MATH 531 Written Homework 3 Radford DUE FRIDAY 10/26/07

In the following exercises F is a field. We follow the notation of the text and that used in class.

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1. Let $L_1, \ldots, L_r$ be Lie algebras over the field F, let $L = L_1 \oplus \cdots \oplus L_r$ be their vector space direct sum, and for each $1 \leq i \leq r$ let $\pi_i : L \longrightarrow L_i$ be the linear map defined by $\pi(\ell_1 \oplus \cdots \oplus \ell_r) = \ell_i$.

(a) Show that L is a Lie algebra over F, where

$$[\ell_1 \oplus \cdots \oplus \ell_r \, \ell'_1 \oplus \cdots \oplus \ell'_r] = [\ell_1 \, \ell'_1] \oplus \cdots \oplus [\ell_r \, \ell'_r].$$

You may assume that this product gives L an algebra structure over F.

(b) Show that $\pi_i : L \longrightarrow L_i$ is a map of Lie algebras for all $1 \leq i \leq r$.

(c) Suppose that L' is a Lie algebra over F and $\pi'_i : L' \longrightarrow L_i$ is a Lie algebra for all $1 \leq i \leq r$. Show that there is one and only one map of Lie algebras $\pi : L' \longrightarrow L$ which satisfies $\pi_i \circ \pi = \pi'_i$ for all $1 \leq i \leq r$. (Thus the system $(L, \{\pi_i\}_{1 \leq i \leq r})$ is product in the category of Lie algebras and Lie algebra maps.)

2. Let L be a finite-dimensional Lie algebra over F and let $\kappa : L \times L \longrightarrow F$ be the killing form of L.

(a) Find the matrix $\begin{pmatrix} \kappa(x,x) & \kappa(x,y) \\ \kappa(y,x) & \kappa(y,y) \end{pmatrix}$, where L is the Lie algebra over F with basis $\{x, y\}$ whose multiplication is determined by $[x\,y] = y$, and find $\mathrm{Rad}\, L$.

(b) Find the matrix $\begin{pmatrix} \kappa(x,x) & \kappa(x,y) & \kappa(x,z) \\ \kappa(y,x) & \kappa(y,y) & \kappa(y,z) \\ \kappa(z,x) & \kappa(z,y) & \kappa(z,z) \end{pmatrix}$, where L is the Lie algebra over F with basis $\{x, y, z\}$ whose multiplication is determined by $[x\,y] = cz$, $[y\,z] = ax$, and $[z\,x] = by$ for some $a, b, c \in F$, and find a basis for $\mathrm{Rad}\, L$.

3. Suppose that the characteristic of F is not 2, $n \geq 2$, and regard $L = sl(2, F)$ as a subalgebra of $gl(n, F)$ with the identification $x = e_{12}$, $y = e_{21}$, and $z = e_{11} - e_{22}$. Let L act on $V = gl(n, F)$ by the adjoint action; that is $\ell \cdot v = [\ell\, v]$ for all $\ell \in L$ and $v \in V$.

(a) Write V as a direct sum of simple L-modules.

(b) Determine the weight spaces, corresponding weights, and a maximal vector for each summand.

4. Let $A = F[x, y]$ be the algebra over polynomials in indeterminates x and y over F.

(a) Show that:

$$\boldsymbol{x} = \ell_x \circ \frac{\partial}{\partial y}, \qquad \boldsymbol{y} = \ell_y \circ \frac{\partial}{\partial x}, \qquad \text{and} \qquad \boldsymbol{z} = [\boldsymbol{x}, \boldsymbol{y}]$$

are derivations of A.

(b) Show that $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ is linearly independent, that

$$[\boldsymbol{z}, \boldsymbol{x}] = 2\boldsymbol{x}, \qquad \text{and that} \qquad [\boldsymbol{z}, \boldsymbol{y}] = -2\boldsymbol{y}.$$

Thus the Lie subalgebra L of $\mathrm{Der}(A)$ is isomorphic to $sl(2, F)$. [Hint: To establish the first equation, consider effect of the derivations $[\boldsymbol{z}, \boldsymbol{x}]$ and $2\boldsymbol{x}$ on algebra generators x, y.]

Let L act on A according by $D{\cdot}v = D(v)$ for all $D \in L$ and $v \in V$. For each $n \geq 0$ let V_n be the span of the monomials $X^\ell Y^{n-\ell}$, where $0 \leq \ell \leq n$. Observe that $\mathrm{Dim} V_n = n + 1$ and $A = \bigoplus_{n=0}^{\infty} V_n$.

(c) Show that V_n is a simple $L = sl(2, F)$-module for all $n \geq 0$. Determine the weight spaces, corresponding weights, and a maximal vector for each V_n.