Hopf Algebras, an Overview

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0. Introduction

Hopf algebras are named after Heinz Hopf who discovered them the last century in the context of algebraic topology [**Hopf 1941**]. They arise in many areas of mathematics. Group algebras, enveloping algebras of Lie algebras, and quantum groups are examples of Hopf algebras. Certain Hopf algebras give rise to invariants of knots and links. We will describe Hopf algebras, discuss basic examples and fundamental results, and trace the development of the theory.

The Hopf algebras we discuss here are technically different from those coming from algebraic topology. See the very interesting discussion in [**Haz 2008**].

Objects are vector spaces over a field $\mathbb{k}$ and maps are $\mathbb{k}$-linear. $\otimes = \otimes_{\mathbb{k}}$. "f-d" = finite-dimensional.

1. A Basic Example and Definitions

G is a group and $A = \Bbbk G$ is the group algebra of G over $\Bbbk$. Let $g, h \in G$. The algebra structure:

$\Bbbk G \otimes \Bbbk G \xrightarrow{m} \Bbbk G$	$m(g \otimes h) = gh$
$k \xrightarrow{\eta} \Bbbk G$	$\eta(1_k) = e = 1_{\Bbbk G}$

The **co**algebra structure:

$\Bbbk G \xrightarrow{\Delta} \Bbbk G \otimes \Bbbk G$	$\Delta(g) = g \otimes g$
$\Bbbk G \xrightarrow{\epsilon} k$	$\epsilon(g) = 1_k$

The map which accounts for inverses:

$\Bbbk G \xrightarrow{S} \Bbbk G$	$S(g) = g^{-1}$

Observe that

$$\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta(g)\Delta(h),$$

$$\epsilon(gh) = 1_{\Bbbk} = 1_{\Bbbk}1_{\Bbbk} = \epsilon(g)\epsilon(h),$$

$$S(gh) = (gh)^{-1} = h^{-1}g^{-1} = S(h)S(g),$$

$$gS(g) = gg^{-1} = 1_{\Bbbk G} = 1_{\Bbbk}1_{\Bbbk G} = \epsilon(g)1_{\Bbbk G},$$

and

$$S(g)g = g^{-1}g = 1_{\Bbbk G} = \epsilon(g)1_{\Bbbk G}.$$

In particular $\boxed{\Delta, \epsilon \text{ are algebra maps}}$ and S is determined by

$$\boxed{gS(g) = \epsilon(g)1_{\Bbbk G} = S(g)g.}$$

We generalize the system $(\Bbbk G, m, \eta, \Delta, \epsilon, S)$.

A **Hopf algebra over** $\Bbbk$ is a tuple $(A, m, \eta, \Delta, \epsilon, S)$, where (A, m, η) is an algebra over $\Bbbk$:

$A \otimes A \xrightarrow{m} A$	$m(a \otimes b) = ab$
$k \xrightarrow{\eta} A$	$\eta(1_k) = 1_A$

(A, Δ, ϵ) is a **coalgebra over** $\Bbbk$:

$A \xrightarrow{\Delta} A \otimes A$	$\Delta(a) = a_{(1)} \otimes a_{(2)}$
$A \xrightarrow{\epsilon} k$	

and $\boxed{A \xrightarrow{S} A}$ is an "antipode" where certain axioms are satisfied.

<u>Comments</u>: $\Delta(a) \in A \otimes A$ is usually a *sum* of tensors; thus $\Delta(a) = a_{(1)} \otimes a_{(2)}$ **is a notation**, called the Heyneman-Sweedler notation. Δ is called the **coproduct** and ϵ the **counit**.

The axioms for a Hopf algebra:

(A, m, η) is an (associative) algebra:

$$\boxed{(ab)c = a(bc), \qquad 1a = a = a1}$$

(A, Δ, ϵ) is a (coassociative) coalgebra:

$$\boxed{\begin{aligned} &a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)}, \\ &\epsilon(a_{(1)})a_{(2)} = a = a_{(1)}\epsilon(a_{(2)}) \end{aligned}}$$

Δ is an algebra map:

$$\boxed{\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}, \qquad \Delta(1) = 1 \otimes 1,}$$

ϵ is an algebra map:

$$\boxed{\epsilon(ab) = \epsilon(a)\epsilon(b), \qquad \epsilon(1) = 1,} \qquad \text{and}$$

$$\boxed{a_{(1)}S(a_{(2)}) = \epsilon(a)1 = S(a_{(1)})a_{(2)}}$$

for all $a, b \in A$. From now on A denotes a Hopf algebra over $\mathbb{k}$.

2. Basic Properties and More Definitions

1. A has a unique antipode and

$\boxed{S(ab) = S(b)S(a), \qquad S(1) = 1.}$ Also

$$\boxed{\Delta(S(a)) = S(a_{(2)}) \otimes S(a_{(1)}), \quad \epsilon(S(a)) = \epsilon(a).}$$

2. $a \in A$ *is cocommutative* if $a_{(1)} \otimes a_{(2)} = a_{(2)} \otimes a_{(1)}$; A *is cocommutative* if all $a \in A$ are. $\Bbbk G$ is cocommutative. A *is commutative* if $ab = ba$ for all $a, b \in A$. $\Bbbk G$ is commutative if and only if G is.

3. $a \in A$ *is grouplike* if $\Delta(a) = a \otimes a$ and $\epsilon(a) = 1$. $1 \in A$ is grouplike. The set $G(A)$ of grouplike elements of A is linearly independent (coalgebra fact). $G(\Bbbk G) = G$.

4. $G(A)$ is a group under multiplication and $S(g) = g^{-1}$ for $g \in G(A)$. Thus if A is f-d then $G(A)$ is a finite group.

5. Let M, N be left A-modules (regard A as an algebra). Then $M \otimes N$ is a left A-module where $\boxed{a \cdot (m \otimes n) = a_{(1)} \cdot m \otimes a_{(2)} \cdot n.}$ For $g \in G \subseteq \Bbbk G$ note $g \cdot (m \otimes n) = g \cdot m \otimes g \cdot n$.

6. If S is bijective then $(A, m^{op}, \eta, \Delta, \epsilon, S^{-1})$, $(A, m, \eta, \Delta^{cop}, \epsilon, S^{-1})$ are Hopf algebras, where $m^{op}(a \otimes b) = ba$ and $\Delta^{cop}(a) = a_{(2)} \otimes a_{(1)}$. $(A, m^{op}, \eta, \Delta^{cop}, \epsilon, S)$ is a Hopf algebra.

7. If A is f-d then S is bijective.

8. If $(A, m, \eta, \Delta, \epsilon, S)$ is f-d $(A^*, \Delta^*, \epsilon^*, m^*, \eta^*, S^*)$ is a f-d Hopf algebra over $\Bbbk$.

9. Let C be a (coassociative) coalgebra. Every f-d subspace of C generates a f-d subcoalgebra. Thus A has simple subcoalgebras, and all are f-d. $\Bbbk 1$, more generally $\Bbbk g$ for $g \in G(A)$, is a simple subcoalgebra of A.

The reader is referred to any basic text on Hopf algebras: [**Swe 1967**], [**Abe 1980**], [**Mont 1993**], [**D–N–Rai 2001**].

3. The Enveloping Algebra

L is a Lie algebra over $\Bbbk$. Then the enveloping algebra $U(L)$ is a cocommutative Hopf algebra over $\Bbbk$ where

$$\boxed{\Delta(\ell) = 1 \otimes \ell + \ell \otimes 1,} \qquad (1)$$

$\epsilon(\ell) = 0$, and $S(\ell) = -\ell$ for $\ell \in L$. An $\ell \in A$ such that (1) holds is *primitive*. The set of primitives $P(A)$ of A is a subspace and a a Lie algebra under associative bracket. $A = U(L)$ is *pointed irreducible* meaning $\Bbbk 1$ is the only simple subcoalgebra of A.

Assume $\Bbbk$ has characteristic 0. Then $P(U(L))) = L$ and the Milnor-Moore Theorem [**Mil−Mo 1965**] characterizes the finitely generated graded pointed irreducible Hopf algebras over the field $\Bbbk$ as the enveloping algebras of f-d Lie algebras over $\Bbbk$. See [**Haz 2008**] also.

4. Affine Algebraic Groups

Coalgebras C over $\Bbbk$ provide many examples of algebras. The linear dual $C^* = \mathrm{Hom}_{\Bbbk}(C, \Bbbk)$ is an algebra over $\Bbbk$ with *convolution product*:

$$\boxed{1_{C^*} = \epsilon \qquad \text{and} \qquad ab(c) = a(c_{(1)})b(c_{(2)})}$$

for all $a, b \in C^*$ and $c \in C$.

Example 1 *C has basis $c_0, c_1, c_2, \ldots$ and*

$$\boxed{\Delta(c_n) = \textstyle\sum_{\ell=0}^{n} c_{n-\ell} \otimes c_\ell, \qquad \epsilon(c_n) = \delta_{n,0}.}$$

As $(ab)(c_n) = \displaystyle\sum_{\ell=0}^{n} a(c_{n-\ell})b(c_\ell)$,

$$C^* \simeq \Bbbk[[x]], \qquad a \mapsto \sum_{n=0}^{\infty} a(c_n)x^n.$$

Example 1 suggests coalgebra connections with combinatorics [**Rom – Rota 1978**], [**Rota 1978**].

Example 2 *Let* $n \geq 1$ *and* $C(n, \mathbb{k})$ *have basis* $x_{i,j}$*,* $1 \leq i, j \leq n$ *and*

$$\boxed{\Delta(x_{i,j}) = \textstyle\sum_{\ell=1}^{n} x_{i,\ell} \otimes x_{\ell,j}, \qquad \epsilon(x_{i,j}) = \delta_{i,j}.}$$

As $(ab)(x_{i,j}) = \displaystyle\sum_{\ell=1}^{n} a(x_{i,\ell})b(x_{\ell,j})$*,*

$C^* \simeq M(n, \mathbb{k}), \quad a \mapsto (a_{i,j}),$ *where* $a_{i,j} = a(x_{i,j})$*.*

We now continue. $B = S(C(n, \mathbb{k}))$ is the free commutative $\mathbb{k}$-algebra on $C(n, \mathbb{k})$. Now let $\Delta : B \longrightarrow B \otimes B$ and $\epsilon : B \longrightarrow \mathbb{k}$ be the algebra maps determined on $x_{i,j}$ as in Example 2. $Alg_{\mathbb{k}}(B, \mathbb{k})$ is closed under the convolution product, contains ϵ, and

$$Alg_{\mathbb{k}}(B, \mathbb{k}) \simeq M(n, \mathbb{k}), \qquad \alpha \mapsto (\alpha(x_{i,j})),$$

as (multiplicative) monoids.

$$\mathrm{Det} = \sum_{\sigma \in S_n} x_{1,\sigma(1)} \cdots x_{n,\sigma(n)} \in G(B),$$

$A = B[\mathrm{Det}^{-1}]$ is a Hopf algebra, $Alg_{\mathbb{k}}(A, \mathbb{k})$ is a group under convolution, and

$$\boxed{Alg_{\mathbb{k}}(A, \mathbb{k}) \simeq GL(n, \mathbb{k}) \qquad \text{as groups.}}$$

An affine algebraic group is a pair (G, A), where G is a group, A is a certain commutative Hopf algebra over $\mathbb{k}$, and $G \simeq Alg_{\mathbb{k}}(A, \mathbb{k})$. We write $A = A(G)$. A determines (G, A).

5. The General Theory of Hopf Algebras Begins

With Sweedler's book [**Swe 1969b**] the study of Hopf algebras in general was just underway. Previously Hopf algebras of interest were either cocommutative or commutative. $\Bbbk G$, $U(L)$ are cocommutative, as are formal groups when thought of as Hopf algebras, and $A(G)$ is commutative. $U(L)$ and $A(G)$ are precursors of quantum groups.

Efforts were made to prove know results, or discover new ones, for affine groups using Hopf algebra methods, see [**Swe 1969a**], [**Sul 1971**], [**Sul 1973**], [**Tak 1972a**], [**Tak 1972b**]. The study of cocommutative Hopf algebras was pursued [**Swe 1967**], [**New–Swe 1979**].

Connections were made with many aspects of algebra. Hopf algebras were seen as rings which were interesting in their own right. There was an effort to generalize results about the group algebras of finite groups to f-d Hopf algebras. For these generalizations would hold for both these group algebras and restricted enveloping algebras.

The Galois group (algebra) was replaced by a Hopf algebra and a general Hopf Galois theory was eventually developed [**Chase–Swe 1969**], [**Krei–Tak 1981**], [**Schauen 2004**]. For an up to date survey see [**Mont 2009**].

Hopf algebras were constructed as vector spaces on certain diagrams, such as rooted trees, which can be combined (accounting for a product) and decomposed (accounting for a coproduct); see [**Gross–Lar 1989**]. See [**Connes–Krei 2001**] for such Hopf algebras related to Feynman graphs.

The antipode was scrutinized since it is such an important part of the structure of a Hopf algebra. In [**Tak 1971**] a Hopf algebra is given where S is not bijective. If A is commutative or cocommutative $S^2 = \mathrm{Id}_A$ and therefore S is bijective.

In [**Taft 1971**] f-d examples T_n, where $n \geq 1$, are given where S^2 has finite order n. T_2 is Sweedler's example. Dim $T_n = n^2$.

Example 3 *Let $n \geq 1$ and suppose $q \in \Bbbk$ is a primitive n^{th} root of unity. T_n is generated as an algebra by a, x subject to the relations*

$$\boxed{xa = qax}, \quad x^n = 0, \quad \textit{and} \quad a^n = 1$$

and the coalgebra structure is determined by

$$\boxed{\Delta(a) = a \otimes a \text{ and } \Delta(x) = 1 \otimes x + x \otimes a.} \quad (2)$$

Necessarily $\epsilon(a) = 1$, thus $A \in G(T_n)$, and $\epsilon(x) = 0$. An $x \in A$ such that (2) is satisfied for some $a \in G(A)$ is skew-primitive. Compare with (1). When $a = 1$ note x is primitive. The boxed expressions are basic types of defining relations for the quantized enveloping algebras (here q is not a root of unity).

In 1975 Kaplansky postulated 10 conjectures [**Kap 1975**] about Hopf algebras some of which are open. They have focused the research of many.

6. Some Fundamental Results for F-D Hopf Algebras

G is a finite group. $\Lambda = \sum_{h \in G} h$ satisfies

$$g\Lambda = \sum_{h \in G} gh = \Lambda = 1_{\Bbbk}\Lambda = \epsilon(g)\Lambda$$

for $g \in G$ and

$$\epsilon(\Lambda) = \sum_{h \in G} \epsilon(h) = \sum_{h \in G} 1_{\Bbbk} = |G|1_{\Bbbk}.$$

Maschke's Theorem can be formulated: All left $\Bbbk G$-modules are completely reducible if and only if $\epsilon(\Lambda) \neq 0$.

$\Lambda \in A$ is a *left (*resp. *right) integral for A* if $\boxed{a\Lambda = \epsilon(a)\Lambda}$ (resp. $\boxed{\Lambda a = \epsilon(a)\Lambda}$) for all $a \in A$. There is non-zero (left) integral Λ for A iff A is f-d [**Swe 1969c**] in which case any (left) integral for A is a scalar multiple of Λ.

Now suppose A is f-d. All left A-modules are completely reducible if and only if $\epsilon(\Lambda) \neq (0)$ [**Lar–Swe 1969c**].

There is a $g \in G(A)$ which relates left and right integrals for A and an $\alpha \in G(A^*)$ which does the same for A^*. Let $A \xrightarrow{\sigma_g} A$, $A^* \xrightarrow{\sigma_\alpha} A^*$ denote conjugation by g, α respectively. Then $\sigma_g, \sigma^*_\alpha$ commute and

$$\boxed{S^4 = \sigma_g \circ \sigma^*_\alpha}.$$

Thus S has finite order. See [**Rad 1976**].

There is a Hopf algebra analog of Lagrange's Theorem for a finite group G. Let H be a subgroup of G. Then $|H|$ divides $|G|$ if and only if $\Bbbk G$ is a free left $\Bbbk H$-module. A most sought after result was finally established in [**Nic–Zel 1989**]:

Theorem 1 *A f-d Hopf algebra is a free (left) module over its sub-Hopf algebras.*

The proof, which was very elusive and is rather subtle, is based on the notion of relative Hopf module, a generalization of Hopf module.

Now let A be any Hopf algebra over $\Bbbk$. Then a left A-Hopf module is a triple (M, μ, ρ), where $A \otimes M \xrightarrow{\mu} M$ is a left A-module, $M \xrightarrow{\rho} A \otimes M$ is a left A-comodule which satisfy a certain compatibility. Writing $\mu(a \otimes m) = am$ and $\rho(m) = m_{(-1)} \otimes m_{(0)}$ this is

$$\rho(am) = a_{(1)}m_{(-1)} \otimes a_{(2)}m_{(0)} = \Delta(a)\rho(m).$$

All left A-Hopf modules are free and have a special basis [**Swe 1969c**]. This result is one of the most important in the theory of Hopf algebras. In particular it accounts for basic results about integrals.

Suppose A is semisimple (as an algebra). Then A is f-d [**Swe 1969c**]. If the characteristic of $\Bbbk$

is 0 then A^* is also semisimple and $S^2 = \mathrm{Id}_A$ [**Lar–Rad 1988a**, **1988b**]. If the characteristic is positive and A, A^* are semisimple $S^2 = \mathrm{Id}_A$ [**Eting–Gel 1998**].

7. With the Advent of Quantum Groups an Explosion of Activity

Drinfel'd's paper [**Drinfel'd 1987**] presented at the Inernational Congress of Mathematicians held at Berkeley, CA, in 1986 described new classes of non-commutative, non-commutative Hopf algebras, which we refer to as quantum groups, which are derived from commutative or cocommutative ones through "quantization". This paper pointed to connections between quantum groups and physics, representation theory, algebra, non-commutative geometry, and topology.

For connections with representation theory see [**Lusztig 1993**] and [**Char–Press 1994**]. For connections with non-commutative geometry see [**Manin 1988, 1991**], and for connections with invariants of knots, links, and 3-manifolds see [**Kassel–Rosso–Tur 1997**].

There are general text books on quantum groups. These include [**Char−Press 1994**], [**Kassel 1995**], and [**Majid 1995**].

Important consequences for Hopf algebras were the introduction of the quantized enveloping algebras, of quasitriangular Hopf algebras, an important example of which is the Drinfel'd double, and later introduction of Lusztig's small quantum groups. The paper [**Majid 1990**] is a good entry point for Hopf algebraists to make first foray into quantum groups.

There was an intense flurry of activity to find quantizations of Hopf algebras associated with certain affine groups. Sometime later quasitriangular Hopf algebras were seen to account for regular isotopy invariants of oriented knots and links in a very concrete manner [**Kauff−Rad 2001**].

For us a *quasitriangular Hopf algebra over* $\Bbbk$ is a pair (A, R), where A is a Hopf algebra over

$\mathbb{k}$, and $R \in A \otimes A$ satisfies certain axioms which guarantee that it satisfies algebraists' Yang–Baxter equation. When A is f-d the Drinfel'd double $(D(A), R)$ can be constructed. Both of A and $A^{*\,cop}$ are subHopf algebras of $D(A)$ and multiplication $A^* \otimes A \longrightarrow D(A)$ is a linear isomorphism.

Thus f-d quasitriangular Hopf algebras abound. The invariants they produce remain a mystery for the most part. Concerning the double, there is a rather remarkable connection between the formula for S^4 and when a certain 3-manifold invariant arises from $D(A), R)$ in [**Kauff–Rad 1993**]. The invariant was first described in [**Henn 1996**].

When A is f-d the category of left $D(A)$-modules is equivalent to the Yetter-Drinfel'd category ${}_A\mathcal{YD}$ [**Yetter 1990**], [**Majid 1991**]. Its objects are triples (M, μ, ρ), where $A \otimes M \xrightarrow{\mu} M$ is a left A-module structure and $M \xrightarrow{\rho} A \otimes M$ is a left A-comodule structure on M which satisfy the compatibility condition

$$a_{(1)}m_{(-1)} \otimes a_{(2)}m_{(0)} = (a_{(1)}m)_{(-1)}a_{(2)} \otimes (a_{(1)}m)_{(0)},$$

or equivalently

$$\rho(am) = a_{(1)}m_{(-1)}S(a_{(2)(2)}) \otimes a_{(2)(1)}m_{(0)},$$

a reflection of the commutation relation for multiplication in $D(A)$.

This condition is quite different from the Hopf module compatibility condition. Certain Hopf algebras in this category are important for the classification of f-d Hopf algebras when $A = \Bbbk G$ is the group algebra of a finite abelian group.

8. Classification of Pointed Hopf Algebras

Let A be any Hopf algebra. A_0 denotes the sum of all the simple subcoalgebras of A and A *is pointed* if these are 1-dimensional. In this case $A_0 = \Bbbk G(A)$ and is a subHopf algebra of A. The quantized enveloping algebras, and the small quantum groups of Lusztig, are pointed.

Suppose A_0 is a subHopf algebra of A. There is a graded pointed irreducible Hopf algebra $\mathrm{gr}(A)$ with $\mathrm{gr}(A)_0 = \mathrm{gr}(A)(0) = A_0$. We now outline the strategy of [**Andrus–Schn 2002**] for determining the structure of A.

Let $\mathrm{gr}(A) \xrightarrow{\pi} A_0$ be the projection. The right covariants $R = \mathrm{gr}(A)^{co\,\pi}$ form a graded pointed irreducible Hopf algebra **in the category** ${}_{A_0}\mathcal{YD}$ and there is an isomorphism of $\mathrm{gr}(A) \simeq R \times A_0$ with a canonical biproduct [**Rad 1985**].

For a discussion of Hopf algebras in ${}_{A_0}\mathcal{YD}$ and related categories see [**Majid 1992**]. We have $R(0) = \Bbbk 1$ and $R(1) = P(R)$. The Nichols algebra associated with $V = P(R)$ is $\mathcal{B}(V)$, the subalgebra of R generated by V. We note $\mathcal{B}(V)$ is analogous the enveloping algebra of a Lie algebra. Steps for classification of A:

(1) Determine the structure of $\mathcal{B}(V)$;

(2) Determine all Hopf algebras B over $\Bbbk$ such that $\mathrm{gr}(B) \simeq \mathcal{B}(V) \times A_0$;

(3) Determine whether or not $\mathcal{B}(V) = R$ (in which case $A = B$ for some B of (2)).

Let B be a Hopf algebra over $\Bbbk$. Then for any object V of ${}_B\mathcal{YD}$ there is a graded pointed irreducible Hopf algebra $\mathcal{B}(V)$ in ${}_B\mathcal{YD}$ which is determined by $\mathcal{B}(V)(1) = V$ and V generates $\mathcal{B}(V)$ as an algebra. These are the Nichols algebras [**Nic 1978**]. They have been described in many ways in important cases which have been studied in [**Lusztig 1993**], [**Rosso 1995, 1998**], [**Heck 2004**]. Basic results about them are nontrivial.

Andruskiewitsch and Schneider have used them in classifying f-d pointed Hopf algebras when $\Bbbk$ is algebraically closed of characteristic 0 and $G(A)$ is commutative with mild restrictions on $|G(A)|$ [**Andrus–Schn 2010**]. The similarities between these Hopf algebras and Lustig's small quantum groups are striking.

9. Semisimple Hopf Algebras

The theory of Hopf algebras and their related structures has developed in many directions. A major one is the classification of the f-d Hopf algebras. For some techniques used for low dimensions see [**Andrus–Nat 2001**]. An indication of how results about group algebras in characteristic 0 can be extended to results on Hopf algebras is given by [**Kas–Som–Zhu 2006**].

Generally classification has focused on two types of Hopf algebras, the pointed Hopf algebras and the semisimple Hopf algebras, when $\Bbbk$ has characteristic 0. In the pointed case efforts are now focused on the case $\Bbbk G(A)$ is not abelian [**Heck–Sch 2008**], [**Andrus–Fan–Gra–Ven 2010**].

There are various types of results concerning semisimple Hopf algebras. Surveys on aspects of semisimple Hopf algebras include [**Mas 1996**], [**Mont 2001**], [**Nat 2007**].

From this point on $\Bbbk$ has characteristic 0 and A is a semisimple Hopf algebra over $\Bbbk$. Recall A must be f-d and cosemisimple.

A consideration for an A is how closely it is related to the group algebra of a finite group or its dual. A is *trivial* if $A \simeq \Bbbk G$, or $A \simeq \Bbbk G^*$, for some finite group G, surely a notion group theorists are not too happy with. [**Ng 2004**] shows that A is trivial if $\operatorname{Dim} A = 2p$, where p is an odd prime. [**Ng 2008**] establishes the same if $\operatorname{Dim} A = pq$, where p, q are primes with $2 < p < q \leq 4p + 11$.

Another measure of how close A is to being a group algebra is whether or not its category of representations is that of the group algebra of a finite group. If this is the case A is said to be *group theoretical*. Just recently an A was discovered which is not group theoretical [**Nyk2008**].

If A is altered by a Drinfel'd twist (dual 2-cocycle twist) then the resulting Hopf algebra A' is semisimple with the same algebra structure. Thus the representations of the algebras A and A' are the same. In [**Eting–Gel 2000**] all semisimple, cosemisimple, and triangular Hopf algebras are shown to be Drinfel'd twists of group algebras (the field need only be algebraically closed).

Another research direction is to classify semisimple Hopf algebras A of a given dimension. See [**Mas 1995a, 1995b, 1995c, 1996a, 1996b**] for a variety of cases. My hope is that this line of research will lead to new techniques for the study of Hopf algebras.

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