MATH 215 # Hour Exam I Solution **Radford 07/14/08**

Name (print) ______________________________

(1) There are *four questions* on this exam. (2) *Return* this exam copy with your test booklet. (3) *You are expected to abide by the University's rules concerning academic honesty.* (4) *Base the proofs* for problems 2(c) and 4 on the axioms for the real number system and:

(a) the product of two positive real numbers is positive,

(b) the product of two negative real numbers is positive,

(c) the product of a positive real number and a negative real number is negative,

(d) the product of zero and any real number is zero, and

(e) if a, b, c are real numbers and $a < b$ then $a - c < b - c$ and $c - a > c - b$.

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1. (25 pts.) Let P and Q be statements.

(a) Write out the truth table for the statement "(Q implies P) implies Q". Include a column for "Q implies P" in your table.

Solution:

P	Q	Q implies P	(Q implies P) implies Q
T	T	T	T
T	F	T	F
F	T	F	T
F	F	T	F

(**8 points**)

(b) Write out the truth table for the statement "(P implies Q) implies Q". Include a column for "P implies Q" in your table.

Solution:

P	Q	P implies Q	(P implies Q) implies P
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

(**8 points**)

(c) Are the statements of part (a) and part (b) logically equivalent? Justify your answer in terms of the truth tables of parts (a) and (b).

Solution: The statements are *not* logically equivalent since the *last columns* (not the whole truth tables) of the truth tables are not the same. (**4 points**)

(d) Does the statement of part (a) imply the statement of part (b)? Does the statement of part (b) imply the statement of part (a)? Justify your answer in terms of the truth tables of parts (a) and (b).

Solution: The statement of part (a) implies the statement of part (b) since whenever the statement of part (a) is true the statement of part (b) is also true. (**3 points**) The statement of part (b) does not imply the statement of part (a) since when P is true and Q is false the statement of part (b) is true and the statement of part (a) is false. (**2 points**)

Comment: To draw conclusions from truth tables by comparing columns the lines of the truth tables should describe the same cases. For example, comparing

P	Q	...
T	T	...
T	F	...
F	T	...
F	F	...

and

Q	P	...
T	T	...
T	F	...
F	T	...
F	F	...

will most likely lead to erroneous conclusions since the case in line 2 of the first table is not the same as the case in line 2 of the second.

2. (30 pts.) Consider the statement "$(a-2)(a-8) > 0$ implies $a \leq 2$ or $8 < a$".

(a) State the converse of the statement and determine whether or not the converse is true.

Solution: The converse is "$a \leq 2$ or $8 < a$ implies $(a-2)(a-8) > 0$". (**5 points**) The converse is false; for with $a = 2$ the hypothesis "$a \leq 2$ or $8 < a$" is true and the conclusion "$(a-2)(a-8) > 0$" is false as $(a-2)(a-8) = 0$ in this case. (**5 points**)

(b) What is the contrapositive of the statement? Write it without "not".

Solution: The contrapositive in the form required is "$2 < a \leq 8$ implies $(a-2)(a-8) \leq 0$". (**5 points**)

(c) Prove the statement by contradiction.

Solution: Suppose that the hypothesis $(a-2)(a-8) > 0$ is true but the conclusion $a \leq 2$ or $8 < a$ is false. Then $(a-2)(a-8) > 0$ and $2 < a \leq 8$. (**5 points**) Now $2 < a$ implies $0 < a-2$ and $a \leq 8$ implies $a - 8 \leq 0$. Thus $a - 8 < 0$, in which case the product $(a-2)(a-8) < 0$ (as it is the product of a positive and negative number), (**5 points**) or $a - 8 = 0$ in which case the product $(a-2)(a-8) = 0$ (as it is the product of a number and zero). (**2 points**) In either case $(a-2)(a-8) \leq 0$, a contradiction. Thus $(a-2)(a-8) > 0$ implies $a \leq 2$ or $8 < a$. (**3 points**)

Comments: If the format of the proof by contradiction was not stated correctly there was an 8 point penalty. The parenthetical remarks in the proof were not required.

3. (25 pts.) Let x be a real number, $x \neq 1$. Prove that the sum $\sum_{i=0}^{n} x^i = 1 + x + \cdots + x^n$ is equal to $\dfrac{x^{n+1}-1}{x-1}$ for all $n \geq 1$ by induction.

Solution: We are to show that the equation $1 + x + \cdots + x^n = \dfrac{x^{n+1}-1}{x-1}$ holds for all $n \geq 1$ by induction. Suppose that $n = 1$. Then the left had side of the equation is $1 + x$ and the right hand side is

$$\frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{x-1} = x + 1.$$

Therefore the equation holds for $n = 1$. (**5 points**)

Suppose that $n \geq 1$ and the equation holds for n. Then

$$\begin{aligned}
&1 + \cdots + x^{n+1} \\
&= (1 + \cdots + x^n) + x^{n+1} \\
&= \frac{x^{n+1}-1}{x-1} + x^{n+1} \\
&= \frac{(x^{n+1}-1) + x^{n+1}(x-1)}{x-1} \\
&= \frac{(x^{n+1}-1) + x^{n+2} - x^{n+1}}{x-1} \\
&= \frac{x^{n+2}-1}{x-1}
\end{aligned}$$

which means that the formula is true for $n+1$. (**3 points per line**) By induction the formula is true for all $n \geq 1$. (**2 points**)

Comments: The formal statement to be proved is P(n) : $1 + x + \cdots + x^n = \dfrac{x^{n+1} - 1}{x - 1}$, not P(n) : $1 + x + \cdots + x^n$ or P(n) : $\dfrac{x^{n+1} - 1}{x - 1}$. The latter two are expressions, not statements as they are neither true not false.

4. (20 pts.) Show directly, by cases, that $a \leq 3$ or $9 \leq a$ implies $a^2 - 12a + 27 \geq 0$.

Solution: Observe that $a^2 - 12a + 27 = (a-3)(a-9)$. (**4 points**) There are two cases to consider.

Case 1: $a \leq 3$. (**4 points**) Here $a-9 < a-3 \leq 0$. If $a-3 < 0$ then the product $(a-3)(a-9) > 0$ (as it is the product of two negative numbers) (**3 points**) and if $a - 3 = 0$ then the product $(a-3)(a-9) = 0$ (as it is the product of zero and a number) (**1 points**). In any event $(a-3)(a-9) \geq 0$.

Case 1: $9 \leq a$. (**4 points**) Here $0 \leq a-9 < a-3$. If $0 < a-9$ then the product $(a-3)(a-9) > 0$ (as it is the product of two positive numbers) (**3 points**) and if $a - 9 = 0$ then the product $(a-3)(a-9) = 0$ (as it is the product of a number and zero) (**1 points**). In any event $(a-3)(a-9) \geq 0$.

Comment: The parenthetical remarks in the proof were not required.