give an explicit construction of a weakly-null sequence of approximate eigenfunctions for $L$, and prove the following theorem.

**Theorem 4.5.** *Let $u(x)$ be a smooth equilibrium solution of the Euler equation on $\mathbb{T}^3$. Suppose that $\mu_{\max}$ is attained at $(x_0, \xi_0, b_0)$ such that $\xi_0 \perp u(x_0)$. Then*

$$\omega(L) = s(L).$$

The full Spectral Mapping Theorem for 3-dimensional Euler remains an open problem even in this special case.

## 5. Nonlinear instability

As we have discussed in the previous sections there many classes of steady Euler flows that are linearly unstable, either due to a nonempty unstable essential spectrum or due to unstable eigenvalues or possible for both reasons. It is natural to ask what this means about the stability/instability of the full nonlinear Euler equations. The issue of nonlinear instability is closely linked with the spectral properties of the linearized equations. However, the problems are complex and subtle and lie outside the scope of this short survey paper concerning results to date on the structure of the spectrum of the linearized Euler operator. To illustrate how special properties can be used to prove nonlinear instability we state several recent results that give partial information about nonlinear Euler instability and refer the reader to more details given in [4, 21, 22, 35, 51, 54, 55] and the references therein.

There are a number of definitions of nonlinear stability and its converse instability. These definitions are related but not always equivalent. We stress the crucial importance of the norm in which instability is measured. Several of the results concerning nonlinear instability are based on an approach using two Banach spaces: a "large" space $Z$ where the spectrum of the linearized operator is studied and a "small" space $X \hookrightarrow Z$ where a local in time existence theorem for the nonlinear Euler equation can be proved. In [23] it is proved that linear instability in $L^2$ implies nonlinear instability in $H^s$, $s > \frac{n}{2} + 1$ ($n$ being the space dimension) provided the spectrum of $e^{tL}$ has a suitable circular gap. However, as we have discussed in Section 4, the essential spectrum is unlikely to have such a gap unless there exists an eigenvalue outside the essential spectral radius (e.g. unstable shear flows). Furthermore, as Yudovich [55] remarks, the $H^s$ definition of instability is probably too strong to appropriately represent hydrodynamic transitions from stability to instability.

In the case of 2D flows Bardos, Guo, and Strauss [4] prove that any flow $u(x)$, for which $L$ has an unstable eigenvalue whose real part is greater than the maximal Lyapunov exponent $\lambda_m$, is nonlinearly unstable with respect to growth of the *vorticity* in $L^2$. This result was extended by Vishik and Friedlander [51] to prove, under the same spectral condition, that such a flow is nonlinearly unstable with respect to growth of the *velocity* in $L^2$. We remark that instability as measured by growth in the energy seems the most physically reasonable concept. When there is no dominant eigenvalue, technical difficulties have, at least to date, precluded a proof of the general result in 2D that linear instability implies nonlinear instability.

We conclude with the observation that the complexity of flows in 3D suggests that most, if not all, such inviscid flows are nonlinearly unstable. As we discussed in Section 4, Lyapunov exponents and stretching on the Lagrangian trajectories are closely connected with instabilities and there are more mechanisms for stretching in 3D as opposed to 2D. However, the challenging issues of nonlinear instability for fully 3D flows are almost completely open.

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