Basic Examples of Rings

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Let S be be a non-empty set and let G be a semigroup. We define a binary operation on the set $\mathrm{Fun}\,(S,G)$ of all functions $f : S \longrightarrow G$ by pointwise multiplication, that is

$$(fg)(s) = f(s)g(s) \tag{1}$$

for all $f, g \in \mathrm{Fun}\,(S,G)$ and $s \in S$. Observe that $\mathrm{Fun}\,(S,G)$ is a semigroup. Further if G is a monoid (respectively group) then $\mathrm{Fun}\,(S,G)$ is a monoid (respectively group).

If the binary operation of S is written as addition then (1) is written

$$(f+g)(s) = f(s) + g(s) \tag{2}$$

for all $f, g \in \mathrm{Fun}\,(S,G)$ and $s \in S$. If S is abelian then $\mathrm{Fun}\,(S,G)$ is abelian.

Lemma 1 *Suppose that $S = G = A$ is an (additive) abelian group. Then the set* $\mathrm{End}\,(A)$ *of group endomorphisms of A is an additive subgroup of* $\mathrm{Fun}\,(A,A)$ *which is a ring with identity* I_A *whose product is composition.* $\square$

A comment on the lemma. Note that $\mathrm{Fun}\,(A,A)$ is an additive abelian group and a monoid under function composition. The distributive law $(f+g)\circ h = f\circ h + g\circ h$ holds for all $f, g, h \in \mathrm{Fun}\,(S,G)$. For $f \in \mathrm{Fun}\,(S,G)$ the other distributive law

$$f\circ(g+h) = f\circ g + f\circ h \tag{3}$$

holds for all $g, h \in \mathrm{Fun}\,(S,G)$ if and only if $f \in \mathrm{nd}\,(A)$. The necessity is seen by taking g, h to be constant functions.

Note the analogy between $\mathrm{End}\,(A)$, where A is an abelian group, and S_A, where A is a non-empty set.

From this point on R is a ring. Then $\mathrm{Fun}\,(S,R)$ is an additive abelian group. We will assume S has additional structure which will give certain additive subgroups $\mathcal{R}$ of $\mathrm{Fun}\,(S,R)$ a multiplication which affords $\mathcal{R}$ a ring structure.

Definition 1 *A* partial semigroup *is a triple* $(S, \mathcal{S}, m)$*, where* $S, \mathcal{S}$ *are non-empty sets,* $\mathcal{S} \subseteq S \times S$*, and* $m : \mathcal{S} \longrightarrow S$*,* $(a, b) \mapsto ab$*, is a function which satisfies:* $(a, b), (ab, c) \in \mathcal{S}$ *if and only if* $(b, c), (a, bc) \in \mathcal{S}$*, in which case* $(ab)c = a(bc)$*, for all* $a, b, c \in S$*.*

Note that a semigroup is a partial semigroup. We will denote a partial semigroup $(S, \mathcal{S}, m)$ by the set S, following the notation convention for semigroups and other algebraic structures.

From this point on S is a partial semigroup. Let $a, b, c \in S$. We say that ab *is defined* if $(a, b) \in \mathcal{S}$. The technical condition in the definition can be restated as: ab and $(ab)c$ are defined if and only if bc and $a(bc)$ are defined, in which case $(ab)c = a(bc)$.

For $f, g \in \text{Fun}\,(S, R)$ and $s \in S$ let

$$S_{f,g,s} = \{(u, v) \in \mathcal{S} \,|\, uv = s, f(u), g(v) \neq 0\}.$$

Observe that if $S_{f,g,s}$ is finite and not empty then

$$(fg)(s) = \sum_{(u,v)\in\mathcal{S},\ uv=s} f(u)g(v) \tag{4}$$

is *well-defined* since the sum has terms and the number of non-zero terms is finite. If $S_{f,g,s} = \emptyset$ then we set $(fg)(s) = 0$.

Proposition 1 *Let* S *be a partial semigroup, let* R *be a ring, and suppose* $\mathcal{R} \subseteq \text{Fun}\,(S, R)$ *be an additive subgroup such that for all* $f, g \in \mathcal{R}$ *the sets* $S_{f,g,s}$ *are finite for all* $s \in S$ *and* $fg \in \mathcal{R}$*, where the product is defined by (4). Then* $\mathcal{R}$ *is a ring under these operations.*

Proof: We begin a proof. Establishing associativity showcases the role of partial semigroups. Suppose $f, g, h \in \mathcal{R}$. Then for $s \in S$ we have

$$\begin{aligned} ((fg)h)(s) &= \sum_{\substack{(x,w)\in\mathcal{S} \\ xw=s}} (fg)(x)g(w) \\ &= \sum_{\substack{(x,w)\in\mathcal{S} \\ xw=s}} \left(\sum_{\substack{(u,v)\in\mathcal{S} \\ uv=x}} (f(u)g(v))g(w) \right) \\ &= \sum_{\substack{(u,v,w)\in S\times S\times S \\ (u,v),(uv,w)\in\mathcal{S},\ (uv)w=s}} (f(u)g(v))g(w). \end{aligned}$$

The last equation follows from the fact that there is a set bijection between

$$\{((x,w),(u,v))\,|\,(x,w),(u,v)\in\mathcal{S},\ xw=s, uv=x\}$$

and

$$\{(u,v,w)\in S{\times}S{\times}S\,|\,(u,v),(uv,w)\in\mathcal{S},\ (uv)w=s\}$$

given by

$$((x,w),\ (u,v))\mapsto(u,v,w)$$

whose inverse is given by

$$(u,v,w)\mapsto((uv,w),\ (u,v)).$$

Likewise

$$(f(gh))(s)=\sum_{\substack{(u,v,w)\in S\times S\times S\\ (v,w),\,(u,vw)\in\mathcal{S},\ u(vw)=s}} f(u)(g(v)g(w)).$$

Thus $((fg)h)(s)=(f(gh))(s)$ for all $s\in S$ which means $(fg)h=f(gh)$ □

Example 1 *Let the set $\mathcal{R}$ consist of all functions $f:S\longrightarrow R$ such that $f(s)=0$ except for finitely many $s\in S$. Then the hypothesis of Proposition 1 is satisfied and therefore $\mathcal{R}$ is a ring with addition and multiplication given by (2) and (4) respectively.*

We can represent elements $f\in\mathcal{R}$ by sums $\sum_{i=1}^{n}a_is_i$, where $s_1,\ldots,s_n\in S$ are distinct, $a_1,\ldots,a_n\in R$, and

$$f(s)=\begin{cases} a_i & : \ s=s_i \ \text{ for some } 1\le i\le n \\ 0 & : \ \text{otherwise}\end{cases}.$$

Suppose that S is a semigroup. Then

$$(ag)(bh)=ab(gh)\ \text{ for all } a,b\in R \text{ and } g,h\in S.$$

In this case

$$\left(\sum_{i=1}^{n}a_ig_i\right)\left(\sum_{i=1}^{n}b_ih_i\right)=\sum_{i=1}^{n}\sum_{j=1}^{m}a_ib_j(g_ih_j).$$

In this case $\mathcal{R}$ is a called the *semigroup ring of S with coefficients in R* and is denoted RS. If S is a monoid (respectively group) then RS is called the *monoid* (respectively *group*) *ring of S with coefficients in R.*

Example 2 *Let* $S = \mathbf{Z}$ *under addition and let* $\mathcal{R}$ *be the set of all functions* $f : \mathbf{Z} \longrightarrow R$ *such that there exists an* $N \in \mathbf{Z}$ *such that* $f(n) = 0$ *for all* $n < N$*. Write such a function as a formal sum*

$$f = \sum_{n=N}^{\infty} a_n x^n, \qquad \textit{where } f(n) = a_n \ \ \forall n \geq N.$$

Then the hypothesis of Proposition 1 is satisfied and therefore $\mathcal{R}$ *is a ring with addition and multiplication given by (2) and (4) respectively.*

Suppose that R is commutative. Then the ring $\mathcal{R}$ of the preceding example is called the *ring of formal Laurent series with coefficients in* R and is denoted $R((x))$. The set of all $f \in R((x))$ of the form $f = \sum_{n=0}^{\infty} a_n x^n$ is a subring of $R((x))$ and is called the *ring of formal power series with coefficients in* R and is denoted $R[[x]]$.

Apropos of Example 1, the set of all $f \in \mathrm{Fun}\,(\mathbf{Z}, R)$ such that $f(n) = 0$ for all but finitely many $n \in \mathbf{Z}$ is a subring of $R[[x]]$ and is called the *ring of polynomials in indeterminate* x *with coefficients in* R and is denoted $R[x]$.

When R is a field $R((x))$ is a field and thus $R[[x]]$, $R[x]$ are integral domains.

Let I be a non-empty set. Then $S = I \times I$ is a partial semigroup, where $(i, j)\cdot(k, \ell)$ is defined if and only if $j = k$, in which case $(i, j)\cdot(k, \ell) = (i, \ell)$.

Example 3 *Let* I *be finite. Then* $\mathcal{R} = \mathrm{Fun}\,(S, R)$ *is a ring with operations given by (2) and (4) respectively by virtue of Example 1.*

The preceding example is very familiar. Identify $f \in \mathcal{R}$ with $(a_{i\,j})$, where $f((i, j)) = a_{i\,j}$. Under this identification $\mathcal{R} = \mathrm{M}_n(R)$, the ring of $n \times n$ matrices with coefficients in R, where $n = |I|$.

When I is not necessarily finite there are interesting variations on $\mathcal{R}$ of the preceding example. For example, $\mathcal{R}$ can be taken to be the ring of all "row finite matrices" with coefficients in R. Row finite matrices are those functions $f = (a_{i\,j})$, where for all $i \in I$ there are only finitely many $j \in I$ such that $a_{i\,j} \neq 0$. Likewise $\mathcal{R}$ can be taken to be the ring of all "column finite matrices" with coefficients in R. Column finite matrices are those functions $f = (a_{i\,j})$, where for all $j \in I$ there are only finitely many $i \in I$ such that $a_{i\,j} \neq 0$.