MATH 531 **Written Homework 4 Solution** **Radford** **11/21/07**

In this exercise set we begin a rather detailed study of $sl(n, F)$.

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1. (**20 points**) For all $1 \leq i, j, k, \ell \leq n$ note $e_{i\,j}e_{k\,\ell} = \delta_{j,k}e_{i\,\ell}$; thus $[e_{i\,j}\,e_{k\,\ell}] = \delta_{j,k}e_{i\,\ell} - \delta_{\ell,i}e_{k\,j}$.

(a) (**5**) This is boring but necessary.

$$\begin{aligned}
(\mathrm{ad}\,e_{i\,j}\circ\mathrm{ad}\,e_{k\,\ell})(e_{u\,v}) &= [e_{i\,j}\,[e_{k\,\ell}\,e_{u\,v}]] \\
&= [e_{i\,j}\,(\delta_{\ell,u}e_{k\,v} - \delta_{v,\ell}e_{u\,\ell})] \\
&= \delta_{\ell,u}[e_{i\,j}\,e_{k\,v}] - \delta_{v,\ell}[e_{i\,j}\,e_{u\,\ell})] \\
&= \delta_{\ell,u}(\delta_{j,k}e_{i\,v} - \delta_{v,i}e_{k\,j}) - \delta_{v,\ell}(\delta_{j,u}e_{i\,\ell} - \delta_{\ell,i}e_{u\,j}) \\
&= \delta_{\ell,u}\delta_{j,k}e_{i\,v} - \delta_{\ell,u}\delta_{i,v}e_{k\,j} - \delta_{k,v}\delta_{j,u}e_{i\,\ell} + \delta_{k,v}\delta_{i,\ell}e_{u\,j}.
\end{aligned}$$

(b) (**5**) In the expression of part (a), the contribution of $ae_{k\,\ell}$, where $a \in F$, to the coefficient of $e_{u\,v}$ is $\delta_{k,u}\delta_{\ell,v}a$. Thus the answer is:

$$\begin{aligned}
&\delta_{i,u}\delta_{v,v}\delta_{\ell,u}\delta_{j,k} - \delta_{k,u}\delta_{j,v}\delta_{\ell,u}\delta_{i,v} - \delta_{i,u}\delta_{\ell,v}\delta_{k,v}\delta_{j,u} + \delta_{u,u}\delta_{j,v}\delta_{k,v}\delta_{i,\ell} \\
&= \delta_{\ell,u}\delta_{j,k}\delta_{i,u} - \delta_{\ell,u}\delta_{i,v}\delta_{k,u}\delta_{j,v} - \delta_{k,v}\delta_{j,u}\delta_{i,u}\delta_{\ell,v} + \delta_{k,v}\delta_{i,\ell}\delta_{j,v}.
\end{aligned}$$

(c) (**5**) Fix $e_{i\,j}, e_{k\,\ell}$. For $e_{u\,v}$ write $(\mathrm{ad}\,e_{i\,j}\circ\mathrm{ad}\,e_{k\,\ell})(e_{u\,v}) = \sum_{1\leq x,y\leq n}\alpha_{(x,y),\,(u,v)}e_{x\,y}$. In light of part (b)

$$\begin{aligned}
\kappa(e_{i\,j}, e_{k\,\ell}) &= \mathsf{Tr}(\mathrm{ad}\,e_{i\,j}\circ\mathrm{ad}\,e_{k\,\ell}) \\
&= \sum_{1\leq u,v\leq n}\alpha_{(u,v),\,(u,v)} \\
&= \sum_{1\leq u,v\leq n}\delta_{\ell,u}\delta_{j,k}\delta_{i,u} - \sum_{1\leq u,v\leq n}\delta_{\ell,u}\delta_{i,v}\delta_{k,u}\delta_{j,v} - \sum_{1\leq u,v\leq n}\delta_{k,v}\delta_{j,u}\delta_{i,u}\delta_{\ell,v} + \sum_{1\leq u,v\leq n}\delta_{k,v}\delta_{i,\ell}\delta_{j,v} \\
&= \sum_{1\leq v\leq n}\delta_{\ell,\ell}\delta_{j,k}\delta_{i,\ell} - \delta_{\ell,\ell}\delta_{i,i}\delta_{k,\ell}\delta_{j,i} - \delta_{k,k}\delta_{j,j}\delta_{i,j}\delta_{\ell,k} + \sum_{1\leq u\leq n}\delta_{k,j}\delta_{i,\ell}\delta_{j,j} \\
&= 2n\delta_{i,\ell}\delta_{j,k} - 2\delta_{i,j}\delta_{k,\ell}.
\end{aligned}$$

(d) (**5**) Let $x = \sum_{1\leq i,j\leq n}a_{i,j}e_{i\,j} \in L$. By part (b), $x \in Rad\,\kappa$ if and only if for all $1 \leq k, \ell \leq n$,

$$0 = \kappa(x, e_{k\,\ell}) = \sum_{1\leq i,j\leq n}a_{i\,j}\kappa(e_{i\,j}, e_{k\,\ell}),$$

or equivalently

$$\sum_{1\leq i,j\leq n} a_{i,j}(2n\delta_{i,\ell}\delta_{j,k} - 2\delta_{i,j}\delta_{k,\ell}) = 0,$$

or equivalently

$$2na_{\ell,k} = 2\left(\sum_{1\leq i\leq n} a_{i,i}\right)\delta_{k,\ell},$$

or equivalently

$$a_{\ell,k} = \delta_{k,\ell}\left(\frac{1}{n}\left(\sum_{1\leq i\leq n} a_{i,i}\right)\right). \tag{1}$$

Thus $x = \sum_{1\leq i\leq n} a_{i,i}e_{i\,i} = a_{1,1}\mathrm{I}_n$. Conversely, if $x = a\mathrm{I}_n = \sum_{1\leq i\leq n} ae_{i\,i}$ for some $a \in F$, then (1) holds, so $x \in Rad\,\kappa$.

2. (**25 points**) Let $\kappa' = \kappa_{gl(n,F)}$. Since $L = sl(n,F)$ is an ideal of $gl(n,F)$ it follows that $\kappa = \kappa'|_{L\times L}$. Now $gl(n,F) = sl(n,F)\oplus FI_n$.

To show that κ is non-degenerate. Let $x \in \mathrm{Rad}\kappa$ and assume that $\kappa(x,y) = 0$ for all $y \in L$. Since $gl(n,F) = L\oplus F\mathrm{I}_n$, any element of $gl(n,F)$ an be written $y + a\mathrm{I}_n$ for some $y \in L$ and $a \in F$. Since $\mathrm{Rad}\kappa' = F\mathrm{I}_n$ by part (d) of Problem 1, the calculation

$$0 = \kappa(x,y) = \kappa'(x,y) = \kappa'(x,y) + \kappa'(x,a\mathrm{I}_n) = \kappa'(x,y+a\mathrm{I}_n)$$

shows that $x \in \mathrm{Rad}\,\kappa'$ (**10**). Thus $x \in L\cap F\mathrm{I}_n = (0)$ which means that $x = 0$. Therefore κ is non-singular (**15**).

3. (**30 points**) Let $\mathbf{\Phi}$ be the set of $\alpha_{k\,\ell}$'s and $h = \sum_{i=1}^{n}\lambda_i e_{i\,i}, \in H$. For $h' \in H$ note that $[h\,h'] = 0$ since both h, h' are diagonal matrices. Thus H is a subalgebra (abelian) of L.

Suppose that $1 \leq k,\ell \leq n$ and are distinct. Then

$$[h\,e_{k\,\ell}] = \sum_{i=1}^{n}\lambda_i[e_{i\,i}\,e_{k\,\ell}] = \lambda_k e_{k\,k}e_{k\,\ell} - \lambda_\ell e_{k\,\ell}e_{\ell\,\ell} = (\lambda_k - \lambda_\ell)e_{k\,\ell} = \alpha_{k\,\ell}(h)e_{k\,\ell}.$$

Since $L = H\oplus(\bigoplus_{1\leq k,\ell\leq n,\ k\neq\ell} Fe_{k\,\ell})$, we have shown that $\mathrm{ad}\,h$ is a diagonalizable operator (thus H is a toral subalgebra of L) and the summand $Fe_{k\,\ell} \subseteq L_{\alpha_{k\,\ell}}$.

It is left as a small exercise to show that $\alpha_{k\,\ell} = \alpha_{k'\,\ell'}$ if and only if $k = k'$ and $\ell = \ell'$. Since

$$H\oplus(\bigoplus_{1\leq k,\ell\leq n,\ k\neq\ell} Fe_{k\,\ell}) = L = \bigoplus_{\alpha\in H^*} L_\alpha, \tag{2}$$

and the summands on the left are subspaces of the summands on the right, the summands on the left are the non-zero summands on the right. In particular $H = L_0$. If H' is a toral subalgebra and $H \subseteq H'$ then $[h\,h'] = 0$ for all $h \in H$ and $h' \in H'$ which implies $h' \in L_0$. Therefore $H = H'$ from which we conclude that (a) H is a *maximal* toral subalgebra of L (**10**). From our comments about the summands of (2) it now follows that (b) $\mathbf{\Phi}$ is the root system of L relative to H (**10**) and (c) $L_{\alpha_{k\,\ell}} = Fe_{k\,\ell}$ for all $1 \leq k,\ell \leq n$ and $k \neq \ell$ (**10**).

4. (**25 points**) Let $\alpha \in \Phi$. Then $\alpha = \alpha_{k\,\ell}$ for some distinct $1 \leq k, \ell \leq n$. Since

$$-\alpha_{k\,\ell} = \alpha_{\ell\,k} \tag{3}$$

we deduce deduce that $S_{\alpha_{k\,\ell}} = Fe_{\ell\,k} \oplus Ft_{\alpha_{k\,\ell}} \oplus Fe_{k\,\ell}$ by Exercise 3. Note that $e_{k\,\ell}$ and $e_{\ell\,k}$ generate $S_{\alpha_{k\,\ell}}$ as a Lie algebra.

Let $\beta \in \Phi$ and suppose that $\beta \neq \pm\alpha_{k\,\ell}$. Then $\beta = \alpha_{u\,v}$, or $\alpha_{u\,\ell}$, or or $\alpha_{u\,k}$, or $\alpha_{k\,v}$, or $\alpha_{\ell\,v}$, where $u, v \not\in \{k, \ell\}$, by (3) and Exercise 3. The α-string $\beta + (-r)\alpha, \ldots, \beta + q\alpha$ through β accounts for a simple $sl(2, F) = S_\alpha$-module $V = L_{\beta - r\alpha} \oplus \cdots \oplus L_{\beta + q\alpha}$ which is therefore generated by $e_{r\,s}$, indeed any non-zero $v \in V$, where $\beta = \alpha_{r\,s}$. Also note that if U is any sl$(2, F)$-module then a subspace U' of U is a submodule if and only if $x{\cdot}U', y{\cdot}U' \subseteq U'$ as then $h{\cdot}U' = [x\ y]{\cdot}U' \subseteq x({\cdot}y{\cdot}U') + y({\cdot}x{\cdot}U') \subseteq U'$ in this case.

Case 1: (**5**) $\beta = \alpha_{u,v}$. Since $[e_{k\,\ell}\ e_{u\,v}] = 0 = [e_{\ell\,k}\ e_{u\,v}]$ it follows that $S_{\alpha_{k\,\ell}}$ acts trivially on $Fe_{u\,v}$. Therefore $V = Fe_{u\,v}$ which means that the $\alpha_{k\,\ell}$-string through β is β ($r = q = 0$ here.)

Case 2: (**5**) $\beta = \alpha_{u\,\ell}$. Since $[e_{k\,\ell}\ e_{u\,\ell}] = 0$, $[e_{\ell\,k}\ e_{u\,\ell}] = -e_{u\,k}$, $[e_{k\,\ell}\ e_{u\,k}] = -e_{u\,\ell}$, and $[e_{\ell\,k}\ e_{u\,k}] = 0$, it follows that $V = Fe_{u\,k} \oplus Fe_{u\,\ell} = L_{\alpha_{u\,k}} \oplus L_{\alpha_{u\,\ell}}$. Since $\alpha_{u\,k} + \alpha_{k\,\ell} = \alpha_{u\,\ell}$ the α-root string through β is $\beta - \alpha, \beta$. ($r = 1, q = 0$ here.)

Case 3: (**5**) $\beta = \alpha_{k\,v}$. Since $[e_{k\,\ell}\ e_{k\,v}] = 0$, $[e_{\ell\,k}\ e_{k\,v}] = e_{\ell\,v}$, $[e_{k\,\ell}\ e_{\ell\,v}] = e_{k\,v}$, and $[e_{\ell\,k}\ e_{\ell\,v}] = 0$, it follows that $V = Fe_{\ell\,v} \oplus Fe_{k\,v} = L_{\alpha_{\ell\,v}} \oplus L_{\alpha_{k\,v}}$. Since $\alpha_{k\,\ell} + \alpha_{\ell\,v} = \alpha_{k\,v}$ the α-root string through β is $\beta - \alpha, \beta$. ($r = 1, q = 0$ here.)

Case 4: (**5**) $\beta = \alpha_{u\,k} = -\alpha_{k\,u}$. Here $V = Fe_{u\,k} \oplus Fe_{u\,\ell}$ as in Case 2. Since $\alpha_{u\,k} + \alpha_{k\,\ell} = \alpha_{u\,\ell}$ the the α-root string through β is $\beta, \beta + \alpha$. ($r = 0, q = 1$ here.)

Case 5: (**5**) $\beta = \alpha_{\ell\,v} = -\alpha_{v\,\ell}$. Here $V = Fe_{\ell\,v} \oplus Fe_{k\,v} = L_{\alpha_{\ell\,v}} \oplus L_{\alpha_{k\,v}}$ as in Case 3. Since $\alpha_{k\,\ell} + \alpha_{\ell\,v} = \alpha_{k\,v}$ the α-root string through β is $\beta, \beta + \alpha$. ($r = 0, q = 1$ here.)

Comment: The hint was meant to lead you on a stroll through the proof involved in determining strings. Many of you used the fact that $\alpha_{i\,j} + \alpha_{k\,\ell}$ is a root if and only if $j = k$ or $i = \ell$ instead. This really required proof, which should have been given. (See §8.4 Proposition (d) for example.)

Cases 4 and 5 fall out quickly from Cases 3 and 2 respectively. Note that if

$$\beta + (-r)\alpha, \ldots, \beta + q\alpha$$

is the α-string through β then

$$-\beta + (-q)\alpha, \ldots, -\beta + r\alpha$$

is the α-string through $-\beta$ as $\gamma \in \Phi$ if and only if $-\gamma \in \Phi$.