MATH 215 **Summer 2009** **Radford**

Written Homework #5

Due at the beginning of class 07/17/2009

For $n \geq 1$ recall we defined the union of sets $A_1, \ldots, A_n$ inductively by

$$A_1 \cup \cdots \cup A_n = \begin{cases} A_1 & : \quad n = 1; \\ (A_1 \cup \cdots \cup A_{n-1}) \cup A_n & : \quad n > 1 \end{cases}$$

and we define the intersection of $A_1, \ldots, A_n$ inductively by

$$A_1 \cap \cdots \cap A_n = \begin{cases} A_1 & : \quad n = 1; \\ (A_1 \cap \cdots \cap A_{n-1}) \cap A_n & : \quad n > 1 \end{cases} .$$

1. Let $A, A_1, \ldots, A_n$ be sets, where $n \geq 1$.

(a) Prove, by induction, that $A \times (A_1 \cup \cdots \cup A_n) = (A \times A_1) \cup \cdots \cup (A \times A_n)$. [You may assume $A \times (B \cup C) = (A \times B) \cup (A \times C)$ for sets A, B, C.]

(b) Prove, by induction, that $A \times (A_1 \cap \cdots \cap A_n) = (A \times A_1) \cap \cdots \cap (A \times A_n)$. [First prove $A \times (B \cap C) = (A \times B) \cap (A \times C)$ for sets A, B, C.]

2. Let A, B be fixed sets. Determine the logical relationships between the following statements:

(a) $\forall a \in A,\ \exists b \in B,\ P(a, b)$;

(b) $\exists b \in B,\ \forall a \in A,\ P(a, b)$;

(c) $\exists a \in A,\ \forall b \in B,\ \text{not } P(a, b)$;

(d) $\exists a \in A,\ \exists b \in B,\ P(a, b)$.

Comment: Here implication means universal implication where we regard the statements $P(a, b)$ as parameters. Thus one statement implies another if the implication holds for all possible $P(a, b)$. To show one statement does not imply another supply a counterexample.

Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a function and $a, b \in \mathbf{R}$. A compact definition of $\lim_{x \longrightarrow a} f(x) = b$ is: $\forall \epsilon > 0,\ \exists \delta > 0,\ P(\epsilon, \delta)$, where $P(\epsilon, \delta) : \forall x \in \mathbf{R},\ (0 < |x - a| < \delta) \Longrightarrow (|f(x) - b| < \epsilon)$.

3. Let $f(x) = \begin{cases} 11x - 3 & : \quad x \neq 4 \\ 47 & : \quad x = 4 \end{cases}$. Prove that $\lim_{x \longrightarrow 4} f(x) = 41$ from the definition of limit.

4. Here is an exercise in what "not ($\lim_{x \longrightarrow a} f(x) = b$)" means.

(a) Use quantifiers to express "not ($\lim_{x \longrightarrow a} f(x) = b$)", expressing "not $P(\epsilon, \delta)$" without using "not".

(b) Show that $\lim_{x \longrightarrow 0} f(x) = b$ is false for all $b \in \mathbf{R}$, where $f(x) = \begin{cases} 1/2 & : \quad x \geq 0 \\ 1/3 & : \quad x < 0 \end{cases}$

5. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be defined by $f(x) = x^2 - 6x + 21$ for all $x \in \mathbf{R}$.

(a) Show that f is not surjective.

(b) Show that f is not injective.