**Math 516** **Fall 2006** **Radford**

# Written Homework # 5

Due at the beginning of class 12/08/06

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*Throughout R is a ring with unity.*

1. Let $M$ be an (additive) abelian group and $\mathrm{End}(M)$ be the set of group homomorphisms $f : M \longrightarrow M$.

(a) Show $\mathrm{End}(M)$ is a ring with unity, where $(f+g)(m) = f(m) + g(m)$ and $(fg)(m) = f(g(m))$ for all $f, g \in \mathrm{End}(M)$ and $m \in M$.

Now suppose that $M$ is a left $R$-module.

(b) For $r \in R$ define $\sigma_r : M \longrightarrow M$ by $\sigma_r(m) = r{\cdot}m$ for all $m \in M$. Show that $\sigma_r \in \mathrm{End}(M)$ for all $r \in R$ and $\pi : R \longrightarrow \mathrm{End}(M)$ defined by $\pi(r) = \sigma_r$ for all $r \in R$ is a homomorphism of rings with unity.

2. Let $M$ be a left $R$-module. For a non-empty subset $S$ of $M$ the subset of $R$ defined by

$$\mathrm{ann}_R(S) = \{r \in R \,|\, r{\cdot}s = 0 \ \ \forall s \in S\}$$

is called the *annihilator of* $S$. If $S = \{s\}$ is a singleton we write $\mathrm{ann}_R(s)$ for $\mathrm{ann}_R(\{s\})$.

(a) Suppose that $N$ is a submodule of $M$. Show that $\mathrm{ann}_R(N)$ is an ideal of $R$.

Now suppose $m \in M$ is fixed.

(b) Show that $\mathrm{ann}_R(m)$ is a left ideal of $R$.

(c) Let $f : R \longrightarrow R{\cdot}m$ be defined by $f(r) = r{\cdot}m$ for all $r \in R$. Show $f$ is a homomorphism of left $R$-modules and $F : R/\mathrm{ann}_R(m) \longrightarrow R{\cdot}m$ given by $F(r + \mathrm{ann}_R(m)) = r{\cdot}m$ for all $r \in R$ is a well-defined isomorphism of left $R$-modules.

3. Let $k$ be a field, $V$ a vector space over $k$, and $T \in \mathrm{End}_k(V)$ be a linear endomorphism of $V$. Then the ring homomorphism $\pi : k[X] \longrightarrow \mathrm{End}_k(V)$ defined by $\pi(f(X)) = f(T)$ for all $f(X) \in k[X]$ determines a left $k[X]$-module structure on $V$ by $f(X){\cdot}v = \pi(f(X))(v) = p(T)(v)$ for all $v \in V$.

(a) Let $W$ be a non-empty subset of $V$. Show that $W$ is a $k[X]$-submodule of $V$ if and only if $W$ is a $T$-invariant subspace of $V$.

(b) Suppose that $V = k[X]{\cdot}v$ is a cyclic $k[X]$-module. Show that $\mathrm{ann}_{k[X]}(V) = (f(X))$, where $f(X)$ is the minimal polynomial of $T$.

4. Let $M$ be a left $R$-module.

(a) Suppose that $\mathcal{N}$ is a non-empty family of submodules of $M$. Show that $L = \bigcap_{N \in \mathcal{N}} N$ is a submodule of $M$.

Since $M$ is submodule of $M$, it follows that any $S$ subset of $M$ is contained in a smallest submodule of $M$, namely the intersection of all submodule containing $S$. This submodule is denoted by $(S)$ and is called the *submodule of* $M$ *generated by* $S$.

(b) Let $\emptyset \neq S \subseteq M$. Show that

$$(S) = \{r_1{\cdot}s_1 + \cdots + r_\ell{\cdot}s_\ell \mid \ell \geq 1,\ r_1, \ldots, r_\ell \in R,\ s_1, \ldots, s_\ell \in S\}.$$

Suppose $f, f' : M \longrightarrow M'$ are $R$-module homomorphisms.

(c) Show that $N = \{m \in M \mid f(m) = f'(m)\}$ is a submodule of $M$.

(d) Suppose that $S$ generates $M$. Show that $f = f'$ if and only if $f(s) = f'(s)$ for all $s \in S$.

5. Use Corollary 2 of "Section 2.3 Supplement" and the equation of Problem 3 of Written Homework 3 to prove the following:

**Theorem 1** *Let $k$ be a field and suppose that $G$ is a finite subgroup of $k^\times$. Then $G$ is cyclic.*