Math 516 **Fall 2006** **Radford**

Written Homework # 3

Due at the beginning of class 10/27/06

You may use results form the book in Chapters 1–4 of the text, from notes found on our course web page, and results of the previous homework.

---

1. Let G be a group and $H, K \leq G$.

(a) Suppose that $HK \leq G$ and let $f : H{\times}K \longrightarrow HK$ be defined by $f((h,k)) = hk$ for all $(h,k) \in H{\times}K$. Show that f is a homomorphism if and only if $hk = kh$ for all $h \in H$ and $k \in K$.

Suppose in addition that $H, K \trianglelefteq G$.

(b) Show that $HK \trianglelefteq G$.

(c) Suppose that $H{\cap}K = (e)$. Show that $hk = kh$ for all $h \in H$ and $k \in K$ and that the homomorphism of part (b) is an isomorphism. [Hint: For $h \in H$ and $k \in K$ consider $hkh^{-1}k^{-1}$.]

2. Use the theory of finite cyclic groups and induction on $|G|$ to prove Cauchy's Theorem for abelian groups:

Theorem 1 *Let G be a finite abelian group and suppose that p is a prime integer which divides $|G|$. Then G as an element of order p.*

[Hint: Let $a \in G$ and set $H = \langle a \rangle$. Then $|G/H||H| = |G|$.]

3. Let G be a finite group. For every positive divisor d of $|G|$ let n_d denote the number of cyclic subgroup of G of order d. Show that

$$|G| = \sum_{d \mid |G|} \varphi(d) n_d,$$

where φ is the Euler phi-function. [Hint: Consider the equivalence relation on G defined by $a \sim b$ if and only if $\langle a \rangle = \langle b \rangle$.]

4. Let G be a finite group of order pqr, where p, q, r are primes and $p < q < r$.

(a) Show that G is not simple.

(b) Show that G has a subgroup of prime index.

[Hint: See the text's discussion of groups of order $30 = 2 \cdot 3 \cdot 5$. If needed, you may use the formula of Exercise 3.]

5. Let G be a finite group of order pqr, where p, q, r are primes, $p < q < r$, and $r \not\equiv 1 \pmod{q}$. Show that G has a subgroup of index p.