Pointed Hopf algebras – from enveloping algebras to quantum groups and beyond

Groups, Rings, Lie and Hopf Algebras, II

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0. Introduction

All objects which we discuss are defined over a field k. Group algebras, enveloping algebras of Lie algebras, quantized enveloping algebras, and the small quantum groups of Lusztig are examples of pointed Hopf algebras. These belong to a rather extensive class of pointed Hopf algebras. The classification of the Hopf algebras in this class is nearing completion and their representation theory is being developed.

We discuss aspects of the theory of pointed Hopf algebras as it has unfolded over the past forty years with emphasis on algebraic techniques which have played an important role in its development. We will describe classification results for the pointed Hopf algebras mentioned above and some of the recent results on their representations.

1. Basic definitions and examples

An associative algebra with unity over k can be thought of as a vector space A over k with linear maps $m : A \otimes A \longrightarrow A$ and $\eta : k \longrightarrow A$ which determine the product and unity 1. The dual notion of associative algebra is coalgebra; a vector space C over k with linear maps

$$\Delta : C \longrightarrow C \otimes C, \;\; \epsilon : C \longrightarrow k$$

which satisfy the "duals" of the axioms for an associative algebra. The maps Δ and ϵ are called the *coproduct* and *counit*.

Let $c \in C$. Then $\Delta(c) \in C \otimes C$ is usually denoted by the Heyneman-Sweedler notation

$$\boxed{\Delta(c) = c_{(1)} \otimes c_{(2)}.}$$

For example, the counit laws

$$(\epsilon \otimes \mathrm{Id}_C)\Delta = \mathrm{Id}_C = (\mathrm{Id}_C \otimes \epsilon)\Delta$$

and coassociative law

$$(\mathrm{Id}_C \otimes \Delta)\Delta = (\Delta \otimes \mathrm{Id}_C)\Delta$$

are expressed as

$$\epsilon(c_{(1)})c_{(2)} = c = c_{(1)}\epsilon(c_{(2)})$$

and

$$c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} = c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}$$

for all $c \in C$. The expression $c_{(1)} \otimes c_{(2)} \otimes c_{(3)}$ is used to represent either side of the preceding equation.

To say that $c \in C$ is *cocommutative* is expressed $c_{(1)} \otimes c_{(2)} = c_{(2)} \otimes c_{(1)}$; the coalgebra C is cocommutative if all of its elements are. In this notation calculations are based on formal manipulations of subscripts.

Example 1 *Let C be a coalgebra. The linear dual C^* is an algebra with unity ϵ, where*

$$ab(c) = a(c_{(1)})b(c_{(2)}) \tag{1}$$

for all $a, b \in C^*$ and $c \in C$.

The product in C^* is a convolution product.

A Hopf algebra is an associative algebra H with unity 1 which has a coalgebra structure, such that Δ, ϵ are algebra maps, and a certain algebra anti-endomorphism $S : H \longrightarrow H$, which is called an *antipode*. *Throughout C denotes a coalgebra* and *H denotes a Hopf algebra*.

Example 2 *Let G be a group. The group algebra $k[G]$ is a cocommutative Hopf algebra where*

$$\Delta(g) = g \otimes g, \;\; \epsilon(g) = 1, \tag{2}$$

and $S(g) = g^{-1}$ for all $g \in G$.

An element $g \in C$ which satisfies (2) is called *grouplike*. The set of grouplike elements $G(C)$ of C is linearly independent and $G(H)$ is a group under multiplication with neutral element 1. The inverse of $g \in G(H)$ is $S(g)$. In particular the linear span of $G(H)$ is the group algebra $k[G(H)]$, thus:

Example 3 $k[G(H)]$ *is a cocommutative sub-Hopf algebra of* H*.*

Also, $G(k[G]) = G$. Thus G is recovered from the coalgebra structure of $k[G]$.

Example 4 *Let* L *be a Lie algebra over* k*. The universal enveloping algebra* $U(L)$ *of* L *is a cocommutative Hopf algebra where*

$$\Delta(e) = 1 \otimes e + e \otimes 1, \ \ \epsilon(e) = 0, \qquad (3)$$

and $S(e) = -e$ *for* $e \in L$*.*

For any Hopf algebra H over k an element $e \in H$ which satisfies (3) is called *primitive*. Generally the set of primitive elements $P(H)$ of H is a subspace which is a Lie algebra under associative bracket.

Also, $P(U(L)) = L$ when L is finite-dimensional and the characteristic of k is zero. Thus L is recovered from the coalgebra structure of $U(L)$ in this case.

A subcoalgebra of H is a subspace D of H such that $\Delta(D) \subseteq D \otimes D$. If $g \in G(H)$ then $D = kg$ is a subcoalgebra of H which is simple, meaning D contains exactly 2 subcoalgebras. The Hopf algebra H is *pointed* if $k[G(H)]$ is the sum of the simple subcoalgebras of H and is *pointed irreducible* if it is pointed and $G(H) = \{1\}$.

Both $U(L)$ and $k[G]$ are pointed; indeed $U(L)$ is pointed irreducible. Note that $U(L)$ generated as an algebra by its primitive elements.

Observe that $k[G]$ is generated by its grouplike elements.

The quantized enveloping algebras and "small quantum groups", fundamental examples of quantum groups, are pointed Hopf algebras generated by their grouplike elements together by their skew-primitive elements. An element e of a Hopf algebra H is called *skew-primitive* if

$$\Delta(e) = g \otimes e + e \otimes h \tag{4}$$

for some $g, h \in G(H)$. Thus primitive elements are skew-primitive with $g = h = 1$. Generally quantum groups are neither commutative nor cocommutative. If H is generated (as an algebra) by grouplike and skew primitive elements then H is pointed.

Open Question 1 *When is a pointed Hopf algebra generated (as an algebra) by its grouplike and skew-primitive elements?*

2. Hopf modules

The theory of (left) Hopf modules is very useful in understanding the structure of finite-dimensional pointed Hopf algebras. A left H-Hopf module is a left H-module M with a left H-comodule structure $\rho : M \longrightarrow H{\otimes}M$ such that

$$\begin{aligned} \rho(hm) &= \Delta(h)\rho(m) \\ &\in (H{\otimes}H)(H{\otimes}M) \\ &= H{\otimes}M \end{aligned} \tag{5}$$

for all $h \in H$ and $m \in M$. Notice the similarity between the multiplicative formula for the coproduct $\Delta(h\ell) = \Delta(h)\Delta(\ell)$ for all $h, \ell \in H$ and the preceding for ρ.

A non-zero left H-Hopf module is always free as an H-module. Any linear basis of

$$M^{\natural} = \{m \in M \,|\, \rho(m) = 1{\otimes}m\}$$

is an H-module basis for M. This is really the sum and substance of the theory of Hopf modules. Its importance lies in applications.

3. The coradical filtration

Let $J = \text{Rad}(C^*)$ be the Jacobson radical of C^*. For $n \geq 0$ let $C_n = (J^{n+1})^{\perp} \subseteq C$ be the set of common zeros of the elements (functionals) of J^{n+1}. Then $C_0, C_1, C_2, \ldots$ is the *coradical filtration of* C. The sum of the simple subcoalgebras of C is C_0,

$$C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq \cup_{n=0}^{\infty} C_n = C,$$

and

$$\Delta(C_n) \subseteq \sum_{i=0}^{n} C_{n-i} \otimes C_i \qquad (6)$$

for all $n \geq 0$. In particular C_n is a subcoalgebra of C for all $n \geq 0$. The coradical filtration can be defined from C_0 inductively by

$$C_n = \Delta^{-1}(C_0 \otimes C + C \otimes C_{n-1}) \qquad (7)$$

for all $n \geq 1$. Since C_n is a subcoalgebra of C it is a left C-comodule under Δ. Let $n \geq 1$. Then the quotient comodule structure $\rho : C_n/C_{n-1} \longrightarrow C{\otimes}(C_n/C_{n-1})$ satisfies $\mathrm{Im}\,\rho \subseteq C_0{\otimes}(C_n/C_{n-1})$ by (6). Therefore C_n/C_{n-1} is a left C_0-comodule.

Suppose that H_0 is a sub-Hopf algebra of H, as is the case when H is pointed by Example 3. Then H_n is a left H_0-submodule under multiplication by (7). The H_0 quotient structures on H_n/H_{n-1} are a left H_0-Hopf module structure. Therefore the quotient is a free left H_0-module; consequently H is a free left H_0-module. As $H_0 = k[G(H)]$ when H is pointed:

Corollary 1 *A pointed Hopf algebra is a free left $k[G(H)]$-module.* $\Box$

There is a Hopf algebra analog of Lagrange's Theorem for finite groups.

Theorem 1 *A finite-dimensional Hopf algebra is a free module over its sub-Hopf algebras.*

The proof, which is quite subtle, is based on the notion of relative Hopf module, a generalization of Hopf module.

Suppose that H is pointed. Then H_1 is spanned by the grouplike and skew-primitive elements of H. Thus Open Question (1) can be reformulated: When is a pointed Hopf algebra H generated as an algebra by H_1?

4. Finite-dimensional simple-pointed Hopf algebras

First some general remarks. Suppose $z \in H$ is skew-primitive. Then

$$\Delta(z) = g\otimes z + z\otimes h$$

for some $g, h \in G(H)$. Since Δ is multiplicative

$$\begin{aligned}\Delta(zg^{-1}) &= \Delta(z)\Delta(g^{-1}) \\ &= (g\otimes z + z\otimes h)(g^{-1}\otimes g^{-1}) \\ &= 1\otimes zg^{-1} + zg^{-1}\otimes hg^{-1}.\end{aligned}$$

Let $x = zg^{-1}$ and $a = hg^{-1} \in G(H)$. Then

$$\Delta(x) = 1\otimes x + x\otimes a. \qquad (8)$$

Let V be the subspace of all $v \in H$ which satisfy (8). If $g \in G(H)$ commutes with a then $gVg^{-1} \subseteq V$ as

$$\Delta(gvg^{-1}) = g1g^{-1}\otimes gvg^{-1} + gvg^{-1}\otimes gag^{-1}$$

for $v \in V$.

Further assume that k is algebraically closed of characteristic zero and H is finite-dimensional. Let $\mathcal{G}$ be a commutative subgroup of $G(H)$

which contains a. Then $\mathcal{G}$ is finite and conjugation by elements of $\mathcal{G}$ on V induces a weight space decomposition $V = \bigoplus_{\chi \in \widehat{\mathcal{G}}} V_\chi$. Replacing x by a weight vector of V_χ we have that $gxg^{-1} = \chi(g)x$, or equivalently $gx = \chi(g)xg$, for all $g \in \mathcal{G}$. In particular

$$\Delta(x) = 1 \otimes x + x \otimes a \quad \text{and} \quad ax = qxa, \tag{9}$$

where $q = \chi(a)$ and is therefore a root of unity.

Let $\mathcal{H}$ be the subalgebra of H generated by $\mathcal{G}$ and x and set $\mathcal{B} = k[\mathcal{G}]$. Then

$$\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{B}x^i.$$

Let $\mathcal{H}_{(n)} = \mathcal{B} + \cdots + \mathcal{B}x^n$ for $n \geq 0$. Then

$$\mathcal{H}_{(0)} \subseteq \mathcal{H}_{(1)} \subseteq \mathcal{H}_{(2)} \subseteq \cup_{n=0}^{\infty} \mathcal{H}_{(n)} = \mathcal{H},$$

$$\mathcal{H}_{(m)}\mathcal{H}_{(n)} \subseteq \mathcal{H}_{(m+n)} \quad \text{and} \quad \text{for all } m, n \geq 0,$$

$$\Delta(\mathcal{H}_{(n)}) \subseteq \sum_{i=0}^{n} \mathcal{H}_{(n-i)} \otimes \mathcal{H}_{(i)}$$

for all $n \geq 0$.

For each $n > 1$ either $\mathcal{H}_{(n)}/\mathcal{H}_{(n-1)}$ is (0) or is a free left module over $\mathcal{H}_{(0)} = \mathcal{B}$ with basis $x^n + \mathcal{H}_{(n-1)}$. The argument follows the one found at the end of Section 3.

Using these ideas, and the nature if the expansion of $\Delta(x^m)$, we can analyze finite-dimensional simple-pointed Hopf algebras. Note that

$$\begin{aligned}
\Delta(x^m) &= (\Delta(x))^m \\
&= (1\otimes x + x\otimes a)^m \\
&= \sum_{i=0}^{m} \binom{m}{i}_q x^{m-i}\otimes a^{m-i}x^i
\end{aligned}$$

has a q-binomial expansion since

$$\begin{aligned}
(x\otimes a)(1\otimes x) &= (x1\otimes ax) \\
&= q(1x\otimes xa) \\
&= q(1\otimes x)(x\otimes a).
\end{aligned}$$

Lemma 1 *Suppose k has characteristic zero and $x \in H$ is a non-zero primitive. Then the set of all non-negative powers $\{1, x, x^2, \ldots\}$ is linearly independent.*

Sketch of proof. Assume there is a minimal dependency relation and apply Δ to it using the q-binomial expansion on the powers of x. Here $q = 1$ and the (binomial) coefficients are not zero.

H is *simple-pointed* if:

(SP.1) pointed,

(SP.2) not cocommutative, and

(SP.3) any proper sub-Hopf algebra of H is contained in $H_0 = k[G(H)]$.

Example 5 *$H_{q,\alpha,n,m}$, where n, m are positive integers, $n > 1$ and $n|m$, $q \in k$ is a primitive n^{th} root of unity, and $\alpha \in k$. As an algebra $H_{q,\alpha,n,m}$ is generated by a, x subject to the relations*

$$xa = qax, \ \ x^n = \alpha(a^n - 1), \ \ \textit{and} \ \ a^m = 1$$

and whose coproduct is determined by

$$\Delta(a) = a \otimes a \ \ \textit{and} \ \ \Delta(x) = 1 \otimes x + x \otimes a.$$

We note that $\mathrm{Dim}\, H_{q,\alpha,n,m} = mn$.

Suppose that k is algebraically closed of characteristic zero. Then the finite-dimensional simple-pointed Hopf algebras are those Hopf algebras of Example 5; we may take $\alpha = 0, 1$.

For a given dimension there are a finite number of isomorphism types of simple-pointed Hopf algebras. This is not true without the simple-pointed assumption. There are examples resulting from a construction of a sequence of Ore extensions.

Finite-dimensional simple pointed Hopf algebras have a finite number of sub-Hopf algebras since $k[G(H)]$ does. There are examples of finite-dimensional pointed Hopf algebras which have infinitely many sub-Hopf algebras.

Example 6 *Let k be any field and H as an algebra be generated by a, x, y subject to the relations*

$$xa = -ax, \;\; ya = -ay, \;\; yx = -xy,$$

$$x^2 = y^2 = 0, \textit{and } a^2 = 1$$

and whose coproduct is determined by

$$\Delta(a) = a \otimes a \;\; \textit{and} \;\; \Delta(z) = 1 \otimes z + z \otimes a,$$

where $z = x, y$. Then $\dim H = 8$ and for $\alpha \in k$ the subalgebra H_α of H generated by a and $x + \alpha y$ is a 4-dimensional sub-Hopf algebra of H. Note that $H_\alpha \simeq H_0$ as Hopf algebras and $H_\alpha = H_{\alpha'}$ implies $\alpha = \alpha'$. Take k to be infinite.

Open Question 2 *Which of the finite-dimensional pointed Hopf algebras have only finitely many sub-Hopf algebras?*

To continue with analogs of results about groups in the theory of pointed Hopf algebras. Regarding Cauchy's Theorem:

Open Question 3 *If p is a positive prime which divides the dimension of a finite-dimensional pointed Hopf algebra is there a sub-Hopf algebra of dimension p?*

In regard to the Sylow Theorems:

Example 7 *Let r, s be distinct positive primes. Then $H = H_{q,\alpha,rs,rs}$ has dimension $(rs)^2$ and sub-Hopf algebras have dimension $1, r, s, rs$ or $(rs)^2$. Therefore H has no sub-Hopf algebras of dimension r^2 or s^2.*

Open Question 4 *Is there a good analog of the Sylow Theorems for finite-dimensional pointed Hopf algebras?*

5. Characters of one-dimensional representations of H

These are simply the algebra homomorphisms $\eta : H \longrightarrow k$. The set $\mathsf{Alg}\,(H,k)$ of them is a multiplicative subgroup of H^* with neutral element ϵ and the restriction map

$$\mathsf{res} : \mathsf{Alg}\,(H,k) \longrightarrow \widehat{G(H)}, \qquad \eta \mapsto \eta|_{G(H)}$$

is a group homomorphism. Note that Ker res consists of all η such that $\eta(g) = 1$ for all $g \in G(H)$.

Many pointed Hopf algebras, including the examples mentioned in Section 1, are generated by grouplikes and skew primitives x such that $gx = \chi(g)xg$ for all $g \in G(H)$, where $\chi \in \widehat{G(H)}$

is non-trivial. Applying η as above to the preceding equation results for all $g \in G(H)$ in $\eta(g)\eta(x) = \chi(g)\eta(x)\eta(g)$, and thus $\eta(x) = \chi(g)\eta(x)$ since g is invertible. As χ is non-trivial $\eta(x) = 0$. Thus for these Hopf algebras $\mathrm{Ker\,res} = \{\epsilon\}$; that is η is determined by its action on $G(H)$.

In light of Open Question 1 the following is interesting.

Proposition 1 *Suppose that H is pointed, finitely generated as a left H_0-module, and k has characteristic zero. Then* $\mathrm{Ker\,res} = \{\epsilon\}$.

Sketch of proof. Let $\eta \in \mathrm{Ker\,res}$. Then η vanishes on $I = \sum_{g \in G(H)} H(g-1)H$ since $\eta(g) = 1 = \eta(1)$ for all $g \in G(H)$. For the same reason ϵ vanishes on I. We show that I has codimension one, and thus $\mathrm{Ker}\,\eta = I = \mathrm{Ker}\,\epsilon$ whence $\eta = \epsilon$.

We next consider the (Hopf algebra) projection $\pi : H \longrightarrow H/I = \mathcal{H}$. Then $\mathcal{H}$ is a finitely generated module over $\pi(H_0) = k1$, hence is finite-dimensional. Now $\mathcal{H}_0 \subseteq \pi(H_0) = k1$ means that $\mathcal{H}$ is pointed irreducible. Since the characteristic of k is zero, $\mathcal{H}$ has no non-zero (skew) primitive elements by Lemma 1. Thus $\mathcal{H}_1 = \mathcal{H}_0$ which means $\mathcal{H} = \mathcal{H}_0 = k1$ by (7). Thus I has codimension one.

6. The graded Hopf algebra associated to a pointed Hopf algebra and "lifting"

Recall $J = \mathrm{Rad}(C^*)$ is the Jacobson radical of C^*. Since J is a two-sided ideal of C^*

$$\mathrm{gr}(C^*) = \bigoplus_{n=0}^{\infty} J^n/J^{n+1}$$

is a graded algebra in the usual way, where $J^0 = C^*$. For $n \geq 0$ the rule

$$\pi_n : J^n/J^{n+1} \longrightarrow \Big((J^{n+1})^{\perp}/(J^n)^{\perp}\Big)^* = (C_n/C_{n-1})^*$$

given by

$$\pi_n(a + J^{n+1})(c + C_{n-1}) = a(c)$$

for all $a \in J^n$ and $c \in C_n$ describes a well-defined linear map whose image is a dense subspace of its codomain. There is a unique coalgebra structure on

$$\mathrm{gr}\,(C) = \bigoplus_{n=0}^{\infty} C_n/C_{n-1}$$

such that the composite

$$\pi : \mathrm{gr}\,(C^*) \longrightarrow \mathrm{gr}(C)^*$$

given by

$$\bigoplus_{n=0}^{\infty} J^n/J^{n+1} \stackrel{\oplus_{n=0}^{\infty}\pi_n}{\longrightarrow} \bigoplus_{n=0}^{\infty} (C_n/C_{n-1})^* \subseteq \mathrm{gr}(C)^*$$

is an algebra homomorphism, where $C_{-1} = (0)$. There is a concrete way of understanding

π. For $n \geq 0$ let V_n be a subspace of C_n such that $C_{n-1} \oplus V_n = C_n$. Then $C = \bigoplus_{n=0}^{\infty} V_n$. The composite of projections

$$j_n : C = \bigoplus_{m=0}^{\infty} V_m \longrightarrow V_n \subseteq C_n \longrightarrow C_n/C_{n-1}$$

satisfies

$$j_n|_{C_n} : C_n \longrightarrow C_n/C_{n-1}$$

is the projection and

$$j_n|_{V_n} : V_n \longrightarrow C_n/C_{n-1}$$

is an isomorphism. In particular

$$\mathrm{Dim}\, C = \mathrm{Dim}\, \mathrm{gr}(C)$$

$$\Delta_{\mathrm{gr}}(c + C_{n-1}) = \sum_{i=0}^{n} (j_{n-i} \otimes j_i)(\Delta(c))$$

and

$$\epsilon_{\mathrm{gr}}(c + C_{n-1}) = \delta_{n,0}\epsilon(c)$$

for all $c \in C_n$. We note that $\mathrm{gr}(C)_0 = C_0$ as coalgebras.

Suppose that H_0 is a sub-Hopf algebra of H, as is the case when H is pointed. Then $H_m H_n \subseteq H_{m+n}$ for all $m, n \geq 0$ by induction on $m+n$ using (6). Thus gr(H) has a natural graded algebra structure which, together with the coalgebra structure above makes gr(H) a Hopf algebra. To study H we first study gr(H).

Suppose further that H is pointed. Then gr(H) is also since gr$(H)_0 = H_0$. We consider how certain algebra relations are affected in replacing H by gr(H). Consider, for example, the defining relations for $H = H_{q,\alpha,n,m}$:

$$xa = qax, \quad \boxed{x^n = \alpha(a^n - 1)}, \quad \text{and} \quad a^m = 1;$$

$$\Delta(a) = a \otimes a \quad \text{and} \quad \Delta(x) = 1 \otimes x + x \otimes a.$$

Set $\mathbf{a} = a$, $\mathbf{1} = 1$, and $\mathbf{x} = x + H_0$. Then in gr(H) we have

$$\mathbf{xa} = q\mathbf{ax}, \quad \boxed{\mathbf{x}^n = \mathbf{0}}, \quad \text{and} \quad \mathbf{a}^m = \mathbf{1};$$

$$\Delta(\mathbf{a}) = \mathbf{a} \otimes \mathbf{a} \quad \text{and} \quad \Delta(\mathbf{x}) = \mathbf{1} \otimes \mathbf{x} + \mathbf{x} \otimes \mathbf{a}.$$

Now suppose $x, y \in H$, $a, b \in G(H)$ commute, and $q \in k\backslash 0$ satisfy:

$$xb = qbx \text{ and } ya = q^{-1}ay;$$

$$\Delta(x) = 1 \otimes x + x \otimes a \text{ and } \Delta(y) = 1 \otimes y + y \otimes b.$$

Then

$$\Delta(xy - qyx) = 1 \otimes (xy - qyx) - (xy - qyx) \otimes ab$$

which is consistent with $\boxed{xy - qyx = \alpha(ab - 1)}$ for some $\alpha \in k$. This relation goes over to $\boxed{\mathbf{xy} - q\mathbf{yx} = \mathbf{0}}$ in gr(H). In practice, $xy - qyx$ is a commutator in a certain context.

Lifting is the process of constructing, or reconstructing, defining relations for H from defining relations for gr(H).

7. Bi-products and Nichols algebras

Suppose that H has bijective antipode, as is the case for pointed Hopf algebras, $\mathcal{H}$ is a sub-Hopf algebra of H, and $\pi : H \longrightarrow \mathcal{H}$ is a Hopf algebra map which satisfies $\pi(h) = h$ for all $h \in \mathcal{H}$. Let

$$R = \{h \in H \mid h_{(1)} \otimes \pi(h_{(2)}) = h \otimes 1\}.$$

Then R is a subalgebra of H; usually *not* a subcoalgebra of H. However there is a coalgebra structure on (R, δ, ε) on R and a surjective map of coalgebras $\Pi : H \longrightarrow R$. Furthermore R has a left $\mathcal{H}$-module structure and a left $\mathcal{H}$-comodule structure (R, ρ) such that

$$\rho(h \cdot m) = h_{(1)} m_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)} \qquad (10)$$

for all $h \in \mathcal{H}$ and $m \in R$, where $\rho(m) \in H \otimes R$ is written

$$\boxed{\rho(m) = m_{(-1)} \otimes m_{(0)}}$$

for $m \in R$. Compare (10) with the Hopf module condition

$$\rho(h \cdot m) = \Delta(h) \rho(m) = h_{(1)} m_{(-1)} \otimes h_{(2)} \cdot m_{(0)}.$$

With the algebra and left $\mathcal{H}$-module structure, and the coalgebra and left $\mathcal{H}$-comodule structure,

$$H = R{\times}\mathcal{H} \ \ (= R{\otimes}\mathcal{H} \text{ as a vector space})$$

is a bi-product. The most important example for our purposes is H pointed and $\mathcal{H} = H_0 = k[G(H)]$. Here the most important structure to understand is R, and R is very subtle.

The compatibility condition mentioned above defines a braided monoidal category $\mathcal{Y}D$, called a *Yetter-Drinfel'd category* whose objects are left $\mathcal{H}$-modules and left $\mathcal{H}$-comodules such that (10) holds. The braiding isomorphisms $\sigma_{M,N} : M{\otimes}N \longrightarrow N{\otimes}M$ are given by

$$\sigma_{M,N}(m{\otimes}n) = m_{(-1)}{\cdot}n{\otimes}m_{(0)}.$$

R is usually *not* a Hopf algebra since the coproduct $\delta : R \longrightarrow R{\otimes}R$ is usually *not* an algebra map.

However, *with the definition of tensor product of algebras in the category $\mathcal{YD}$, R is a Hopf algebra in $\mathcal{YD}$.*

For algebras A, B over k, the product of $A \otimes B$ is a composition of linear maps described by

$$\begin{array}{rcl} (a \otimes b) \otimes (a' \otimes b') & \mapsto & a \otimes b \otimes a' \otimes b' \\ & \mapsto & a \otimes (b \otimes a') \otimes b' \\ & \overset{\mathrm{I} \otimes \tau \otimes \mathrm{I}}{\mapsto} & a \otimes (a' \otimes b) \otimes b' \\ & \mapsto & (a \otimes b) \otimes (a' \otimes b') \\ & \mapsto & aa' \otimes bb' \end{array}$$

For algebras A, B in $\mathcal{YD}$ we define the tensor product algebra structure in the same way, replacing the "twist map" τ, which is usually not a morphism, by the braiding isomorphism $\sigma_{R,R}$. Writing $A \underline{\otimes} B$ for this structure setting $a \underline{\otimes} b = a \otimes b$, note

$$(a \underline{\otimes} b)(a' \underline{\otimes} b') = a(b_{(-1)} \cdot a') \underline{\otimes} b_{(0)} b'.$$

The appropriate notion of "commutator" in A is that of *braided commutator*. For $a, b \in A$ the usual commutator

$$\mathrm{ad}\, a(b) = [a, b] = ab - ba = m(\mathrm{Id}_A - \tau)(a \otimes b)$$

where m and τ are the product and "twist" maps respectively. Replacing τ by the braiding isomorphism $\sigma_{A,A}$ we have

$$\mathrm{ad}_c\, a(b) = [a, b]_c = ab - (a_{(-1)} \cdot b)a_{(0)}$$

Now we turn to to the important case H is pointed and $G(H)$ is a *commutative* group. Our discussion applies to $\mathrm{gr}(H)$ and $\mathcal{H} = k[G(H))]$ with

$$\pi : \mathrm{gr}(H) \longrightarrow \mathrm{gr}(H)_0 = H_0 = k[G(H)]$$

the projection onto the degree zero term. In this case:

(N.1) $R = \bigoplus_{n=0}^{\infty} R(n)$ is a graded pointed irreducible Hopf algebra in $\mathcal{YD}$;

(N.2) $P(R) = R(1)$;

and possibly

(N.3) $R(1)$ generates R as an algebra.

An algebra in $\mathcal{YD}$ which satisfies (N.1)–(N.3) is called a *Nichols algebra*. The subalgebra of R generated by $V = R(1)$ is a Nichols algebra denoted $\mathcal{B}(V)$. It is determined by the object V of $\mathcal{YD}$.

Suppose further that V is finite-dimensional and k is algebraically closed of characteristic zero. Then there is a basis $\{x_1, \ldots, x_n\}$ for

V, elements $g_1, \ldots, g_n \in G(G)$, and characters $\chi_1, \ldots, \chi_n \in \widehat{G(H)}$, such that

$$\rho(x_i) = g_i \otimes x_i \text{ and } g x_i g^{-1} = \chi_i(g) x_i$$

for all $1 \leq i \leq n$ and $g \in G$. The module action of $\mathcal{H}$ on V is given by

$$g x_i = \chi_i(g) x_i$$

for all $g \in G(H)$ and $1 \leq i \leq n$. In particular V is an object of $\mathcal{Y}D$. The braiding $c = \sigma_{V,V}$ is given by

$$c(x_i \otimes x_j) = g_i \cdot x_j \otimes x_i = \chi_j(g_i) x_j \otimes x_i = q_{ij} x_j \otimes x_i,$$

where

$$q_{ij} = \chi_j(g_i).$$

The matrix (q_{ij}) is called the *infinitesimal braiding matrix*. The endomorphism $c : V \otimes V \longrightarrow V \otimes V$ is called an *infinitesimal braiding*. Note that

$$\begin{aligned} \mathrm{ad}_c\, x_i(x_j) &= [x_i, x_j]_c \\ &= x_i x_j - (g_i \cdot x_j) x_i \\ &= x_i x_j - q_{ij} x_j x_i. \end{aligned}$$

The Nichols algebra $\mathcal{B}(V)$ has a more intrinsic description which does not involve $\mathcal{YD}$. Luzstig and Rosso studied these algebras in this light and found generators and relations in important cases. These are deep results from quantum groups and are used heavily in the classification program for pointed Hopf algebras.

We end with the simple example of the Taft algebra $H = H_{q,1,n,m}$, where $n > 1$ and $q \in k$ is a primitive n^{th} root of unity. Recall that H is generated by a, x subject to the relations

$$xa = qax, \;\; x^n = a^n - 1, \;\text{ and }\; a^m = 1$$

and whose coproduct is determined by

$$\Delta(a) = a \otimes a \;\text{ and }\; \Delta(x) = 1 \otimes x + x \otimes a.$$

Here gr(H) as above with $x^n = a^n - 1$ replaced by $x^n = 0$. Note $V = R(1) = kx$,

$$a \cdot x = q^{-1}x, \;\; \rho(x) = a^{-1} \otimes x, \;\text{ and }\; c(x \otimes x) = qx \otimes x.$$

Now $\delta(x) = 1 \otimes x + x \otimes 1$. Thus R is *not* a Hopf algebra over k since $\{1, x, x^2, \ldots\}$ is not linearly independent. To calculate

$$\delta(x^m) = (1 \underline{\otimes} x + x \underline{\otimes} 1)^m$$

we note that

$$\begin{aligned}(1 \underline{\otimes} x)(x \underline{\otimes} 1) &= 1(x_{(-1)} \cdot x) \underline{\otimes} x_{(0)} 1 \\ &= qx \underline{\otimes} x \\ &= q(x \underline{\otimes} 1)(1 \underline{\otimes} x)\end{aligned}$$

as $c(1 \otimes 1) = 1$. Thus

$$\delta(x^m) = \sum_{i=0}^{m} \binom{m}{i}_q x^{m-i} \underline{\otimes} x^i$$

whence $\delta(x^n) = 1 \underline{\otimes} x^n + x^n \underline{\otimes} 1$, as

$$\binom{n}{i}_q = 0 \ \text{ for all } \ 1 \leq i \leq n-1,$$

which is compatible with $x^n = 0$.

8. Classification results; the finite-dimensional case

We assume that k is an algebraically closed field of characteristic zero. Suppose that H is finite-dimensional and pointed. Under some mild assumptions on the infinitesimal braiding matrix and $G(H)$, $H \simeq u(\mathcal{D}, \lambda, \mu)$ as a Hopf algebras.

$\boxed{\mathcal{D} = (\Gamma, \{g_i\}, \{\chi_i\}, (a_{ij}))}$, where $1 \le i, j \le n$, is a **datum of finite Cartan type**: Γ is an abelian group, $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$, (a_{ij}) is a matrix of finite Cartan type and

$$\chi_j(g_i)\chi_i(g_j) = \chi(q_{ii})^{a_{ij}} \text{ and } \chi_{ii}(g_i) \neq 1. \quad (11)$$

Set $\boxed{q_{ij} = \chi_j(g_i).}$ Let χ be components of the Dynkin diagram, $\sim$ the equivalence relation which defines them. For $J \in \chi$ then common order of the q_{ii}'s, $i \in J$, is N_J.

$\boxed{\lambda = \{\lambda_{ij}\}}$, where $1 \leq i, j \leq n$ is a **family of linking parameters**: $\lambda_{ij} \in k$, $\lambda_{ji} = -q_{ij}^{-1}\lambda_{ij}$, and $\lambda_{ij} \neq 0$ implies

$$i \not\sim j, \;\; g_i \neq g_j^{-1}, \;\text{ and }\; \chi_i = \chi_j^{-1},$$

.

$\boxed{\mu = \{\mu_\alpha\}_{\alpha\in\Phi+}}$: $\mu_\alpha \in k$, for all $J \in \chi$, $\alpha \in J$, $\mu_\alpha \neq 0$ implies

$$g_\alpha^{N_J} \neq 1 \;\text{ and }\; \chi_\alpha^{N_J} = \epsilon.$$

Let V have basis $\{x_1, \ldots, x_n\}$. Then V is an object of $\mathcal{Y}D$ with

$$g{\cdot}x_i = \chi_i(g)x_i \;\text{ and }\; \rho(x_i) = g_i{\otimes}x_i$$

for all $g \in \Gamma$ and $1 \leq i \leq n$. $T(V)$ is an algebra in $\mathcal{Y}D$.

$U(\mathcal{D}, \lambda) = T(V){\times}k[\Gamma]$ modulo the ideal generated by:

(QSR) $\mathrm{ad}_c\,(x_i)^{1-a_{ij}}(x_j) = 0,\ i \sim j, i \neq j;$

(LR) $\mathrm{ad}_c\,(x_i)^{1-a_{ij}}(x_j) = \mathrm{ad}_c\,(x_i)(x_j) =$
$x_i x_j - q_{ij} x_j x_i = \lambda_{ij}(1 - g_i g_j),\ i \not\sim j;$

and $u(\mathcal{D}, \lambda, \mu)$ is $U(\mathcal{D}, \lambda)$ modulo the ideal generated by

(RVR) $x_\alpha^{N_J} = u_\alpha(\mu),\ \alpha \in \Phi_J^+,\ J \in \chi.$

Certain constructions produce $g_\alpha \in \Gamma$, $\chi_\alpha \in \widehat{\Gamma}$, $x_\alpha \in T(V)$, and $u_\alpha(\mu) \in k[< g_1^{N_1}, \ldots, g_n^{N_n} >]^+$. Identifying cosets with representatives,

$$\Delta(g) = g \otimes g, \ \ \Delta(x_i) = g_i \otimes x_i + x_i \otimes 1$$

for all $g \in \Gamma$ and $1 \leq i \leq n$. The $U(\mathcal{D}, \lambda)$'s account for the quantized enveloping algebras.

A huge result from quantum groups (basically):

Theorem 2 *$\mathcal{D}$ as above, assume q_{ij} has odd order for all $1 \leq i,j \leq n$, and q_{ii} has order prime to 3 if i lies in a component of G_2. Then $\mathcal{B}(V) = T(V)/I$, where I is generated by*

$$\mathrm{ad}_c\,(x_i)^{1-a_{ij}}(x_j) \text{ for all } 1 \leq i,j \leq n, i \neq j$$

$$x_\alpha^{N_J} \text{ for all } \alpha \in \Phi_J^+, J \in \chi.$$

Theorem 3 *H finite-dimensional pointed. Assumptions on the infinitesimal braiding above, and the prime divisors of $|G(H)|$ exceed 7. Then $H \simeq u(\mathcal{D}, \lambda, \mu)$ for some data as above.*

Corollary 2 $\mathrm{gr}(u(\mathcal{D}, \lambda, \mu)) \simeq \mathcal{B}(V) \times k[\Gamma]$.

9. Generalized doubles

Our representation theory applies to $U(\mathcal{D}, \lambda)$ and $u(\mathcal{D}, \lambda, 0)$. In the latter $x_\alpha^{N_J} = 0$ for all $\alpha \in \Phi_J^+$, $J \in \chi$.

Suppose that $\mathcal{D}$ is *generic*, that is q_{ii} is not a root of unity for all $1 \leq i \leq n$. Then the linking graph is bipartite. The vertices of the linking graph are elements of χ and J, J' are joined by an edge if $\lambda_{ii'} \neq 0$ for some $i \in J$ and $i' \in J'$.

As a result there there are bi-products

$$U = \mathcal{B}(U) \times k[\Lambda] \text{ and } A = \mathcal{B}(V) \times k[\Gamma]$$

and a surjective Hopf algebra map

$$(U \otimes A)_\sigma \longrightarrow U(\mathcal{D}, \lambda)$$

whose kernel is generated by differences of central grouplikes. $\sigma : (U \otimes A) \otimes (U \otimes A) \longrightarrow k$ is a certain type of two-cocycle twist. As a coalgebra $(U \otimes A)_\sigma = U \otimes A$ and the product satisfies

$$(u \otimes a)(u' \otimes a') = uu' \otimes aa' \text{ if } a = 1 \text{ or } u' = 1. \tag{12}$$

Suppose further that k is algebraically closed of characteristic zero. Then the finite-dimensional irreducible representations of U and A are one-dimensional. *A generalized double* is a Hopf algebra of the form $(U\otimes A)_\sigma$, where U, A are Hopf algebras.

10. The foundation of a representation theory

Suppose U, A are algebra and $H = U\otimes A$ has an algebra structure which satisfies (12). Let

$$\rho \in \text{Alg}(U, k) \ \text{ and } \ \chi \in \text{Alg}(A, k).$$

Classification results lead to a very general highest weight theory which accounts for a parameterization of simple modules in many cases. The context of this theory is the class of algebras of the form $U\otimes A$, where U and A are algebras over k and $(u\otimes a)(u'\otimes a') = uu'\otimes aa'$ whenever $a = 1$ or $u' = 1$. The pointed Hopf algebras H of interest to us are quotients of $U\otimes A$.

Which simple H-modules are finite-dimensional has been solved and the complete reducibility of finite-dimensional representations of H is under investigation.