The Isomorphism Theorems

09/25/06 Radford

The isomorphism theorems are based on a simple basic result on homomorphisms. For a group G and $N \trianglelefteq G$ we let $\pi : G \longrightarrow G/N$ be the projection which is the homomorphism defined by $\pi(a) = aN$ for all $a \in G$.

Proposition 1 *Let $f : G \longrightarrow G'$ be a group homomorphism and suppose $N \trianglelefteq G$ which satisfies $N \subseteq \operatorname{Ker} f$. Then there is a unique homomorphism $F : G/N \longrightarrow G'$ which satisfies $F \circ \pi = f$.*

PROOF: Uniqueness. Suppose that $F, F' : G/N \longrightarrow G'$ are functions which satisfy $F \circ \pi = f$ and $F' \circ \pi = f$. Then $F(\pi(a)) = F'(\pi(a))$ for all $a \in G$. Since π is surjective, any element of G/N has the form $\pi(a)$ for some $a \in G$. Therefore $F = F'$. We note that $F(aN) = f(a)$ for all $a \in G$.

Existence. We define F as follows. Let $S \in G/N$; that is let S be a left coset of N in G. Choose any $x \in S$ and set $F(S) = f(x)$. We need to show that $F(S)$ does not depend on our choice of x; that is, F is *well defined.*

We may write $S = aN$ for some $a \in G$. Suppose that $x \in S$. Then $x = an$ for some $n \in N$. Since $N \subseteq \operatorname{Ker} f$ by assumption, the calculation

$$f(x) = f(an) = f(a)f(n) = f(a)e' = f(a)$$

shows that $F(S)$ does not depend on x and $F(S) = F(aN) = f(a)$. That $F \circ \pi = f$ and F is a homomorphism are easy to see at this point. $\square$

Our versions of the isomorphism theorems are slight variants of the ones found in in Dummit and Foote.

Theorem 1 (First Isomorphism Theorem) *Suppose $f : G \longrightarrow G'$ is a homomorphism. Then $\operatorname{Ker} f \trianglelefteq G$, $\operatorname{Im} f \leq G'$, and there is an isomorphism $G/\operatorname{Ker} f \longrightarrow \operatorname{Im} f$ given by $a(\operatorname{Ker} f) \mapsto f(a)$ for all $a \in G$.*

PROOF: We have seen that $\operatorname{Ker} f \trianglelefteq G$ and $\operatorname{Im} f \leq G'$. Without loss of generality we may assume that $G' = \operatorname{Im} f$; that is f is surjective. With $N = \operatorname{Ker} f$ we deduce from Proposition 1 that there is a homomorphism $F : G/N \longrightarrow G'$ given by $F(aN) = f(a)$ for all $a \in G$. Since

$$\operatorname{Ker} F = \{aN \mid f(a) = e'\} = \{aN \mid a \in N\} = \{N\}$$

is the trivial subgroup of G/N, it follows F is injective. Since f is surjective, F is surjective. □

The Second Isomorphism Theorem is formulated in terms of subgroups of the normalizer. Suppose that $A, B \leq G$ and $A \leq \mathrm{N}_G(B)$. Then $aBa^{-1} = B$, or equivalently $aB = Ba$, for all $a \in A$. Therefore $AB = BA$ which means $AB \leq G$. From $A, B \leq \mathrm{N}_G(B) \leq G$ we deduce $AB \leq \mathrm{N}_G(B)$. Since $B \trianglelefteq \mathrm{N}_G(B)$ and $B \leq AB$ the relation

$$B \trianglelefteq AB$$

follows. For $a \in A$ the calculation $a(A \cap B)a^{-1} \subseteq A \cap aBa^{-1} = A \cap B$ shows that

$$A \cap B \trianglelefteq A.$$

Theorem 2 (Second Isomorphism Theorem) *Suppose that G is a group and $A, B \leq G$ satisfy $A \leq \mathrm{N}_G(B)$. Then $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$, and there is an isomorphism $A/A \cap B \longrightarrow AB/B$ given by $a(A \cap B) \mapsto aB$ for all $a \in A$.*

PROOF: We have established the normality assertions. Since the projection $\pi : AB \longrightarrow AB/B$ is a homomorphism, and $A \leq AB$, the restriction $f = \pi|_A : A \longrightarrow AB/B$ is a homomorphism. For all $a \in A$ and $b \in B$ note that $abB = a(bB) = aB = f(a)$. Thus f is surjective. Since

$$\operatorname{Ker} f = \{a \in A \mid aB = B\} = \{a \in A \mid a \in B\} = A \cap B$$

the theorem follows by Theorem 1 □

Theorem 3 (Third Isomorphism Theorem) *Suppose that G is a group and suppose that $N, H \trianglelefteq G$ satisfy $N \leq H$. Then there is an isomorphism $(G/N)/(H/N) \longrightarrow G/H$ given by $(aN)(H/N) \mapsto aH$ for all $a \in G$.*

PROOF: Since $N \leq \operatorname{Ker} \pi$, where $\pi : G \longrightarrow G/H$ is the projection, by Proposition 1 there is a homomorphism $f : G/N \longrightarrow G/H$ which is given by $f(aN) = \pi(a) = aH$. Note that f is surjective. Since $\operatorname{Ker} f = H/N$ the theorem follows by Theorem 1. □

The conclusion of the preceding theorem can be remembered by interpreting quotient groups as fractions and applying the usual rules of arithmetic for fractions:

$$\frac{\left(\begin{matrix} G \\ - \\ N \end{matrix}\right)}{\left(\begin{matrix} H \\ - \\ N \end{matrix}\right)} = \left(\begin{matrix} G \\ - \\ N \end{matrix}\right)\left(\begin{matrix} N \\ - \\ H \end{matrix}\right) = \left(\begin{matrix} G \\ - \\ H \end{matrix}\right).$$

Suppose that $f : G \longrightarrow G'$ is a surjective homomorphism. The next isomorphism theorem gives a one-one correspondence between the subgroups of G' and a certain subset of subgroups of G.

Theorem 4 (Fourth Isomorphism Theorem) *Let $f : G \longrightarrow G'$ be a surjective homomorphism. Then:*

(a) *There is an inclusion preserving bijection*

$$\{A \mid \operatorname{Ker} f \leq A \leq G\} \longrightarrow \{B \mid B \leq G'\}$$

given by $A \mapsto f(A)$ with inverse given by $B \mapsto f^{-1}(B)$.

(b) *Let $\operatorname{Ker} f \leq N \leq G$. Then $N \trianglelefteq G$ if and only if $f(N) \trianglelefteq G'$ in which case there is an isomorphism $G/N \longrightarrow G'/f(N)$ given by $aN \mapsto f(a)f(N)$ for all $a \in G$.*

PROOF: We sketch a proof. As for part (a), suppose that $f : G \longrightarrow G'$ is *any* homomorphism. Then $\operatorname{Ker} f \subseteq f^{-1}(B)$ and $B \subseteq f(f^{-1}(B))$ for all $B \leq G'$. If f is surjective then $B = f(f^{-1}(B))$. Let $A \leq G$. Then $A \subseteq f^{-1}(f(A))$. If $\operatorname{Ker} f \subseteq A$ then $A = f^{-1}(f(A))$. As for part (b), observe that the composite $G \longrightarrow G' \longrightarrow G'/f(N)$ of f followed by the projection is surjective and has kernel $f^{-1}(f(N)) = N$; apply Theorem 1. □