Math 516 **Fall 2006** **Radford**

Written Homework # 4

Due at the beginning of class 11/17/06

You may use results form the book in Chapters 1–6 of the text, from notes found on our course web page, and results of the previous homework.

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1. Let R be a ring with unity (identity). Show that every element of R is either a unit or a zero divisor if

(a) R is finite or

(b) $R = \mathrm{M}_n(k)$, where k is a field.

[Hint: Let $a \in R$ and consider the sequence $1, a, a^2, a^3, \ldots$, noting that its terms belong to a finite set or a finite-dimensional vector space.]

2. Let R be a commutative ring with unity and let N be the set of nilpotent elements of R.

(a) Show that N is an ideal of R. [Hint: Let $a, b \in R$. You may assume that the binomial theorem holds for a, b and that $(ab)^n = a^n b^n$ for all $n \geq 0$.]

(b) Let $U = \{1 + n \,|\, n \in N\}$. Show that $U \trianglelefteq R^\times$. [Hint: Show that $U = \{1 - n \,|\, n \in N\}$ also. If $n^\ell = 0$ then $1 - n^\ell = 1$.]

(c) Find a ring with unity whose set of nilpotent elements is *not* an ideal. Justify your answer. [Hint: Consider $\mathrm{M}_2(k)$ where k is a field.]

3. Let R be a commutative ring with unity and set $\mathcal{R} = R[[X]]$.

(a) Show that $f : \mathcal{R} \longrightarrow R$ defined by $f(\sum_{n=0}^{\infty} a_n X^n) = a_0$ is a ring homomorphism.

(b) Show that $\sum_{n=0}^{\infty} a_n X^n \in \mathcal{R}^\times$ if and only if $a_0 \in R^\times$.

(c) Show that $\mathcal{R}$ is an integral domain if and only if R is an integral domain.

4. Let R be ring with unity.

(a) Suppose that $\mathcal{I}$ is a non-empty family of ideals of R. Show that $J = \bigcap_{I \in \mathcal{I}} I$ is an ideal of R. (Since R is an ideal of R, it follows that any S subset of R is contained in a smallest ideal of R, namely the intersection of all ideals containing S. This ideal is denoted by (S) and is called the ideal of R generated by S.)

(b) Suppose that R is commutative and $S = \{a_1, \ldots, a_r\}$ is a finite subset of R. Show that

$$(S) = Ra_1 + \cdots + Ra_r.$$

5. Let R by any ring with unity 1 and $\mathcal{R} = \mathrm{M}_n(R)$. Let J be an ideal of R.

(a) Show that $\mathrm{M}_n(J)$ is an ideal of $\mathcal{R}$ and all ideals of $\mathcal{R}$ have this form.

(b) Show that $\mathcal{R}$ is simple if and only if R is simple.

[Hint: For part (a) let $E_{i\,j} \in \mathrm{M}_n(R)$ be defined by $(E_{i\,j})_{k\,\ell} = \delta_{i,k}\delta_{j,\ell}$, where $\delta_{u,v} = \begin{cases} 1 & : \quad u = v \\ 0 & : \quad u \neq v \end{cases}$. Work out the formula for $E_{i\,j}E_{k\,\ell}$. Show that any $A = (A_{u\,v}) \in \mathrm{M}_n(R)$ can be written $A = \sum_{u,v=1}^{n} A_{u\,v}E_{u\,v}$ and consider $E_{i\,j}AE_{k\,\ell}$.]