

Kaplansky's Ten Hopf Algebra Conjectures

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0. Introduction

Hopf algebras were discovered by Heinz Hopf 1941 as structures in algebraic topology **[Hopf]**. Hopf algebras we consider are in the category of vector spaces over a field \mathbb{k} . A few examples: group algebras, enveloping algebras of Lie algebras, and quantum groups. Some quantum groups produce invariants of knots and links.

A general theory of Hopf algebras began in the late 1960s **[Swe]**. In a 1975 publication **[Kap]** Kaplansky listed 10 conjectures on Hopf algebras. These have been the focus of a great deal of research. Some have not been resolved.

We define Hopf algebra, describe examples in detail, and discuss the significance and status of each conjecture. In discussing conjectures more of the nature of Hopf algebras will be revealed.

$\otimes = \otimes_{\mathbb{k}}$. “f-d” = finite-dimensional. “n-d” = n -dimensional. *What is a Hopf algebra?*

1. A Basic Example and Definition

G is a group, $A = \mathbb{k}[G]$ is the group algebra of G over \mathbb{k} . Let $g, h \in G$. The algebra structure:

$\mathbb{k}[G] \otimes \mathbb{k}[G] \xrightarrow{m} \mathbb{k}[G]$	$m(g \otimes h) = gh$
$\mathbb{k} \xrightarrow{\eta} \mathbb{k}[G]$	$\eta(1_{\mathbb{k}}) = e = 1_{\mathbb{k}[G]}$

The coalgebra structure:

$\mathbb{k}[G] \xrightarrow{\Delta} \mathbb{k}[G] \otimes \mathbb{k}[G]$	$\Delta(g) = g \otimes g$
$\mathbb{k}[G] \xrightarrow{\epsilon} \mathbb{k}$	$\epsilon(g) = 1_{\mathbb{k}}$

The map which accounts for inverses:

$\mathbb{k}[G] \xrightarrow{S} \mathbb{k}[G]$	$S(g) = g^{-1}$
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Observe that

$$\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta(g)\Delta(h),$$

$$\epsilon(gh) = 1_{\mathbb{k}} = 1_{\mathbb{k}}1_{\mathbb{k}} = \epsilon(g)\epsilon(h),$$

$$gS(g) = gg^{-1} = e = 1_{\mathbb{k}}e = \epsilon(g)1_{\mathbb{k}[G]},$$

and

$$S(g)g = g^{-1}g = e = \epsilon(g)1_{\mathbb{k}[G]}.$$

Also $\Delta(1_{\mathbb{k}[G]}) = 1_{\mathbb{k}[G]} \otimes 1_{\mathbb{k}[G]}$ and $\epsilon(1_{\mathbb{k}[G]}) = 1_{\mathbb{k}}$; thus

$$\boxed{\Delta, \epsilon \text{ are algebra maps}}$$

and S is determined by

$$\boxed{gS(g) = \epsilon(g)1_{\mathbb{k}[G]} = S(g)g.}$$

We generalize the system $(\mathbb{k}[G], m, \eta, \Delta, \epsilon, S)$.

Hopf algebra over \mathbb{k} , a tuple $(A, m, \eta, \Delta, \epsilon, S)$, where (A, m, η) is an algebra over \mathbb{k} :

$A \otimes A \xrightarrow{m} A$	$m(a \otimes b) = ab$
$\mathbb{k} \xrightarrow{\eta} A$	$\eta(1_{\mathbb{k}}) = 1_A$

(A, Δ, ϵ) is a coalgebra over \mathbb{k} :

$A \xrightarrow{\Delta} A \otimes A$	$\Delta(a) = a_{(1)} \otimes a_{(2)}$
$A \xrightarrow{\epsilon} \mathbb{k}$	

and $\boxed{A \xrightarrow{S} A}$ is an "antipode" where certain axioms are satisfied.

Comments: $\Delta(a) \in A \otimes A$ is usually a **sum** of tensors; thus $\Delta(a) = a_{(1)} \otimes a_{(2)}$ **is a notation**, called the *Heyneman-Sweedler notation*. Δ is called the *coproduct* and ϵ the *counit*.

The axioms for a Hopf algebra over \mathbb{k} :

(A, m, η) is an (associative) algebra over \mathbb{k}

$$(ab)c = a(bc), \quad 1a = a = a1;$$

(A, Δ, ϵ) is a (coassociative) coalgebra over \mathbb{k}

$$a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)},$$
$$\epsilon(a_{(1)})a_{(2)} = a = a_{(1)}\epsilon(a_{(2)});$$

Δ is an algebra map

$$\Delta(ab) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}, \quad \Delta(1) = 1 \otimes 1;$$

ϵ is an algebra map

$$\epsilon(ab) = \epsilon(a)\epsilon(b), \quad \epsilon(1) = 1;$$

and

$$a_{(1)}S(a_{(2)}) = \epsilon(a)1 = S(a_{(1)})a_{(2)}$$

for all $a, b, c \in A$. From now on A denotes a Hopf algebra over \mathbb{k} .

2. Basic Properties and More Definitions

1. A has a unique antipode.

For $g, h \in \mathbb{k}[G]$ observe

$$S(gh) = (gh)^{-1} = h^{-1}g^{-1} = S(h)S(g)$$

and

$$S(e) = e^{-1} = e, \text{ or } S(1_{\mathbb{k}[G]}) = 1_{\mathbb{k}[G]}.$$

Also

$$\epsilon(S(g)) = \epsilon(g^{-1}) = 1_{\mathbb{k}} = \epsilon(g).$$

2. For $a, b \in A$

$$S(ab) = S(b)S(a); \quad S(1) = 1.$$

Also

$$\Delta(S(a)) = S(a_{(2)}) \otimes S(a_{(1)}), \quad \epsilon(S(a)) = \epsilon(a).$$

Explanation of notation:

$$a_{(2)} \otimes a_{(1)} = \tau(a_{(1)} \otimes a_{(2)}),$$

where $\tau : A \otimes A \longrightarrow A \otimes A$ is given by

$$\tau(a \otimes b) = b \otimes a.$$

3. $a \in A$ is cocommutative if $\tau(\Delta(a)) = \Delta(a)$, or $a_{(2)} \otimes a_{(1)} = a_{(1)} \otimes a_{(2)}$. A is cocommutative if $\tau \circ \Delta = \Delta$, or if all $a \in A$ are cocommutative.

$\mathbb{k}[G]$ is cocommutative ($\Delta(g) = g \otimes g$ for $g \in G$).

4. A is commutative if $m \circ \tau = m$, or $ba = ab$ for all $a, b \in A$.

$\mathbb{k}[G]$ is commutative if and only if G is an abelian group.

Variation on $ba = ab$; $ba = qab$, $q \in \mathbb{k}$.

5. $a \in A$ is grouplike if $\Delta(a) = a \otimes a$ and $\epsilon(a) = 1$. $G(A)$ denotes the set of grouplike elements of A .

6. $G(A)$ is linearly independent (a coalgebra fact).

$$G(\mathbb{k}[G]) = G.$$

7. $G(A)$ is a group under multiplication and $S(g) = g^{-1}$ for $g \in G(A)$. Thus if A is f-d then $G(A)$ is a finite group.

8. $(A, m^{op}, \eta, \Delta^{cop}, \epsilon, S)$ is a Hopf algebra over \mathbb{k} , where $m^{op} = m \circ \tau$ and $\Delta^{cop} = \tau \circ \Delta$.

9. $(A^*, \Delta^*, \epsilon^*, m^*, \eta^*, S^*)$ is a f-d Hopf algebra over \mathbb{k} if $(A, m, \eta, \Delta, \epsilon, S)$ is.

10. If A is f-d then S is bijective.

11. If A is commutative or cocommutative then $S^2 = I_A$. In this case S is bijective.

12. If S is bijective then $(A, m^{op}, \eta, \Delta, \epsilon, S^{-1})$, $(A, m, \eta, \Delta^{cop}, \epsilon, S^{-1})$ are Hopf algebras over \mathbb{k} .

13. Let M, N be left A -modules (regard A as an algebra). Then $M \otimes N$ is a left A -module where $a \cdot (m \otimes n) = a_{(1)} \cdot m \otimes a_{(2)} \cdot n$.

For $g \in G(A)$ note $g \cdot (m \otimes n) = g \cdot m \otimes g \cdot n$.

14. Let C be a coalgebra over \mathbb{k} . Every f-d subspace of C generates a f-d subcoalgebra.

15. A has simple subcoalgebras, and all are f-d. $\mathbb{k}1$, more generally $\mathbb{k}g$ for $g \in G(A)$, is a simple subcoalgebra of A .

The reader is referred to any basic text on Hopf algebras: **[Swe]**, **[Abe]**, **[Mont]**, **[DNR]**, **[R]**.

3. The Enveloping Algebra

Let L be a Lie algebra over \mathbb{k} . The enveloping algebra $U(L)$ is a cocommutative Hopf algebra over \mathbb{k} where

$$\Delta(\ell) = 1 \otimes \ell + \ell \otimes 1, \quad (1)$$

$\epsilon(\ell) = 0$, and $S(\ell) = -\ell$ for $\ell \in L$. An $\ell \in A$ such that (1) holds is *primitive*.

The set of primitives $P(A)$ of A is a subspace and a Lie algebra under associative bracket.

$A = U(L)$ is *pointed irreducible* meaning $\mathbb{k}1$ is the only simple subcoalgebra of A .

Variation on primitive: $\Delta(\ell) = g \otimes \ell + \ell \otimes h$, where $g, h \in G(A)$.

Such an ℓ is a *skew primitive element*.

4. The 10 Kaplansky conjectures

1. Hopf algebras are free modules over their Hopf subalgebras (under multiplication).

Let G be a group. The Hopf subalgebras of $\mathbb{k}[G]$ are the group algebras $\mathbb{k}[L]$, where L is a subgroup of G . True in this case.

When A is f-d the conjecture can be thought of as generalization of **Lagrange's Theorem** for finite groups.

Infinite-dimensional counterexamples: Takeuchi in 1972 [**Tak**], Oberst and Schneider in 1974 [**ObSch**], the speaker 1980 [**Rad**], Schneider in 1981. [**Sch1**].

Proved when A is f-d by Nichols and Zoeller [**NicZel1**] in 1989. Implications when A is f-d:

If B is a Hopf subalgebra of A then $\text{Dim}(B)$ divides $\text{Dim}(A)$.

$|G(A)| = \text{Dim}(\mathbb{k}[G(A)])$ divides $\text{Dim}(A)$.

Suppose B is a Hopf subalgebra of A . Then A semisimple implies B is also.

Nichols and Zoeller used representation theory of algebras to establish relative Hopf modules are free when A is f-d.

A relative Hopf module is a structure (M, μ, ρ) , where

$$\mu : B \otimes M \longrightarrow M$$

is a left B -module and

$$\rho : M \longrightarrow A \otimes M$$

is a left A -comodule, which have a certain compatibility. With $(b \otimes b')(a \otimes m) = ba \otimes b' \cdot m$,

$$\rho(b \cdot m) = \Delta(b)\rho(m).$$

When $B = A$ then (M, μ, ρ) is a Hopf module. Such are always free with A -basis any linear basis of

$$M^{co\ inv} = \{m \in M \mid \rho(m) = 1 \otimes m\}.$$

Freeness of Hopf modules is elementary in that no representation theory needed.

Implications of freeness of Hopf modules:

If A contains a non-zero f-d left ideal then A is f-d.

If A is semisimple then A is f-d.

If A is f-d then A contains a 1-d ideal.

A definition for our next conjecture.

Definition: A coalgebra is admissible if there is an algebra structure which makes it a Hopf algebra.

2. A coalgebra is admissible iff every f-d subspace lies in a f-d admissible subcoalgebra.

Assume \mathbb{k} has characteristic 0. If $0 \neq \ell \in P(A)$ the Hopf subalgebra generated by ℓ is $\mathbb{k}[\ell]$, the free \mathbb{k} -algebra on $\mathbb{k}\ell$.

$\mathbb{k}[x]$ is a Hopf algebra where x is primitive. Any Hopf subalgebra contains x or is $\mathbb{k}1$. Thus:

The Hopf subalgebras of $\mathbb{k}[x]$ are $\mathbb{k}1$ and $\mathbb{k}[x]$.

Hence the conjecture is false. This example is due to Larson. Note $\mathbb{k}[x] = U(\mathbb{k}x)$.

Generally the Hopf subalgebras of $U(L)$ are $U(M)$, where M is a Lie subalgebra of L , when the characteristic of \mathbb{k} is zero.

3. A Hopf algebra in characteristic 0 has no non-zero nilpotent elements.

Let G be a finite group, $\Lambda = \sum_{g \in G} g \in \mathbb{k}[G]$. Then $g\Lambda = \Lambda$, or $g\Lambda = \epsilon(g)\Lambda$, for all $g \in G$. Thus

$$a\Lambda = \epsilon(a)\Lambda \quad (2)$$

for all $a \in \mathbb{k}[G]$. Since

$$\epsilon(\Lambda) = \sum_{g \in G} \epsilon(g) = |G| \mathbf{1}_{\mathbb{k}} :$$

Maschke's Theorem Let G be a finite group. Then $\mathbb{k}[G]$ is semisimple if and only if $\epsilon(\Lambda) \neq 0$.

Let A be f-d. Then A contains a non-zero Λ which satisfies (2) for all $a \in A$. Further

A is semisimple if and only if $\epsilon(\Lambda) \neq 0$.

If A is not semisimple $\Lambda^2 = \epsilon(\Lambda)\Lambda = 0\Lambda = 0$; thus Λ is a non-zero nilpotent element.

There are f-d Hopf algebras over \mathbf{C} which are not semisimple. Thus the conjecture is false.

Example 1: A as an algebra is generated by a, x subject to the relations

$$a^2 = 1, \quad x^2 = 0, \quad xa = -ax$$

and as a coalgebra its structure is given by

$$\Delta(a) = a \otimes a, \quad \Delta(x) = x \otimes a + 1 \otimes x$$

and

$$\epsilon(a) = 1 \quad \epsilon(x) = 0.$$

$$S(a) = a, \quad \text{and} \quad S(x) = ax.$$

Note

$S^2(x) = S(ax) = S(x)S(a) = (ax)a = -x$;
thus $S^2 \neq I_A$. $\dim(A) = 4$. We can take

$$\Lambda = (1 + a)x.$$

A is not semisimple since

$$\epsilon(\Lambda) = \epsilon(1 + a)\epsilon(x) = \epsilon(1 + a)0 = 0.$$

4. An element x in a Hopf algebra is in its center if $a_{(1)}xS(a_{(2)}) = \epsilon(a)x$ for all $a \in A$.

$x \in \mathbb{k}[G]$ is central if and only if

$$gxg^{-1} = x \quad (= \epsilon(g)x)$$

for all $g \in G$. The formula

$$gxg^{-1} = \epsilon(g)x$$

translates to

$$a \cdot x := a_{(1)}xS(a_{(2)}) = \epsilon(a)x$$

for all $a, x \in A$. The conjecture rewritten:

$x \in A$ is central if and only if $a \cdot x = \epsilon(a)x$

for all $a \in A$. If x is central then

$$a \cdot x = a_{(1)}S(a_{(2)})x = a_{(1)}\epsilon(a_{(2)})1x = \epsilon(a)x.$$

Observe $a \cdot x$ defines a left A -module action on itself.

For $g, x \in \mathbb{k}[G]$ the formula

$$gx = (g x g^{-1})g = gxS(g)g$$

translates to

$$\begin{aligned} ax &= a_{(1)}xS(a_{(2)(1)})a_{(2)(2)} \\ &= a_{(1)(1)}xS(a_{(1)(2)})a_{(2)} \\ &= (a_{(1)} \cdot x)a_{(2)}; \end{aligned}$$

thus the commutation relation

$$\boxed{ax = (a_{(1)} \cdot x)a_{(2)}}$$

holds for all $a, x \in A$.

If $a \cdot x = \epsilon(a)x$ for all $a \in A$ then

$$ax = \epsilon(a_{(1)})xa_{(2)} = x\epsilon(a_{(1)})a_{(2)} = xa$$

for all $a \in A$ and therefore x is central.

The conjecture is true.

From now on suppose A is f-d.

5. If A or A^* is semisimple then $S^2 = I_A$.

Established by Larson and the speaker [RL1, RL2] when \mathbb{k} has characteristic 0 in 1988.

Over any field

$$\mathrm{Tr}(S^2) = \mathrm{Dim}(A) \mathrm{Tr}(S^2|_Ax), \quad (3)$$

where $x \in A$ is defined by $p(x) = \mathrm{Tr}(\ell(p))$ for all $p \in A^*$ and $\ell(p)(q) = pq$ for all $q \in A^*$.

First assume A and A^* are semisimple. Then $S^4 = I_A$. Now assume the characteristic of \mathbb{k} is 0. Then S^2 is diagonalizable and eigenvalues are ± 1 . But -1 is not an eigenvalue of S^2 by (3). Thus $S^2 = I$. Now A or A^* semisimple implies A and A^* are in characteristic 0.

The characteristic $p > 0$ case is still open.

Etingof and Gelaki showed in 1998 if A and A^* are semisimple then $S^2 = I_A$ by “lifting” to the characteristic 0 case [EtGel1].

6. The size of matrices occurring in any matrix constituent of A divides $\text{Dim}(A)$.

The conjecture is interpreted:

If A is a f -d Hopf algebra over an algebraically closed field \mathbb{k} and M is a simple A -module then $\text{Dim}(M) \mid \text{Dim}(A)$.

True when $A = \mathbb{k}[G]$ is semisimple, false for group algebras in general.

Assume A is semisimple and that M is a simple A -module. Then $\text{Dim}(M) \mid \text{Dim}(A)$ when:

(1) $\text{Dim}(M) = 2$, due to Nichols and Zoeller [NicZel2] in 1996;

(2) $A = D(B)$, where B is a f -d semisimple, due to Etingof and Gelaki [EtGel2] in 1998;

(3) A is cosemisimple and quasitriangular, due to Etingof and Gelaki [EtGel2] 1998;

(4) $\text{Dim}(A)$ is a prime power, this is due to Montgomery and Witherspoon [MW] in 1998.

Regarding (2), [EtGel2] builds on important work of Zhu [Zhu1]. He obtains partial results for $D(B)$.

Regarding (3), every f-d quasitriangular Hopf algebra A is a quotient of $D(A)$. Therefore (2) implies (3).

Recent progress on conjecture six is through classification of semisimple Hopf algebras. We single out Natale's 2007 paper [Nat].

There are many contributors to classification. See the 2008 survey by Masuoka [Mas].

7. If A and A^* are both semisimple then the characteristic of \mathbb{k} does not divide $\text{Dim}(A)$.

Settled in 1988. Let $\Lambda \in A$ satisfy (2), that is

$$a\Lambda = \epsilon(a)\Lambda$$

for all $a \in A$ and $\lambda \in A^*$ satisfy

$$\lambda p = p(1)\lambda$$

for all $p \in A^*$. We may assume $\lambda(\Lambda) = 1$. Then

$$\text{Tr}(S^2) = \epsilon(\Lambda)\lambda(1). \quad (4)$$

Both A and A^* are semisimple if and only if $\text{Tr}(S^2) \neq 0$.

Now we use (4) and (3), which is

$$\text{Tr}(S^2) = \text{Dim}(A)\text{Tr}(S^2|_{Ax}),$$

to prove the conjecture. Equation (4) is found in the 1988 paper [LR1] by Larson and the speaker.

When $\mathbb{k} = \mathbb{Z}_p$ note $\mathbb{k}[\mathbb{Z}_p]^*$ is semisimple, $\mathbb{k}[\mathbb{Z}_p]$ is not, and $\text{Dim}(\mathbb{k}[\mathbb{Z}_p]) = p$.

8. If $\text{Dim}(A)$ is prime then A is commutative and cocommutative.

We may assume that \mathbb{k} is algebraically closed. Suppose $\text{Dim}(A)$ is prime. Then $S^4 = I_A$.

Assume \mathbb{k} has characteristic 0. $\text{Tr}(S^2) \neq 0$. Thus A (and also A^*) is semisimple. We use Zhu's class equation in the 1994 paper [Zhu2]. $C_{\mathbb{k}}(A) \subseteq A^*$ is the span of the characters of A . It is a semisimple algebra. For semisimple A :

Theorem 1 *Let $\{e_1, \dots, e_n\}$ be a complete set of orthogonal idempotents for $C_{\mathbb{k}}(A)$. Then*

$$\text{Dim}(A) = \sum_{i=1}^n \text{Dim}(e_i A^*),$$

$\text{Dim}(e_i A^) \mid \text{Dim}(A)$ for all i , and $\text{Dim}(e_i A^*) = 1$ for some i .*

There is a bijective correspondence between $G(A) \cap Z(A)$ and the set of 1-dimensional $e_i A^*$'s. Thus $p = \text{Dim}(A)$ prime implies $A = \mathbb{k}[Z_p]$.

Zhu established the conjecture in this case in 1993 in [Zhu1]. The class equation is based on work of Kac [Kac].

The characteristic $p > 0$ case has not been completely decided.

Etingof and Gelaki showed A is commutative and cocommutative when both A and A^* are semisimple by “lifting” to the characteristic zero case in 1998 [EtGel2].

For the two remaining conjectures suppose the characteristic of \mathbb{k} does not divide $\text{Dim}(A)$.

9. The radicals of A and A^* have the same dimension.

We reformulate the conjecture. A_0 is the sum of the simple subcoalgebras of A .

$$\text{Rad}(A^*) = A_0^\perp := \{p \in A^* \mid p(A_0) = (0)\}.$$

Reformulation: $\boxed{\text{Dim}(A_0) = \text{Dim}(A^*_0)}$

Example 2: Consider A of Example 1. $A_0 = \mathbb{k}1 \oplus \mathbb{k}a$ and thus $\text{Dim}(A_0) = 2$. The Drinfel'd double, a very special quantum group,

$$D(A) = A^{*cop} \otimes A = A \otimes A$$

as a coalgebra. Thus $\text{Dim}(D(A)_0) = 4$ since

$$D(A)_0 = (A \otimes A)_0 = A_0 \otimes A_0.$$

As $D(A)$ has a 2-d simple module, $D(A)^*$ has a 4-d simple subcoalgebra C . Therefore the conjecture is false since $D(A)^*_0 \supseteq \mathbb{k}1 \oplus C$ and hence $\text{Dim}(D(A)_0) = 4 < \text{Dim}(D(A)^*_0)$.

Counterexamples were given by Schneider in 1995 [Sch2] and Sommerhäuser in 1998 [Som].

10. For a given dimension there are only finitely many different isomorphism types of Hopf algebras.

Established for semisimple cosemisimple Hopf algebras by Ştefan in 1997 [Stef].

Generally false, shown by Andruskiewitsch and Schneider in 2000 [AS], by Beattie, Dăscălescu, and Grünenfelder in 1999 [BDG], by Gelaki in 1998 [Gel], and by Müller in 2000 [Mue]. These were independent efforts.

We describe the family of $H_q(a)$'s, where $a \in \mathbb{k}$, of [BDG]. $\dim(H_q(a)) = p^4$, where p is an odd prime. Assume $q \in \mathbb{k}$ is a primitive p^{th} root of unity and \mathbb{k} is infinite.

Example 3: $H_q(a)$ is generated as a \mathbb{k} -algebra by c, x, y subject to the relations

$$c^{n^2} = 1, \quad x^n = c^n - 1, \quad y^n = c^n - 1,$$

$$xc = q^{-1}cx, \quad yc = qcy, \quad yx = qxy + a(c^2 - 1).$$

As a \mathbb{k} -coalgebra $H_q(a)$ is determined by

$$\Delta(c) = c \otimes c, \quad \Delta(z) = c \otimes z + z \otimes 1,$$

$$\epsilon(c) = 1, \quad \epsilon(z) = 0,$$

where $z = x, y$.

$H_q(b) \simeq H_q(a)$ if and only if $b = ua$ for some p^{th} root of unity $u \in \mathbb{k}$.

Remark: As an algebra $H_q(a)$ is a quotient of iterated Ore extensions.

A fuller account of the conjectures is found in Sommerhäuser's excellent 2000 expository paper [Som]. Appended is an expanded list of references.

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