Math 215 Written Homework 6 Solution (REVISION) 07/28/08

Slightly revised and more detailed point distributions are given and several more comments are included.

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1. (**20 points total**) Here A is any subset of the set of real numbers **R** and is not necessarily finite.

(a) Suppose that $a_1, a_2 \in A$ are maxima for A. Then $a \leq a_1$ and $a \leq a_2$ for all $a \in A$. Since $a_2 \in A$, $a_2 \leq a_1$. Since $a_1 \in A$, $a_1 \leq a_2$. Therefore $a_2 \leq a_1 \leq a_2$ which means $a_1 = a_2$. (**6 points**)

Comment: The assertion of part (a) is a *uniqueness* statement, a statement which asserts "at most one". An *existence* statement is one which asserts "at least one". An *existence and uniqueness* statement asserts "exactly one".

(b) $a \in A$ by assumption.

"Only if". Suppose that a is a minimum for A. Since $a \in A$, $-a \in -A$. Let $x \in -A$. Then $x = -b$ for some $b \in A$. Therefore $b \leq a$ which means $x = -b \geq -a$. We have shown that $-a$ is a maximum for $-A$. (**4 points**)

"If". Suppose that $-a$ is a maximum for $-A$. Let $b \in A$. Then $-b \in -A$. Therefore $-b \leq -a$ which means $b \geq a$. Therefore a is a maximum for A. (**4 points**)

Comment: Part (b) relates maxima and minima.

(c) Suppose that a_1, a_2 are minima for A. Then $-a_1, -a_2$ are maxima for $-A$ by part (b). Therefore $-a_1 = -a_2$ by part (a). From this equation $a_1 = a_2$ follows. Thus A has at most one minimum. (**6 points**)

2. (**20 points total**) We investigate when $A \cup B$ has a maximum.

(a) Suppose A, $B \subseteq$ **R** and $A \cup B$ has a maximum c. Since $c \in A \cup B$, by definition $c \in A$ or $c \in B$.

Assume first of all that $c \in A$ (the first set listed in $A \cup B$). Let $a \in A$. Since $a \in A \cup B$, $a \leq c$. Therefore c is a maximum for A.

If $c \notin A$ then $c \in B$. As $A \cup B = B \cup A$, and thus c is a maximum for $B \cup A$, the preceding argument shows that c is a maximum for B. (**8 points**)

(b) Suppose that $a \in A$ is a maximum for A and $b \in B$ is a maximum for B. Let c be the maximum of a, b. Since $a, b \in A \cup B$, and $c = a$ or $c = b$, it follows that $c \in A \cup B$.

Suppose that $d \in A \cup B$. Then $d \in A$, in which case $d \leq a \leq c$ and hence $d \leq c$, or $d \in B$, in which case $d \leq b \leq c$, and consequently $d \leq c$. Therefore c is a maximum for $A \cup B$. (**12 points**)

Comment: Problems 1 and 2 are good exercises in simple proofs, ones which follow from definitions and a few basic axioms.

3. (**20 points total**)

(a) From the table

$x \in A$	$x \in B$	$x \in A$	$x \in B$	$x \in A \cap B$
T	T	T	T	T
T	F	T	F	F
F	T	F	T	F
F	F	F	F	F

we derive the table

$x \in A$	$x \in B$	$\chi_A(x)$	$\chi_B(x)$	$\chi_{A \cap B}(x)$	$\chi_A(x)\chi_B(x)$
T	T	1	1	1	$1 \cdot 1 = 1$
T	F	1	0	0	$1 \cdot 0 = 0$
F	T	0	1	0	$0 \cdot 1 = 0$
F	F	0	0	0	$0 \cdot 0 = 0$

from which we deduce that $\chi_{A \cap B}(x) = \chi_A(x)\chi_B(x)$ for all $x \in U$. Therefore $\chi_{A \cap B} = \chi_A \chi_B$. (**7 points**)

Comment: The preceding proof is somewhat elaborate; it shows the connection between the tables involved in showing that two sets are equal and the equality of characteristic functions. In any event, a proof should involve various cases. Let $x \in U$. For example; $x \in A$ and $x \notin B$. Thus $\chi_A(x) = 1$ and $\chi_B(x) = 0$ which means $\chi_A \chi_B(x) = \chi_A(x)\chi_B(x) = 1 \cdot 0 = 0$. Now $x \notin B$ means $x \notin A \cap B$. Therefore $\chi_{A \cap B}(x) = 0$. We have shown

$\chi_A\chi_B(x) = 0 = \chi_{A\cap B}(x)$; hence $\chi_A\chi_B(x) = \chi_{A\cap B}(x)$ in this case. (It would *not* be correct to write $\chi_A\chi_B = \chi_{A\cap B}$ to summarize this case.) This comment applies to part (b) and to part (a) of Problem 4 as well.

Comment: Some solutions were of the form: Case 1 ... Therefore for all $x \in U$, $\chi_{A\cap B}(x) = 1$ if and only if $\chi_A\chi_B(x) = 1$. Case 2 .... Therefore for all $x \in U$, $\chi_{A\cap B}(x) = 0$ if and only if $\chi_A\chi_B(x) = 0$. The conclusion of Case 2 is equivalent to conclusion of Case 1 as "P if and only if Q" is logically equivalent to "(not P) if and only if (not Q)"; consider the contrapositives. Thus Case 1 (or Case 2) is sufficient for showing that $\chi_{A\cap B} = \chi_A\chi_B$.

(b) From the table

$x \in A$	$x \in A$	$x \in A^c$
T	T	F
F	F	T

we derive the table

$x \in A$	$\chi_A(x)$	$\chi_{A^c}(x)$	$1 - \chi_A(x)$
T	1	0	$1 - 1 = 0$
F	0	1	$1 - 0 = 1$

which shows that $\chi_{A^c}(x) = 1 - \chi_A(x)$ for all $x \in U$. Therefore $\chi_{A^c} = 1 - \chi_A$. (**7 points**)

(c) Note that $A - B = A\cap B^c$ (a short proof would be good). Thus

$$\chi_{A-B} = \chi_{A\cap B^c} = \chi_A\chi_{B^c} = \chi_A(1 - \chi_B).$$

(**6 points**)

4. (**20 points total**)

(a) Let $x \in U$. From the table

$x \in A$	$x \in B$	$x \in A$	$x \in B$	$x \in A\cap B$	$x \in A\cup B$
T	T	T	T	T	T
T	F	T	F	F	T
F	T	F	T	F	T
F	F	F	F	F	F

we derive the table

$x \in A$	$x \in B$	$\chi_A(x)$	$\chi_B(x)$	$\chi_{A\cap B}(x)$	$\chi_{A\cup B}(x)$	$\chi_A(x)+\chi_B(x)-\chi_{A\cap B}(x)$
T	T	1	1	1	1	$1+1-1=1$
T	F	1	0	0	1	$1+0-0=1$
F	T	0	1	0	1	$0+1-0=1$
F	F	0	0	0	0	$0+0-0=0$

which shows that $\chi_{A\cup B}(x) = \chi_A(x)+\chi_B(x)-\chi_{A\cap B}(x)$ for all $x \in U$. Therefore $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$. (**8 points**)

(b) By part (b) of Problem 3 and part (a)

$$\chi_{(A\cup B)^c} = 1 - \chi_{A\cup B} = 1 - \chi_A - \chi_B + \chi_A\chi_B$$

and by parts (a) and (b) of Problem 3

$$\chi_{A^c\cap B^c} = \chi_{A^c}\chi_{B^c} = (1-\chi_A)(1-\chi_B) = 1 - \chi_A - \chi_B + \chi_A\chi_B.$$

Thus $\chi_{(A\cup B)^c} = \chi_{A^c\cap B^c}$ which implies $(A\cup B)^c = A^c\cap B^c$. (**12 points**)

Comment: If one of De Morgan's laws (they are equivalent to each other) was used to prove part (a), then one can not *prove* the conclusion of part (b) as required. For then the proof would be a tautology; De Morgan's Law implies De Morgan's Law.

5. (**20 points total**) In each case we compute $D(a)$ for the smaller value of a of the pair. Note that if $0 < b < a$ and divides a then $b \leq a/2$.

(a) $D(22) = \{1, 2, 11, 22, -1, -2, -11, -22\}$. Thus the greatest common divisor of 22 and 234 is 1, 2, 11, or 22. Since 2 divides 234 and 11 does not, and therefore 22 does not, the greatest common divisor of 22 and 234 is 2. (**7 points**)

(b) $D(39) = \{1, 3, 13, 39, -1, -3, -13\}$. Thus the greatest common divisor of 39 and 385 is 1, 3, 13, or 39. Since 1 divides 385 and 3, 13 do not, and therefore 39 does not, the greatest common divisor of 39 and 385 is 1. (**7 points**)

(c) $D(16) = \{1, 2, 4, 8, 16, -1, -2, -4, -8, -16\}$. Thus the greatest common divisor of 16 and 120 is 1, 2, 4, 8, or 16. Since 8 divides 120 and 16 does not, 8 is the greatest common divisor of 16 and 120. (**6 points**)