MATH 215 # Hour Exam I Solution **Radford** **07/07/2009**

Base the proofs for problems 2c) and 4 on the axioms for the real number system and:

a) the product of two positive real numbers is positive,

b) the product of two negative real numbers is positive,

c) the product of a positive real number and a negative real number is negative,

d) the product of zero and any real number is zero, and

e) if a, b, c are real numbers and $a < b$ then $a - c < b - c$ and $c - a > c - b$.

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1. (**25 points total**) Let P and Q be statements.

a) Write out the truth table for the statement "P implies (Q and P)".

Solution:

P	Q	P implies (Q and P)
T	T	T
T	F	F
F	T	T
F	F	T

(**5**)

b) Write out the truth table for the statement "(P implies Q) and P".

P	Q	(P implies Q) and P
T	T	T
T	F	F
F	T	F
F	F	F

(**5**)

c) Does the statement of part a) imply the statement of part b)? Justify your answer in terms of the truth tables of parts a) and b).

Solution: No (**2**), line 3 (or 4) of the table of a) has a "T" in the last column and the corresponding position in the table of part b) has an "F". (**5**)

Solution:

d) Does the statement of part b) imply the statement of part a)? Justify your answer in terms of the truth tables of parts a) and b).

Solution: Yes (**3**), in the table of part b) the only "T" which occurs is in the last column in row 1 and there is a "T" in the corresponding position in the table of part a). (**5**)

Comment: One could also out the full truth table of "(P implies (Q and P)) implies ((P implies Q) and P)" and "((P implies Q) and P) implies (P implies (Q and P))" to answer parts c) and d) respectively.

2. (**30 points total**) Consider the statement "$(a+3)(a-5) > 0$ implies $a \leq -3$ or $5 < a$".

a) Write down the converse of the statement and determine whether or not it is true.

Solution: The converse is "$a \leq -3$ or $5 < a$ implies $(a+3)(a-5) > 0$ ". (**5**) The converse is false since with $a = -3$, the relation $-3 \leq a < 5$ holds and $(a+3)(a-5) = 0 \not> 0$. (**5**)

b) What is the contrapositive of the statement? Write it without "not".

Solution: $-3 < a \leq 5$ (**3**) $\implies$ (**3**) $(a+3)(a-5) \leq 0$ (**4**).

c) Prove the statement by contradiction.

Solution: Suppose the hypothesis $(a+3)(a-5) > 0$ is true and the conclusion $a \leq -3$ or $5 < a$ is false. (**2**) Then $(a+3)(a-5) > 0$ and $-3 < a \leq 5$. (**2**) Now $-3 < a$ implies $0 < a+3$ and $a \leq 5$ implies $a - 5 \leq 0$. (**2**) Therefore $a+3$ is positive and $a-5$ is negative, in which case $(a+3)(a-5) < 0$, or $a+3$ is positive and $a-5 = 0$, in which case $(a+3)(a-5) = 0$. In either case we contradict $(a+3)(a-5) > 0$. (**2**) Thus if the hypothesis is true the conclusion must be also. (**2**)

3. (**25 points total**) Let x be a real number, $x \neq \pm 1$. Prove by induction that the sum of the odd powers

$$x + x^3 + x^5 + \cdots + x^{2n-1} = \frac{x^{2n+1} - x}{x^2 - 1}$$

for all $n \geq 1$.

Solution: The equation holds when $n = 1$ since the left hand side is $x^1 = x$ and the right hand side is $\dfrac{x^3 - x}{x^2 - 1} = \dfrac{x(x^2-1)}{x^2 - 1} = x$. (**5**) Suppose that $n \geq 1$ and the equation holds. We must show that it holds for $n+1$; that is

$$x + x^3 + x^5 + \cdots + x^{2(n+1)-1} = \frac{x^{2(n+1)+1} - x}{x^2 - 1}.$$

Now

$$\begin{aligned} & x + x^3 + x^5 + \cdots + x^{2(n+1)-1} \\ &= x + x^3 + x^5 + \cdots + x^{2n-1} + x^{2(n+1)-1} \quad (\mathbf{4}) \end{aligned}$$

$$
\begin{aligned}
&= \frac{x^{2n+1} - x}{x^2 - 1} + x^{2n+1} \quad (\mathbf{4}) \\
&= \frac{x^{2n+1} - x + x^{2n+3} - x^{2n+1}}{x^2 - 1} \quad (\mathbf{4}) \\
&= \frac{-x + x^{2n+3}}{x^2 - 1} \quad (\mathbf{4}) \\
&= \frac{x^{2(n+1)+1} - x}{x^2 - 1} \quad (\mathbf{4})
\end{aligned}
$$

Thus if the formula holds for $n \geq 1$ it holds for $n + 1$. By induction the formula holds for all $n \geq 1$.

4. (**20 points total**) Show directly that $a < 4$ or $9 \leq a$ implies $a^2 - 13a + 36 \geq 0$.

Solution: First of all notice that $a^2 - 13a + 36 = (a - 4)(a - 9)$; thus we are to show $a < 4$ or $9 \leq a$ implies $(a - 4)(a - 9) \geq 0$.

Suppose $a < 4$. Then $a - 9 < a - 4 < 0$ which implies that $(a - 4)(a - 9)$ is the product of two negative numbers and thus $(a - 4)(a - 9) > 0$. (**8**)

Suppose $9 \leq a$. Then $0 \leq a - 9 < a - 4$ which implies $(a - 4)(a - 9)$ is the product of zero and a positive number, and is thus zero, (**4**) or is the product of two positive numbers and thus positive. In either case $(a - 4)(a - 9) \geq 0$. (**8**)