MATH 531 **Notes on Simple $sl(2, F)$-modules** **Radford 11/09/07**

Throughout F is an algebraically closed field of characteristic zero. Let $\lambda \in F$ and $\mathcal{Z}(\lambda)$ be the vector space over F with basis $\{v_\ell\}_{\ell \in \mathbf{Z}}$. We define endomorphisms $\mathbf{x}, \mathbf{y}, \mathbf{h}$ of $\mathcal{Z}(\lambda)$ by

$$\mathbf{x}(v_\ell) = (\lambda - \ell + 1)v_{\ell-1}, \quad \mathbf{y}(v_\ell) = (\ell + 1)v_{\ell+1}, \quad \text{and} \quad \mathbf{h}(v_\ell) = (\lambda - 2\ell)v_\ell$$

for all $\ell \in \mathbf{Z}$. Let $\mathcal{L} = gl(\mathcal{Z}(\lambda))$.

Lemma 1 *The span L of $\{\mathbf{x}, \mathbf{y}, \mathbf{h}\}$ is $sl(2, F)$; indeed $[\mathbf{x}\ \mathbf{y}] = \mathbf{h}$, $[\mathbf{h}\ \mathbf{x}] = 2\mathbf{x}$, $[\mathbf{h}\ \mathbf{y}] = -2\mathbf{y}$.*

Proof: Suppose that $a\mathbf{x} + b\mathbf{y} + c\mathbf{z} = 0$, where $a, b, c \in F$. Applying both sides of this equation to v_ℓ we have

$$a(\lambda + \ell - 1)v_{\ell-1} + b(\ell + 1)v_{\ell+1} + c(\lambda - 2\ell)v_\ell = 0.$$

Therefore $a(\lambda + \ell - 1) = 0$, $b(\ell + 1) = 0$, and $c(\lambda - 2\ell) = 0$ for all $\ell \in \mathbf{Z}$. This means $a = b = c = 0$. Thus $\{\mathbf{x}, \mathbf{y}, \mathbf{h}\}$ is a basis for L.

We will show that $[\mathbf{x}\ \mathbf{y}] = \mathbf{h}$ and leave the verification of the two remaining equations to the reader. Suppose $\ell \in \mathbf{Z}$. Then

$$\begin{aligned}
[\mathbf{x}\ \mathbf{y}](v_\ell) &= \mathbf{x}(\mathbf{y}(v_\ell)) - \mathbf{y}(\mathbf{x}(v_\ell)) \\
&= \mathbf{x}((\ell + 1)v_{\ell+1}) - \mathbf{y}((\lambda - \ell + 1)v_{\ell-1}) \\
&= (\lambda - (\ell + 1) + 1)(\ell + 1)v_\ell - \ell(\lambda - \ell + 1)v_\ell \\
&= [\lambda(\ell + 1) - \ell\lambda - \ell(\ell + 1) + \ell(\ell - 1)]v_\ell \\
&= (\lambda - 2\ell)v_\ell \\
&= \mathbf{h}(v_\ell)
\end{aligned}$$

from which $[\mathbf{x}\ \mathbf{y}] = \mathbf{h}$ follows. $\square$

Regard $\mathcal{Z}(\lambda)$ as a left $gl(\mathcal{Z}(\lambda))$-module by $\ell \cdot v = \ell(v)$ for all $\ell \in gl(\mathcal{Z}(\lambda))$ and $v \in \mathcal{Z}(\lambda)$. Then the span $\mathcal{Y}(\lambda)$ of $\{v_\ell\}_{\ell \leq -1}$ is a submodule of $\mathcal{Z}(\lambda)$. The critical calculation for seeing this is $\mathbf{y} \cdot v_{-1} = \mathbf{y}(v_{-1}) = (-1 + 1)v_0 = 0$.

Let $Z(\lambda) = \mathcal{Z}(\lambda)/\mathcal{Y}(\lambda)$. Identifying cosets with representatives $Z(\lambda)$ has basis $\{v_\ell\}_{\ell \geq 0}$ and

$$\mathbf{x} \cdot v_\ell = (\lambda - \ell + 1)v_{\ell-1}, \quad \mathbf{y} \cdot v_\ell = (\ell + 1)v_{\ell+1}, \quad \mathbf{h} \cdot v_\ell = (\lambda - 2\ell)v_\ell$$

for all $\ell \geq 0$. By convention $v_{-1} = 0$.

Proposition 1 *Let $\lambda \in F$. Then:*

(a) *If $\lambda \notin \{0, 1, 2, 3, \ldots\}$ then $Z(\lambda)$ is a simple left $sl(2, F)$-module.*

(b) *Suppose that $\lambda = m$ is a non-negative integer. Then $Z(\lambda)$ has a unique proper submodule $Y(\lambda)$ which is the span of $\{v_\ell\}_{\ell \geq m+1}$.*

(c) *$V(m) = Z(m)/Y(m)$ is a simple $sl(2, F)$-module of dimension $m + 1$.*

Proof: A proof can be based on the following lemma, where $T = \boldsymbol{h}\cdot$ is left multiplication by $\boldsymbol{h}$. The details are left as a good exercise for the reader. □

Lemma 2 *Let $T : V \longrightarrow V$ be a linear endomorphism of a vector space V over any field F and suppose that V is the sum of eigenspaces of T. Suppose that W is a non-zero subspace of V and $T(W) \subseteq W$. Then W is the (direct) sum of eigenspaces of $T|_W$.*

Proof: The sum of eigenspaces is direct. Let $0 \neq w \in W$. By assumption $w = v_1 + \cdots + v_n$, where $v_1, \ldots, v_n$ are eigenvectors of T, which we may assume belong to distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $V' = Fv_1 + \cdots + Fv_n$. Then $T(V') = T(Fv_1) + \cdots + T(Fv_n) \subseteq Fv_1 + \cdots + Fv_n = V'$. Set $f(x) = (x - \lambda_1) \cdots (x - \lambda_n)$. Then $f(T|_{V'}) = 0$, and thus $f(T|_{V' \cap W}) = 0$ which means that $T|_{V' \cap W}$ is a diagonalizable endomorphism of $V' \cap W$. Since $w \in V' \cap W$, it follows that $V' \cap W \neq (0)$ and is therefore a sum of eigenspaces of $T|_{V' \cap W}$. We have shown that W is the sum of eigenspaces of $T|_W$. □