Math 516 **Fall 2008** **Radford**

Written Homework # 5 Comments

12/02/08

Some comments, most of which are not completely new.

1. (b) Let R be a ring and I, J ideals of R. The additive analog of internal direct product is internal direct sum.

The notation $R = I \oplus J$ means that R is the internal direct sum of additive groups; that is $R = I + J$ and $I \cap J = (0)$. Note for $a \in I$ and $b \in J$ that $ab, ba \in I \cap J = (0)$; thus $ab = 0 = ba$. Now it east to see that the isomorphism of additive groups $I \times J \longrightarrow R$ given by $(a, b) \mapsto a + b$ is in fact a ring isomorphism.

2. Let A be an additive group. Then multiple is the analog of power in a multiplicative group. Thus for $a \in A$ and $n \in \mathbf{Z}$

$$n \cdot a = \begin{cases} 0 & : \quad n = 0 \\ (n-1) \cdot a + a & : \quad n > 0 \\ -((-n) \cdot a) & : \quad n < 0 \end{cases}$$

4. (b) We have seen products of two groups. There is an natural generalization to product of any number of groups, technically any indexed set of groups. Of relevance here is the analog for modules.

Suppose R is a ring, $I \neq \emptyset$, and $\{M_i\}_{i \in I}$ is an indexed family of left R-modules. A product of the family is a pair $(\{\pi_i\}_{i \in I}, P)$,

- where P is a left R-module and $\pi_i : P \longrightarrow M_i$ is a module homomorphism for all $i \in I$,

such that for any other pair $(\{\pi'_i\}_{i \in I}, P')$,

- where P' is a left R-module and $\pi'_i : P' \longrightarrow M_i$ is a module homomorphism for all $i \in I$,

there is a homomorphism of left R-modules $F : P' \longrightarrow P$ determined by $\pi_i \circ F = \pi'_i$ for all $i \in I$.

5. The notion of direct sum, or "coproduct", is "dual" to that of product in the sense that the arrows are formally reversed in its definition.

Suppose R is a ring, $I \neq \emptyset$, and $\{M_i\}_{i \in I}$ is an indexed family of left R-modules. A direct sum (coproduct) of the family is a pair $(\{\jmath_i\}_{i \in I}, C)$,

- where C is a left R-module and $\jmath_i : M_i \longrightarrow C$ is a module homomorphism for all $i \in I$,

such that for any other pair $(\{\jmath'_i\}_{i \in I}, C')$,

- where C' is a left R-module and $\jmath'_i : M_i \longrightarrow C'$ is a module homomorphism for all $i \in I$,

there is a homomorphism of left R-modules $F : C \longrightarrow C'$ determined by $F \circ \jmath_i = \jmath'_i$ for all $i \in I$.