MATH 215 **Final Examination Solution** **Radford** **08/07/2009**

Name (print) ______________________________

(1) With the exception of part (a) of Problem 1, write your answers in the exam booklet provided. (2) *Return* this exam copy with your test booklet. (3) *You are expected to abide by the University's rules concerning academic honesty.*

---

1. (**25 points**) Let P and Q be statements.

(a) (**8**) Complete the following three truth tables (on this sheet if you wish):

P	Q	P implies (Q implies P)
T	T	
T	F	
F	T	
F	F	

P	Q	P or (not Q)
T	T	
T	F	
F	T	
F	F	

P	Q	(not P) and Q
T	T	
T	F	
F	T	
F	F	

Solution:

P	Q	P implies (Q implies P)
T	T	T
T	F	T
F	T	T
F	F	T

P	Q	P or (not Q)
T	T	T
T	F	T
F	T	F
F	F	T

P	Q	(not P) and Q
T	T	F
T	F	F
F	T	T
F	F	F

(b) (**11**) What are the logical relationships (implication, equivalence, negation) between the statements "P implies (Q implies P)", "P or (not Q)", and "(not P) and Q"? You may refer to these statements as A, B, and C respectively. Justify your answer in terms of the truth tables of part (a).

Solution: Let A, B, and C denote the three statements respectively. Each of these statements implies itself. Note that B and C are negations of each other since the last column of one is the last column of the other with the Ts and Fs exchanged.

Take any two of A, B, and C. The first does not imply the second if there a "T" in the last column of the truth table of the first and an "F" in the last column of the corresponding row of the second; otherwise the first implies the second. Thus since A is always true, all statements imply A; in particular B, C imply A. A does not imply B, C; B does not imply C, C does not imply B.

(c) (**3**) What is the negation of "P and Q" in terms of "not P", "not Q"?

Solution: "(not P) or (not Q)".

(d) (**3**) What is the negation of "P or Q" in terms of "not P", "not Q"?

Solution: "(not P) and (not Q)".

2. (**30 points**) Consider the statement "$a^2 - 7a + 6 > 0$ implies $a < 1$ or $6 \leq a$".

(a) (**6**) What is the *converse* of the statement? Show that it is false.

Solution: "$a < 1$ or $6 \leq a$ implies $a^2 - 7a + 6 > 0$". With $a = 6$ the premise is true as $6 \leq 6$ but the conclusion is false since $6^2 - 7{\cdot}6 + 6 = 0 \not> 0$.

(b) (**5**) What is the *contrapositive* of the statement? Write it without "not".

Solution: "$1 \leq a < 6$ implies $a^2 - 7a + 6 \leq 0$".

(c) (**10**) Prove the statement by contradiction. Clearly state your assumptions.

Solution: Note that $a^2 - 7a + 6 = (a-1)(a-6)$. Suppose the hypothesis is true but the conclusion is false; that is $a^2 - 7a + 6 > 0$ and $1 \leq a < 6$. Then $a = 1$ or $1 < a < 6$. If $a = 1$ then $a^2 - 7a + 6 = (1-1)(1-6) = 0$, a contradiction. If $1 < a < 6$ then $a - 1 > 0$ and $a - 6 < 0$ which means $a^2 - 7a + 6 = (a-1)(a-6) < 0$, a contradiction. Therefore if the hypothesis is true the conclusion must be also.

(d) (**9**) Prove the *contrapositive*. Clearly state your assumptions.

Solution: Suppose $1 \leq a < 6$. To show $a^2 - 7a + 6 \leq 0$. Since $1 \leq a < 6$ either $a = 1$ or $1 < a < 6$. If $a = 1$ then $a^2 - 7a + 6 = 0$. If $1 < a < 6$ then $a - 1 > 0$ and $a - 6 < 0$; therefore $a^2 - 7a + 6 = (a-1)(a-6) < 0$. In either case $a^2 - 7a + 6 \leq 0$.

Base proofs for parts (c) and (d) in Problem 2 on the axioms for the real number system $\mathbf{R}$ and for all for $a, b, c \in \mathbf{R}$: if $a, b > 0$ or $a, b < 0$ then $ab > 0$; if $a > 0$ and $b < 0$, or $a < 0$ and $b > 0$, then $ab < 0$; $a0 = 0 = 0a$; and if $a < b$ then $a - c < b - c$ and $c - a > c - b$.

3. (**20 points**) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a function and $a, L \in \mathbf{R}$.

(a) (**6**) "$\lim_{x \longrightarrow a} f(x) = L$" means "$\forall \epsilon > 0, \exists \delta > 0, \forall x \in \mathbf{R}, 0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$". (It is understood that $\epsilon, \delta \in \mathbf{R}$.) Write the negation of "$\lim_{x \longrightarrow a} f(x) = L$" in terms of quantifiers.

Solution: "$\exists \epsilon > 0, \forall \delta > 0, \exists x \in \mathbf{R}, 0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$"

(b) (**8**) Let $f(x) = \begin{cases} 41x - 30 & x \neq 2; \\ 93.1 & x = 2 \end{cases}$ Use the definition of (a) to show $\lim_{x \longrightarrow 2} f(x) = 52$.

Solution: Let $\epsilon > 0$. Note for $x \neq 2$ that $|f(x) - 52| = |(41x - 30) - 52| = |41x - 82| = 41|x - 2|$. Thus $|f(x) - 52| < \epsilon$ if $x \neq 2$ and $41|x - 2| < \epsilon$ if $0 < |x - 2| < \epsilon/41$. Take $\delta = \epsilon/41$. Then $\delta > 0$ since $\epsilon > 0$ and our calculations show $0 < |x - 2| < \delta$ implies $|f(x) - 52| < \epsilon$.

(c) (**6**) Express the statement "$\lim_{n \longrightarrow \infty} f(n) = L$" in English without using quantifier symbols ($n \in \mathbf{Z}$ and we regard $f(1), f(2), f(3), \ldots$ as a sequence).

Solution: For all (real numbers) $\epsilon > 0$ there exists (an integer) $N \geq 1$ such that $n \geq N$ ($n \in \mathbf{Z}$) implies $|f(n) - L| < \epsilon$.

Comment: The parenthetical parts were not necessary to include. The usual notation conventions imply them.

4. (**25 points**) Show, by induction, that the sum of the cubes of first n positive integers is given by the formula $\sum_{\ell=1}^{n} \ell^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$ for all $n \geq 1$.

Solution: When $n = 1$ the formula is true since the left hand side is $1^3 = 1$ and the right hand side is $\dfrac{1^2(1+1)^2}{4} = \dfrac{1(4)}{4} = 1$.

Suppose $n \geq 1$ and the formula is true. Then

$$
\begin{aligned}
& 1^3 + 2^3 + \cdots + (n+1)^3 \\
&= (1^3 + 2^3 + \cdots + n^3) + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\
&= \frac{(n+1)^2\,(n^2 + 4(n+1))}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \frac{(n+1)^2((n+1)+1)^2}{4}.
\end{aligned}
$$

Thus the formula is true for $n+1$. By induction the formula the formula is true for all $n \geq 1$.

5. (**25 points**) Let X, Y, and Z be sets.

(a) (**8**) Give the conditional definitions of $X - Y$ and $X \times Y$.

Solution: $X - Y = \{x \,|\, x \in X \text{ and } x \notin Y\}$ and $X \times Y = \{(x, y) \,|\, x \in X, y \in Y\}$.

(b) (**8**) Working from definitions, show that $Y \subseteq X \cap Y$ if and only if $Y \subseteq X$.

Solution: *Only if*: Suppose $Y \subseteq X \cap Y$. We need to show that an element of Y is an element of X. Let $z \in Y$. Since $Y \subseteq X \cap Y$, $z \in X \cap Y$. Thus $z \in X$ and $z \in Y$ from which $z \in X$ follows.

If: Suppose $Y \subseteq X$. We need to show that an element of Y is an element of $Y \cap X$. Let $z \in Y$. Since $Y \subseteq X$ it follows $z \in X$. Therefore $z \in X \cap Y$.

(c) (**9**) Working from definitions, show that $(X - Y) - Z \subseteq X - (Y \cup Z)$.

Solution: We need to show that an element of $(X - Y) - Z$ is an element of $X - (Y \cup Z)$.

Let $x \in (X - Y) - Z$. Then $x \in X - Y$ and $x \notin Z$. Therefore $x \in X$, $x \notin Y$, and $x \notin Z$. The later two conditions are equivalent to $x \notin (Y \cup Z)$. Thus $x \in X - (Y \cup Z)$.

6. (**25 points**) Suppose that $f : X \longrightarrow Y$ is a function.

(a) (**6**) Define, using quantifiers, what it means for f to be an *injection.*

Solution: Either

"$\forall x, x' \in X, f(x) = f(x')$ implies $x = x'$."

or

"$\forall x, x' \in X, x \neq x'$ implies $f(x) \neq f(x')$."

(b) (**6**) Define, using quantifiers, what it means for f to be a *surjection.*

Solution: "$\forall y \in Y, \exists x \in X, y = f(x)$."

(c) (**13**) Now suppose $g : Y \longrightarrow X$ is a function and $g{\circ}f = I_X$. Show that f is an injection and that g is a surjection.

Solution: Note that $x = g(f(x))$ for all $x \in X$ as $x = I_X(x) = (g{\circ}f)(x) = g(f(x))$ for all $x \in X$.

f is an injection. Suppose that $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$. Then $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$ and therefore $x_1 = x_2$.

g is a surjection. Suppose $x \in X$. Then $x = g(f(x))$; therefore $x = g(y)$, where $y = f(x)$.

7. (**30 points**) *In this problem all binomial symbols must be computed.* A committee of 5 is to be formed from a group of 10 people.

(a) (**5**) How many such committees are there?

Solution: $\binom{10}{5} = \dfrac{10{\cdot}9{\cdot}8{\cdot}7{\cdot}6}{5{\cdot}4{\cdot}3{\cdot}2{\cdot}1} = 3{\cdot}2{\cdot}7{\cdot}6 = 252.$

(b) (**5**) Suppose a certain 3 individuals from this group are to be *excluded.* How many such committees are there?

Solution: $\binom{10-3}{5} = \binom{7}{5} = \dfrac{7{\cdot}6}{2{\cdot}1} = 21.$

(c) (**5**) Suppose that a certain 2 individuals from this group are to be *included.* How many such committees are there?

Solution: $\binom{10-2}{5-2} = \binom{8}{3} = \dfrac{8{\cdot}7{\cdot}6}{3{\cdot}2{\cdot}1} = 8{\cdot}7 = 56.$

Suppose A and B are different individuals in this group. Let X be the set of committees which include A and let Y the set of committees which include B.

(d) (**7**) Use the Principle of Inclusion-Exclusion to calculate the number of elements of $X \cup Y$, the set of committees which *include* A or B.

Solution:

$$\begin{aligned}
|X \cup Y| &= |X| + |Y| - |X \cap Y| \\
&= \binom{10-1}{5-1} + \binom{10-1}{5-1} - \binom{10-2}{5-2} \\
&= \binom{9}{4} + \binom{9}{4} - \binom{8}{3} \\
&= 126 + 126 - 56 \\
&= 196.
\end{aligned}$$

(e) (**8**) Let U be the set of all committees of 5 which can be formed from the group of 10 people. Express the set of committees which *exclude* both A and B in terms of X and Y and use De Morgan's Law to compute the number of such committees.

Solution: This set of committees is $X^c \cap Y^c$ and thus

$$|X^c \cap Y^c| = |(X \cup Y)^c| = |U| - |(X \cup Y)| = 252 - 196 = 56.$$

8. (**20 points**) Give the definition of

(a) (**3**) equipotent sets,

Solution: Two sets X and Y are equipotent if there is bijection $f : X \longrightarrow Y$.

(b) (**3**) finite set,

Solution: A set X is finite if $X = \emptyset$ or there is a bijection $f : \mathbf{N}_n \longrightarrow X$ for some $n \geq 1$.

(c) (**3**) denumerable set,

Solution: A set X is denumerable if there is a bijection $f : \mathbf{Z}^+ \longrightarrow X$.

and (d) (**11**) use the Euclidean Algorithm to find the greatest common divisor of 213 and 63.

Solution:

$$\begin{aligned}
\mathbf{213} &= 3 \cdot \mathbf{63} + \mathbf{24} \\
\mathbf{63} &= 2 \cdot \mathbf{24} + \mathbf{15} \\
\mathbf{24} &= 1 \cdot \mathbf{15} + \mathbf{9} \\
\mathbf{15} &= 1 \cdot \mathbf{9} + \mathbf{6} \\
\mathbf{9} &= 1 \cdot \mathbf{6} + \mathbf{3} \\
\mathbf{6} &= 2 \cdot \mathbf{3} + 0;
\end{aligned}$$

therefore the greatest common divisor of 213 and 63 is 3.