Comparing Nash and IESDS equilibria

We consider a finite two player game.

Proposition 1 *If (a^*, b^*) is a strictly dominant strategy equilibrium, then (a^*, b^*) is a Nash equilibrium.*

Proof The strategy a^* strictly dominates each strategy in A_1. Thus

$$v_1(a^*, b^*) > v(a, b^*)$$

for all $a \in A_1$ with $a \neq a^*$. Thus a^* is the unique best response to b^*, i.e.,

$$BR_1(b) = \{a^*\}.$$

Similarly, b^* is the unique best response to a^*. Thus (a^*, b^*) is a Nash equilibrium. □

The game *Battle of the Sexes* shows that the converse fails as there are games with Nash equilibria which are not strictly dominant.

Proposition 2 *If (a^*, b^*) is a Nash equilibrium, then (a^*, b^*) is an IESDS-equilibrium.*

Proof We prove this by contradiction. Suppose (a^*, b^*) is not an IESDS-equilibrium. Then one of the strategies is removed at some stage of the construction. Let's suppose that a^* is removed no later than b^* (the other case is similar). Consider the stage when a^* is eliminated. At this stage of the construction, we have a game where a^* and b^* are possible strategies and, because it is about to be eliminated, there is a strategy $a' \in A_1$ such that a' strictly dominates a^*. But then

$$v_1(a', b^*) > v_1(a^*, b^*)$$

and a^* is not a best response for Player 1 to b^*. This contradicts our assumption that (a^*, b^*) is a Nash equilibrium. □

The converse fails. For example, in Battle of the Sexes, (B, S) and (S, B) are IESDS-equilibria that are not Nash equilibria. Similarly, in *Heads/Tails* every strategy pair is an IESDS-equilibrium but no strategy pair is a Nash equilibrium.

On the other hand, the converse is true in the special case when the IESDS equilibrium is unique.

Proposition 3 *If* (a^*, b^*) *is the unique IESDS equilibrium, then* (a^*, b^*) *is a Nash equilibrium.*

Proof For purposes of contradiction suppose $a^* \notin BR_1(b^*)$–the other case is similar. Let

$$X = \{a \in A_1 : v_1(a, b^*) > v_1(a^*, b^*)\}.$$

If a^* is not a best response to b^*, then X is non-empty. Since X is finite. we can find $a' \in X$ such that $v(a', b^*)$ is maximal.

Since (a^*, b^*) is the unique strategy profile surviving, we must eliminate a' at some stage. But $v_1(a', b^*)$ is maximal. Thus at no stage will be find a strategy strictly dominating a' and a' will never be eliminated, a contradiction. □

Corollary 4 *If there is a unique IESDS equilibrium, it is also the unique pure strategy Nash equilibrium.*

Proof By Proposition 3 the unique IESDS equilibrium is a Nash equilibrium. By Proposition 2, if there was a second Nash equilibrium it would also be an IESDS equilibrium. □

Corollary 5 *If there is a strictly dominant strategy equilibrium, it is the unique Nash equilibrium.*

Proof If (a^*, b^*) is a strictly dominant strategy equilibrium, then in the IESDS process at stage 1 would eliminate all strategies except a^* and b^*, so (a^*, b^*) is the unique IESDS-equilibrium and hence the unique Nash-equilibrium. □