Some Remarks on Cosets

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Our discussion is predicated on general results about group actions which were discussed earlier. Suppose that G is a group and $H \leq G$. Then a left action of H on G is defined by

$$h{\cdot}a = ha$$

for all $h \in H$ and $a \in G$. The relation on G defined by $a \sim b$ if and only if $b = h{\cdot}a$ for some $h \in h$ is an equivalence relation on G. The equivalence class containing a is

$$[a] = H{\cdot}a = Ha,$$

the H-orbit of a which is also the right coset of H in G containing a. Thus:

$$\text{The right cosets of } H \text{ in } G \text{ partition } G. \tag{1}$$

Fix $a, b \in G$. Then the map $Ha \longrightarrow Hb$ defined by $ha \mapsto hb$ is well-defined and bijective. That the map is well-defined and injective follow from right cancellation. Thus all right cosets of H have the same cardinality. Observe that $H^{op} \leq G^{op}$ and

$$H^{op}{\cdot}^{op}a = aH. \tag{2}$$

Thus right cosets of H^{op} in G^{op} are the left cosets of H in G. We have shown in (1) that the right cosets of a subgroup of a group partition the group. By (2) therefore:

$$\text{The left cosets of } H \text{ in } G \text{ partition } G. \tag{3}$$

The function $G \longrightarrow G$ given by $g \mapsto g^{-1}$ for all $g \in G$ is bijective; indeed it is its own inverse. This bijection induces a bijection $2^G \longrightarrow 2^G$ of the set of all subsets of G to itself defined by $S \mapsto S^{-1}$, where the latter is the set of all inverses of elements of S. Now $H^{-1} = H$ since $H \leq G$. Noting that $(Ha)^{-1} = a^{-1}H^{-1} = a^{-1}H$ it is easy to see that there is a bijection

$$\{ \text{ right cosets of } H \text{ in } G \} \longrightarrow \{ \text{ left cosets of } H \text{ in } G \} \tag{4}$$

given by $Ha \mapsto a^{-1}H$. Since $Ha \longrightarrow a^{-1}H$ given by $ha \mapsto (ha)^{-1}$ is a bijection, using the bijection of (4) we conclude that:

$$\text{All cosets, left or right, of } H \text{ in } G \text{ have the same cardinality.} \tag{5}$$

$|G : H|$, the index of H in G, is the cardinality of the set of right cosets of H in G. Since the map of (4) is a bijection, $|G : H|$ is also the cardinality of the set of left cosets of H in G. When G is finite, in light of (3) and (5) we have

$$|G| = |G : H||H| \tag{6}$$

from which Lagrange's Theorem follows.

Suppose that H is proper subgroup of G. Then the smallest possible value of $|G : H|$ is 2. If H is not trivial then H has at least 2 elements. These extreme cases are interesting.

Proposition 1 *Let G be a group and suppose that $H \leq G$.*

(a) *Suppose that $|G : H| = 2$. Then $H \trianglelefteq G$.*

(b) *Suppose that $|H| = 2$ and $H \trianglelefteq G$. Then $H \leq \mathrm{Z}(G)$.*

Proof: Suppose that $|G : H| = 2$. Since $H = eH$ is a left coset of H in G, by (3) the other left coset of H in G is $G\backslash H$, the complement of H in G. Likewise the right cosets of H in G are $H = He$ and $G\backslash H$. Since the set of left cosets of H in G is the set of right cosets of H in G, it follows that $H \trianglelefteq G$. We have shown part (a).

Suppose that $|H| = 2$, write $H = \{e, a\}$, and let $g \in G$. Since $H \trianglelefteq G$ we have

$$\{ge, ga\} = gH = Hg = \{eg, ag\}.$$

Since $ge = g = eg$, and the two preceding cosets have two elements each, necessarily $ga = ag$. Thus $a \in \mathrm{Z}(G)$ and consequently $H \leq \mathrm{Z}(G)$. □