Math 516 **Fall 2008** **Radford**

Written Homework # 4

Due at the beginning of class 11/14/08

Throughout R is a ring with unity $1 \neq 0$ and F is a field.

---

1. Let G be an additive group and S be a non-empty set. Then $\mathcal{G} = \mathrm{Fun}(S, G)$, the set of all functions $f : S \longrightarrow G$, is an additive group, where $(f + g)(s) = f(s) + g(s)$ for all $f, g \in \mathcal{G}$ and $s \in S$. Let $\mathrm{A}(S, G)$ be the subset of $\mathcal{G}$ consisting of all $f \in \mathcal{G}$ which satisfy $f(s) = 0$ for all but finitely many $s \in S$.

   (a) Show that $\mathrm{A}(S, G) \leq \mathcal{G}$.

   (b) Let $\imath : S \longrightarrow \mathrm{A}(S, \mathbf{Z})$ be the function defined by $\boxed{\imath(s)(s')} = \begin{cases} 1 & : \ s' = s; \\ 0 & : \ s' \neq s \end{cases}$. Show that $(\imath, \mathrm{A}(S, \mathbf{Z}))$ is a free abelian group on S.

2. Let $R^\times$ be the $\boxed{\text{group}}$ of units of R.

   (a) Suppose that $a \in R$ and $\{1, a, a^2, a^3, \ldots\}$ is a finite set. Show that $a \in R^\times$ or $ab = 0 = ba$ for some non-zero $b \in R$.

   (b) Suppose that R is finite. Show that any element of R is a unit or is a zero divisor.

   (c) Let $R = \mathrm{M}_n(F)$, where $n \geq 1$, and let $a \in R$. Show that Show that $a \in R^\times$ or $ab = 0 = ba$ for some non-zero $b \in R$. [Hint: Is $\{1, a, a^2, a^3, \ldots\}$ linearly independent?]

3. You may assume that if $a, b \in R$ commute then the binomial theorem holds for them; that is $(a + b)^n = \sum_{\ell=0}^{n} \binom{n}{\ell} a^{n-\ell} b^\ell$ for all $n \geq 0$. Also, you may assume the exponent laws.

   (a) Suppose $a, b \in R$ are nilpotent and $ab = ba$. Show that $a \pm b$ is nilpotent.

   (b) Suppose $a, r \in R$, where a is nilpotent and $ar = ra$. Show that ar is nilpotent.

   (c) Suppose that R is commutative and N is the set of nilpotent elements of R. Show that N is an ideal of R.

   (d) Suppose $R = \mathrm{M}_2(F)$. Find nilpotent $a, b \in R$ such that $a + b$ and ab are not nilpotent. Justify your answer.

4. Let $F((x)) = \{\sum_{n \geq N} a_n x^n \mid N \in \mathbf{Z},\ a_n \in F \ \ \forall\, n \geq N\}$ be the (commutative) ring of formal Laurent series with coefficients in F. Writing elements of $F((x))$ as $\sum a_n x^n$ we have

$$\sum a_n x^n + \sum b_n x^n = \sum (a_n + b_n) x^n \ \text{ and } \ \left(\sum a_n x^n\right)\left(\sum b_n x^n\right) = \sum c_n x^n,$$

where $c_n = \sum_{i+j=n} a_i b_j$ for all $n \in \mathbf{Z}$. Observe that $F[[x]]$ is a subring of $F((x))$.

(a) Show that a non-zero element of $F((x))$ has a multiplicative inverse.

(b) Let $0 \neq f(x) = \sum_{n=0}^{\infty} a_n x^n \in F[[x]]$. Show that $f(x)^{-1} \in F[[x]]$ if and only if $a_0 \neq 0$.

5. Suppose that R is commutative and for all $a \in R$ there is a positive integer $n > 1$ such that $a^n = a$. Show that every prime ideal of R is maximal.