MATH 215 **Written Homework 3 Solution** **07/14/08**

1. **20 points total** We prove the formula $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ holds for all $n \geq 1$ by induction on n.

Suppose that $n = 1$. Then $\sum_{i=1}^{n} i^3 = \sum_{i=1}^{1} i^3 = 1^3 = 1$ and $\frac{n^2(n+1)^2}{4} = \frac{1^2(1+1)^2}{4} = \frac{1 \cdot 4}{4} = 1$. Therefore the formula holds when $n = 1$. (**5 points**)

Suppose that $n \geq 1$ and the assertion holds for n; that is $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$. We will show that the assertion holds for $n+1$; that is $\sum_{i=1}^{n+1} i^3 = \frac{(n+1)^2((n+1)+1)^2}{4}$, or equivalently $\sum_{i=1}^{n+1} i^3 = \frac{(n+1)^2(n+2)^2}{4}$. Since

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \sum_{i=1}^{n} i^3 + (n+1)^3 \qquad (\textbf{5 points}) \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{(n+1)^2[n^2 + 4(n+1)]}{4} \\
&= \frac{(n+1)^2[n^2 + 4n + 4]}{4} \qquad (\textbf{5 points}) \\
&= \frac{(n+1)^2(n+1)^2}{4}. \qquad (\textbf{5 points})
\end{aligned}$$

We have shown that if the formula is true for $n \geq 1$ then it is true for $n+1$. Therefore the formula is true for all $n \geq 1$.

2. **20 points total** Part (a) is to make part (b) easier.

(a) The statement we are to prove is $n+1 \leq 2^{n-1}$ for $n \geq 3$. When $n = 3$, $n+1 = 4 = 2^2 = 2^{3-1} = 2^{n-1}$; thus the statement is true in the base case. (**2 points**)

Suppose $n \geq 3$ and the statement is true. The statement for $n+1$ is $(n+1)+1 < 2^{(n+1)-1}$ or $(n+1)+1 < 2^n$. The calculation

$$(n+1)+1 < (n+1)+(n+1) \leq 2^{n-1} + 2^{n-1} = 2 \cdot 2^{n-1} = 2^n \quad (\textbf{3 points})$$

shows that the statement holds for $n+1$. Therefore the statement is true for all $n \geq 3$. (**3 points**)

(b) The statement we are to prove is $n^2 < 2^n$ for $n > 4$, or equivalently $n \geq 5$. When $n = 5$, $n^2 = 5^2 = 25 < 32 = 2^5$; thus the statement holds for $n = 5$, the base case. (**2 points**)

Suppose that $n \geq 5$ and the statement is true. The statement for $n+1$ is $(n+1)^2 < 2^{n+1}$. Using part (a) (**2 points**) we calculate

$$(n+1)^2 = n^2 + 2n + 1 < n^2 + 2(n+1) < 2^n + 2{\cdot}2^{n-1} = 2{\cdot}2^n = 2^{n+1} \quad (\textbf{3 points})$$

which means that the statement is true for $n + 1$. Thus the statement holds for all $n \geq 5$. (**2 points**)

(c) $n = 1$: $1^2 < 2 = 2^1$, true; $n = 2$: $2^2 = 4 \not< 4 = 2^2$, false; $n = 3$: $3^2 = 9 \not< 8 = 2^3$, false; $n = 34$: $4^2 = 16 \not< 16 = 2^4$, false; $n \geq 5$, true by part (a). $n = 2, 3, 4$. (**3 points**)

3. **20 points total** "$A \cup B \subseteq A$ only if $B \subseteq A$" is the same as "$A \cup B \subseteq A$ implies $B \subseteq A$".

Assume that $A \cup B \subseteq A$. The implication follows once we show $B \subseteq A$. Let $x \in B$. (**5 points**) Then $x \in A$ or $x \in B$ which means $x \in A \cup B$ by definition of union. (**5 points**) Since $A \cup B \subseteq A$, $x \in A$ by definition of set inclusion. We have shown that $x \in B$ implies $x \in A$. (**5 points**) Therefore $B \subseteq A$. (**5 points**)

4. **40 points total** "$B \subseteq A \cap B$ if and only if $B \subseteq A$".

Only if: '$B \subseteq A \cap B$ implies $B \subseteq A$". (**5 points**)

Assume $B \subseteq A \cap B$ and let $x \in B$. (**5 points**) Since $B \subseteq A \cap B$, $x \in A \cap B$. Thus $x \in A$ and $x \in B$; in particular $x \in A$. (**5 points**) We have shown $x \in B$ implies $x \in A$. Therefore $B \subseteq A$. (**5 points**)

If: "$B \subseteq A$" implies "$B \subseteq A \cap B$. (**5 points**)

Assume that $B \subseteq A$ and let $x \in B$. (**5 points**) Then $x \in A$, since $x \in B$, which means $x \in A$ and $x \in B$. Therefore $x \in A \cap B$. (**5 points**) We have shown $x \in B$ implies $x \in A \cap B$. Therefore $B \subseteq A \cap B$. (**5 points**)