1. (**20 points total**) We construct truth table for the statements of parts a) and b), adding an extra column for convenience.

P	Q	not P	(not P) or Q
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

and

P	Q	not Q	P and (not Q)
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

.

(**4** points for each truth table.) Since the last column of the second table is the last column of the first with T's and F's interchanged, the statements of a) and b) are negations of each other. (**3**)

The statement of a) does not imply that of b) since the first table has a T in the first row, last column and the second table has a F in the first row, last column, for example. (**3**)

The statement of b) does not imply that of a) since the second table has a T in the second row, last column and the first table has a F in the second row, last column. (**3**)

Since the statement of a) does not imply that of b), the statements are not equivalent. (**3**)

2. (**20 points total**) We construct truth table for the statements of parts a) and b), adding extra columns for convenience.

P	Q	P or Q	not (P or Q)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

and

P	Q	not P	not Q	(not P) or (not Q)
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

.

(**4** points for each truth table.) Since the last column of the second table is not the last column of the first with T's and F's interchanged, the statements of a) and b) are not negations of each other. (**3**)

The statement of a) does not imply that of b) since the first table has a T in the fourth row, last column and the second table has a F in the fourth row, last column. (**3**)

The statement of b) does not imply that of a) since the second table has a T in the second row, last column and the first table has a F in the second row, last column, for example. (**3**)

Since the statement of a) does not imply that of b), the statements are not equivalent. (**3**)

3. (**20 points total**) We construct truth table for the statements of parts a) and b), adding extra columns for convenience.

P	Q	R	P ⇒ Q	not R	(P ⇒ Q) ⇒ (not R)
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	F	F	T
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	T	F	F
F	F	F	T	T	T

and

P	Q	R	not Q	(not Q) ⇒ R	P ⇒ ((not Q) ⇒ R)
T	T	T	F	T	T
T	T	F	F	T	T
T	F	T	T	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	T	F	F	T	T
F	F	T	T	T	T
F	F	F	T	F	T

.

(**4** points for each truth table.) Since the last column of the second table is not the last column of the first with T's and F's interchanged, the statements of a) and b) are not negations of each other. (**3**)

The statement of a) does not imply that of b) since the first table has a T in the fourth row, last column and the second table has a F in the fourth row, last column. (**3**)

The statement of b) does not imply that of a) since the second table has a T in the first row, last column and the first table has a F in the first row, last column, for example. (**3**)

Since the statement of a) does not imply that of b), the statements are not equivalent. (**3**)

4. (**20 points total**) Note that $x^2 < x$ is equivalent to $0 < x - x^2 = x(1-x)$. Observe that the graph of $y = x - x^2$ is a parabola opening upward and crossing the x-axis at $x = 0, 1$. For all real numbers x let P(x) be the statement "$x \geq 0$" and Q(x) be the statement "$x^2 < x$".

a) We construct the table

	P(x)	Q(x)	P(x) ⇒ $Q(x)$
$x < 0$	F	F	T
$x = 0$	T	F	F
$0 < x < 1$	T	T	T
$x = 1$	T	F	F
$1 < x$	T	F	F

.

(**6** points for the truth table.) Since the last column contains at least one F the universal implication is false. (**4**)

b) Using the preceding table we construct

	P(x)	Q(x)	Q(x) $\Rightarrow$ P(x)
$x < 0$	F	F	T
$x = 0$	T	F	T
$0 < x < 1$	T	T	T
$x = 1$	T	F	T
$1 < x$	T	F	T

(**6** points for the truth table.) Since there are no F's in the last column the converse is true. (**4**)

5. (**20 points total**) P and Q are really a universal statements. Let a be a real number and P(a) and Q(a) be the statements "$a < -1$" and $a^2 - 2a - 3 \geq 0$" respectively. Then P $\Rightarrow$ Q is true exactly when P(a) $\Rightarrow$ Q(a) is true for all real number a and likewise Q $\Rightarrow$ P is true exactly when Q(a) $\Rightarrow$ P(a) is true for all real number a.

The graph of $y = x^2 - 2x - 3$ is a parabola which opens upward and crosses the x-axis at $x = -1$ and $x = 3$. Thus $x^2 - 2x - 3 > 0$ exactly when $x < -1$ or $3 < x$. Thus P(a) true, that is $a < -1$, implies $a^2 - 2a - 3 > 0$ and therefore $a^2 - 2a - 3 \geq 0$, or Q(a) is true. We have shown:

a) P $\Rightarrow$ Q. (**5**)

Now Q(3) is true since $3^2 - 2{\cdot}3 - 3 = 0$, and thus $3^2 - 2{\cdot}3 - 3 \geq 0$, but P(3) is false as $3 < -1$ is false. We have shown Q(3) does not imply P(3) and therefore

b) Q does not imply P. (**5**)

c) Note "P only if Q" is logically equivalent to "P $\Rightarrow$ Q". Thus "P only if Q" is true by a). (**3**)

d) Note "P is necessary for Q" is logically equivalent to "Q $\Rightarrow$ P". Thus "P is necessary for Q" is false by b). (**3**)

e) Note "P is sufficient for Q" is logically equivalent to "P $\Rightarrow$ Q". Thus "P is sufficient for Q" is true by a). (**2**)

f) Note "P if and only if Q" is logically equivalent to "(P $\Rightarrow$ Q) and (Q $\Rightarrow$ P)". Since "Q $\Rightarrow$ P" is false by b) the preceding compound statement is false and thus "P if and only if Q" is false. (**2**)