MATH 215 Final Examination with Solution Radford

12/05/05

Name (print) ______________________________

(1) With the exception of part a) of Problem 1, write yours answers in the exam booklet provided. (2) The last problem, Problem 8, has two versions. **Do one or the other but not both**. (3) *Return* this exam copy with your test booklet. (4) *You are expected to abide by the University's rules concerning academic honesty.*

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1. (25 pts. total) Let P and Q be statements.

a) (12 pts.) Complete the following three truth tables (on this sheet if you wish):

P	Q	P implies Q
T	T	
T	F	
F	T	
F	F	

P	Q	P and Q
T	T	
T	F	
F	T	
F	F	

P	Q	not P	(not P) or Q
T	T		
T	F		
F	T		
F	F		

Solution:

P	Q	P implies Q
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	not P	(not P) or Q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(**4 points each**)

b) (8 pts.) What are the logical relationships (implication, equivalence, negation) between the statements "P implies Q", "P and Q", and "(not P) or Q"? Justify your answer in terms of the truth tables of parts a).

Solution: Since the last columns of the first and third truth tables are the same, "P implies Q" and "(not P) or Q" are logically equivalent. (**4 points**) Whenever "P and Q" is true "P implies Q" is true. Therefore "P and Q" implies "P implies Q" and "(not P) or Q". (**4 points**) This is sufficient for an answer.

c) (5 pts.) Given the logical equivalence of "P" and "not (not P)", of "not (P or Q)" and "(not P) and (not Q)", deduce that "P and (not Q)" is the negation of "(not P) or Q".

Solution: Writing "≡" for logical equivalence we deduce

$$\text{not }((\text{not P})\text{ or Q}) \equiv (\text{not }(\text{not P}))\text{ and }(\text{not Q}) \equiv \text{P and }(\text{not Q}),$$

2. (30 pts. total) Consider the statement "$(a-3)(a-5) > 0$ implies $a < 3$ or $5 < a$".

a) (5 pts.) What is the converse of the statement?

Solution: "$a < 3$ or $5 < a$ implies $(a-3)(a-5) > 0$".

b) (5 pts.) What is the contrapositive of the statement? Write it without "not".

Solution: "$3 \leq a \leq 5$ implies $(a-3)(a-5) \leq 0$".

c) (10 pts.) Prove the statement by contradiction.

Solution: Suppose the the hypothesis is true and the conclusion is false; that is $(a-3)(a-5) > 0$ and $(a-3)(a-5) \leq 0$". (**3 points**) If $a = 3$ or $a = 5$ then $a - 3 = 0$ or $a - 5 = 0$ in which case $(a-3)(a-5) = 0$, contradiction. (**3 points**) Since $3 \leq a \leq 5$ necessarily Thus $3 < a < 5$. But then $a - 3 > 0$ and $a - 5 < 0$ which means $(a-3)(a-5) < 0$, a contradiction. (**4 points**) Therefore the hypothesis implies the conclusion after all.

d) (10 pts.) Prove the *converse* of the statement directly.

Solution: Suppose that $a < 3$ or $5 < a$.

Case 1: $a < 3$. Then $a - 3 < 0$ and $a - 5 < a - 3 < 0$. Therefore $(a-3)(a-5) > 0$ as the left hand side is the product of two negative numbers. (**5 points**)

Case 1: $5 < a$. Then $a - 5 > 0$ and $a - 3 > a - 5 > 0$. Therefore $(a-3)(a-5) > 0$ as the left hand side is the product of two positive numbers. (**5 points**)

Base the proofs for parts c) and d) of Problem 2 on the axioms for the real number system and: the product of two positive real numbers is positive; the product of two negative real numbers is positive; the product of a positive real number and a negative real number is negative; the product of zero and any real number is zero; and if a, b, c are real numbers and $a < b$ then $a - c < b - c$ and $c - a > c - b$.

3. (15 pts. total) Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers and $\ell \in \mathbf{R}$. The statement $\lim_{n \longrightarrow \infty} a_n = \ell$ expressed in terms of quantifiers is $\forall \epsilon > 0, \exists N > 0, \forall n \geq N, |a_n - \ell| < \epsilon$. (It is understood that $\epsilon \in \mathbf{R}^+$, $N \in \mathbf{Z}^+$, and $n \in \{N, N+1, \ldots\}$.)

a) (5 pts.) Express the statement $\lim_{n\longrightarrow\infty} a_n = \ell$ in English with the quantifiers translated.

Solution: $\lim_{n\longrightarrow\infty} a_n = \ell$ means that for all positive numbers ϵ there exists a positive integer N such that $|a_n - \ell| < \epsilon$ for all integers $n \geq N$.

b) (10 pts.) Express the statement $\lim_{n\longrightarrow\infty} a_n \neq \ell$ (the negation of $\lim_{n\longrightarrow\infty} a_n = \ell$) in terms of quantifiers, without the use of "not".

Solution: $\exists \epsilon > 0, \forall N > 0, \exists n \geq N, |a_n - \ell| \geq \epsilon$. (**2, 2, 2, 4 points for parts**)

4. (25 pts. total) Let U be a universal set.

a) (7 pts.) State De Morgan's Laws for subsets A, B of U.

Solution:

$(A\cup B)^c = A^c\cap B^c$ and $(A\cap B)^c = A^c\cup B^C$. (**4 points one, 7 points both**)

b) (18 pts.) For sets $A_1, \ldots, A_n$ we define the union $A_1\cup\cdots\cup A_n$ and intersection $A_1\cap\cdots\cap A_n$ inductively by

$$A_1\cup\cdots\cup A_n = \begin{cases} A_1 & : \ n = 1; \\ (A_1\cup\cdots\cup A_{n-1})\cup A_n & : \ n > 1 \end{cases}$$

and

$$A_1\cap\cdots\cap A_n = \begin{cases} A_1 & : \ n = 1; \\ (A_1\cap\cdots\cap A_{n-1})\cap A_n & : \ n > 1 \end{cases}.$$

Now suppose that $A_1, \ldots, A_n \subseteq U$. Use De Morgan's Laws and the definitions above to construct a proof by induction that

$$(A_1\cap\cdots\cap A_n)^c = A_1^c\cup\cdots\cup A_n^c$$

for $n \geq 1$. [Comment: You may assume $A_1\cup\cdots\cup A_m, A_1\cap\cdots\cap A_m \subseteq U$ for all $A_1, \ldots, A_m \subseteq U$. *The steps of your proof must at least be implicitly justified.*]

Solution: We wish to prove $(A_1\cap\cdots\cap A_n)^c = A_1^c\cup\cdots\cup A_n^c$ for $n \geq 1$. Suppose $n = 1$. Then $(A_1\cap\cdots\cap A_n)^c = A_1^c = A_1^c\cup\cdots\cup A_n^c$ by the definition of intersection and union above. Thus the assertion is true for $n = 1$. (**4 points**)

Suppose that $n \geq 1$ and the assertion is true. Let $A_1, \ldots, A_{n+1} \subseteq U$. Then $A_1, \ldots, A_n \subseteq U$ and

$$\begin{aligned} (A_1\cap\cdots\cap A_{n+1})^c &= ((A_1\cap\cdots\cap A_n)\cap A_{n+1})^c \quad \textbf{(3 points)} \\ &= (A_1\cap\cdots\cap A_n)^c\cup A_{n+1}^c \quad \textbf{(4 points)} \\ &= (A_1^c\cup\cdots\cup A_n^c)\cup A_{n+1}^c \quad \textbf{(4 points)} \\ &= A_1^c\cup\cdots\cup A_{n+1}^c. \quad \textbf{(3 points)} \end{aligned}$$

Thus the assertion holds for $n+1$, By induction the assertion holds for all $n \geq 1$. (**14 points**)

5. (30 pts. total) Let X and Y be sets.

   a) (8 pts.) Give the conditional definitions of $X \cup Y$, $X \cap Y$, and $X \times Y$.

   Solution: $X \cup Y = \{z \,|\, z \in X \;\; or \;\; z \in Y\}$ (**2 points**), $X \cap Y = \{z \,|\, z \in X \;\; and \;\; z \in Y\}$ (**2 points**), $X \times Y = \{(x, y) \,|\, x \in X \;\; and \;\; y \in Y\}$ (**4 points**).

   b) (4 pts.) Give the definition of $X \subseteq Y$.

   Solution: $X \subseteq Y$ if and only if $x \in X$ implies $x \in Y$.

   c) (2 pts.) What does it mean for $x \notin X \cap Y$?

   Solution: $x \notin X$ or $x \notin Y$.

   d) (12 pts.) Construct a proof of the equivalence of: (1) $X = X \cap Y$, (2) $X \subseteq X \cap Y$, and (3) $X \subseteq Y$ based on the definitions in parts a) and b).

   Solution: It suffices to show that (1) implies (2), (2) implies (3), and (3) implies (1).

   (1) implies (2). Assume $X = X \cap Y$. Then $x \in X$ implies $x \in X \cap Y$ which means $X \subseteq X \cap Y$ by definition.

   (2) implies (3). Suppose $X \subseteq X \cap Y$. Let $x \in X$. Then $x \in X \cap Y$ which means $x \in X$ and $x \in Y$. In particular $x \in Y$. Therefore $X \subseteq Y$ by definition.

   (3) implies (1). Suppose $X \subseteq Y$. Let $x \in X$. Then $x \in Y$ which means $x \in X \cap Y$. Conversely, if $x \in X \cap Y$ then $x \in X$ and $x \in Y$; in particular $x \in X$. By definition of equality of sets, $X = X \cap Y$.

   e) (4 pts.) Now suppose that $X = \{2, \pi, -6\}$. Find $P(X)$.

   Solution: $P(X) = \{\emptyset, \{2\}, \{\pi\}, \{-6\}, \{2, \pi\}, \{2, -6\}, \{\pi, -6\}, X\}$.

6. (25 pts. total) Suppose that $f : A \longrightarrow B$ is a function.

   a) (10 pts.) Suppose further $g : B \longrightarrow A$ is a function and $g \circ f = I_A$. Show that f is an injection and g is a surjection.

   Solution: Let $a \in A$. Then $g(f(a)) = (g \circ f)(a) = I_A(a) = a$.

   f is an injection. Suppose that $a, a' \in A$ and $f(a) = f(a')$. Then $a = g(f(a)) = g(f(a')) = a'$ by the preceding equations. Therefore f is a an injection. (**5 points**)

   g is a surjection. Suppose $a \in A$ and set $b = f(a)$. Then $g(b) = a$ by the preceding equations. Therefore g is a surjection. (**5 points**)

For the remainder of this problem $A = B = \mathbf{Z}^+$ and $f(n) = 2n$ for all $n > 0$.

b) (10 pts.) Find $\overrightarrow{f}(\{1,2,3,4\})$ and $\overleftarrow{f}(\{1,2,3,4\})$.

Solution: $\overrightarrow{f}(\{1,2,3,4\}) = \{f(1), f(2), f(3), f(4)\} = \{2,4,6,8\}$ (**5 points**) and

$\overleftarrow{f}(\{1,2,3,4\}) = \{n \in \mathbf{Z}^+ \,|\, f(n) \in \{1,2,3,4\}\} = \{n \in \mathbf{Z}^+ \,|\, 2n \in \{1,2,3,4\}\} = \{1,2\}$.

(**5 points**)

c) (5 pts.) *Define* a function $g : B \longrightarrow A$ such that $g \circ f = I_A$. [You may assume these basic facts about the integers: any integer n can be written $n = 2m$ for some integer m or $n = 2m+1$ for some integer m, but not both, and in either case the integer m is unique.]

Solution: What is required for g is simply $g(f(n)) = n$ for all $n \in \mathbf{Z}^+$; that is $g(2n) = n$ for all $n \in \mathbf{Z}^+$. Thus

$$g(n) = \begin{cases} m & : \quad n = 2m \text{ for some } m \in \mathbf{Z}^+; \\ 1: & : \quad n = 2m+1 \text{ for some } m \in \mathbf{Z}^+ \end{cases},$$

or more informally

$$g(n) = \begin{cases} n/2 & : \quad n > 0 \text{ even}; \\ 1: & : \quad n > 0 \text{ odd} \end{cases},$$

works.

7. (25 pts. total) A committee of 5 is to be formed from a group of 11 people.

a) (8 pts.) Suppose a certain 3 individuals from this group are to be *excluded*. How many such committees are there?

Solution: $\binom{11-3}{5} = \binom{8}{5} = \dfrac{8{\cdot}7{\cdot}6}{3{\cdot}2{\cdot}1} = 56.$

b) (5 pts.) Suppose that a certain 2 individuals from this group are to be *included*. How many such committees are there?

Solution: $\binom{11-2}{5-2} = \binom{9}{3} = \binom{9}{6} = \dfrac{9{\cdot}8{\cdot}7}{3{\cdot}2{\cdot}1} = 84.$

c) (12 pts.) Suppose *at least one* of 2 certain individuals from this group is to be included. Let X be the set of committees which include the first and Y the set of committees which include the second. Use the principle of inclusion-exclusion to calculate the number of elements of $X \cup Y$, the set of committees which include at least one of the two.

Solution: $|X| = |Y| = \binom{11-1}{5-1} = \binom{10}{4} = \binom{10}{6} = \frac{10{\cdot}9{\cdot}8{\cdot}7}{4{\cdot}3{\cdot}2{\cdot}1} = 210$. Thus

$$|X \cup Y| = |X| + |Y| - |X \cap Y| = 210 + 210 - 84 = 336$$

which follows from part b).

Do ONE version of Problem 8 but NOT both.

8. (version 1) (25 pts. total) Let A, B be sets.

a) (4 pts.) Give the definition for "A and B are equipotent".

Solution: A and B are equipotent if and only if there is a bijection $f : A \longrightarrow B$.

b) (4 pts.) Give the definition for "A is a finite set".

Solution: A is a finite set if and only if $A = \emptyset$ or for some $m \geq 1$ there is a bijection $f : \mathbf{N}_m \longrightarrow A$, where $\mathbf{N}_m = \{1, 2, \ldots, m\}$.

c) (4 pts.) Give the definition for "A is a denumerable set".

Solution: A is a denumerable set if and only if there is a bijection $f : \mathbf{Z}^+ \longrightarrow A$, where $\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$.

d) (3 pts.) Give the definition for "A is a countable set".

Solution: A is a countable set if it is either finite or denumerable.

e) (10 pts.) Let $A = \{7, 9, 11, 13, 15, \ldots\}$. Show that A is denumerable by showing that the definition of denumerable set is satisfied.

Solution: Note that $2n + 5$ is an odd integer for all $n \geq 1$, and $n \geq 1$ implies $2n + 5 \geq 7$, or equivalently $2n + 5 \in A$. Define $f : \mathbf{Z}^+ \longrightarrow A$ by $f(n) = 2n + 5$ for all $n \geq 1$.

First of all f is an injection. Let $n, n' \in \mathbf{Z}^+$ and suppose that $f(n) = f(n')$. Then $2n + 5 = 2n' + 5$ which implies $2n = 2n'$ and therefore $n = n'$.

f is a surjection. Let $n \in A$. Then $n = 2m + 1$ for some $m \in \mathbf{Z}$. Since $n \geq 7$ it follows that $2m + 1 \geq 7$. Thus $2m \geq 6$ and therefore $m \geq 3$. Thus $m - 2 \geq 1$, which implies $m - 2 \in \mathbf{Z}^+$, and $f(m-2) = 2(m-2) + 5 = 2m + 1 = n$.

8. (version 2) (25 pts. total) Determine:

a) (12 pts.) the rational number $32.7\overline{38}$;

Solution: Let $r = 32.7\overline{38}$. Then $10r = 327.\overline{38}$ and $1000r = 32738\overline{38}$. Therefore

$$990r = 1000r - 10r = 32738 - 327 = 34111$$

which means that $r = \dfrac{34111}{990}$.

b) (13 pts.) the greatest common divisor of 344 and 60 by the Euclidean Algorithm.

Solution:

$$\begin{aligned} 344 &= 5{\cdot}60 + 44 \\ 60 &= 1{\cdot}44 + 16 \\ 44 &= 2{\cdot}16 + 12 \\ 16 &= 1{\cdot}12 + 4 \\ 12 &= 3{\cdot}4 + 0; \end{aligned}$$

thus

$$\gcd(144, 60) = \gcd(60, 44) = \gcd(44, 16) = \gcd(16, 12) = \gcd(12, 4) = \gcd(4, 0) = 4.$$