Proofs of "Three Hard Theorems"

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Chapter §7 of Spivak's *Calculus* focuses on three of the most important theorems in Calculus. In this note I will give alternative proofs of these results.

1 Preliminaries

We review a few important facts we have seen about sequences.

Lemma 1 *If $x_n \in [a,b]$ for all $n \in \mathbb{N}$ and $(x_n)_{n=1}^{\infty}$ converges to x. Then $x \in [a,b]$.*

Proof We first show $a \leq x$. Suppose $x < a$. Choose $\epsilon > 0$ with $\epsilon < a - x$. Then no element of the sequence is in the interval $(x-\epsilon, x+\epsilon)$, a contradicton. A similar argument shows $b \geq x$.

Lemma 2 *If $f : [c,d] \to \mathbb{R}$ is continuous, $a_n \in [c,d]$ for $n \in \mathbb{N}$ and $(a_n)_{n=1}^{\infty}$ converges to $a \in [c,d]$. Then $(f(a_n))_{n=1}^{\infty}$ converges to $f(a)$.*

Proof Let $\epsilon > 0$. Since f is continuous, there is $\delta > 0$ such that if $|x-a| < \delta$, then $|f(x) - f(a)| < \epsilon$. Since $(a_n)_{n=1}^{\infty} \to a$, there is $N \in \mathbb{N}$ such that $|a_n - a| < \delta$ for all $n \geq N$. Thus $|f(a_n) - f(a)| < \epsilon$ for all $n \geq N$ and $(f(a_n))_{n=1}^{\infty} \to f(a)$.

Theorem 3 (Nested Interval Theorem) *Suppose $I_n = [a_n, b_n]$ where $a_n < b_n$ for $n \in \mathbb{N}$ and $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$. Then*

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Proof Note that we have

$$a_1 \le a_2 \le a_3 \ldots \le a_n \le \ldots\ldots \le b_n \le \ldots b_2 \le b_1.$$

Then each b_i is an upper bound for the set $A = \{a_1, a_2, \ldots\}$. By the Completeness Axiom, we can find α a least upper bound for A.

We claim that $\alpha \in I_n$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Since α is an upper bound for A, $a_n \le \alpha$. But b_n is an upper bound for A and α is the least upper bound. Thus $\alpha \le b_n$. Hence $\alpha \in I_n$ for all $n \in \mathbb{N}$ and $\alpha \in \bigcap_{n=1}^{\infty} I_n$.

Theorem 4 (Bolzano–Weierstrass Theorem) *Every bounded sequence has a convergent subsequence.*

Proof Let $(x_i)_{i=1}^{\infty}$ be bounded. There is $M \in \mathbb{R}$ such that $|x_i| \le M$ for all $i \in N$. We inductively construct a sequence of intervals

$$I_0 \supset I_1 \supset I_2 \supset \ldots$$

such that:

i) I_n is a closed interval $[a_n, b_n]$ where $b_n - a_n = \frac{2M}{2^n}$;

ii) $\{i : x_i \in I_n\}$ is infinite.

We let $I_0 = [-M, M]$. This closed interval has length $2M$ and $x_i \in I_0$ for all $i \in \mathbb{N}$.

Suppose we have $I_n = [a_n, b_n]$ satisfying i) and ii). Let c_n be the midpoint $\frac{a_n+b_n}{2}$. Each of the intervals $[a_n, c_n]$ and $[c_n, b_n]$ is half the length of I_n. Thus they both have length $\frac{1}{2}\frac{2M}{2^n} = \frac{2M}{2^{n+1}}$ If $x_i \in I_n$, then $x_i \in [a_n, c_n]$ or $x_i \in [c_n, b_n]$, possibly both. Thus at least one of the sets

$$\{i : x_i \in [a_n, c_n]\} \text{ and } \{i : x_i \in [c_n, b_n]\}$$

is infinite. If the first is infinite, we let $a_{n+1} = a_n$ and $b_{n+1} = c_n$. If the second is infinite, we let $a_{n+1} = c_n$ and $b_{n+1} = b_n$. Let $I_{n+1} = [a_{n+1}, b_{n+1}]$ Then i) and ii) are satisfied.

By the Nested Interval Theorem, there is $\alpha \in \bigcap_{n=1}^{\infty} I_n$. We next find a subsequence converging to α.

Choose $i_1 \in \mathbb{N}$ such that $x_{i_1} \in I_1$. Suppose we have i_n. We know that $\{i : x_i \in I_{n+1}\}$ is infinite. Thus we can choose $i_{n+1} > i_n$ such that $x_{i_{n+1}} \in I_{n+1}$. This allows us to construct a sequence of natural numbers

$$i_1 < i_2 < i_3 < \dots$$

where $i_n \in I_n$ for all $n \in \mathbb{N}$.

We finish the proof by showing that the subsequence $(x_{i_n})_{n=1}^{\infty} \to \alpha$. Let $\epsilon > 0$. Choose N such that $\epsilon > \frac{2M}{2^N}$. Suppose $n \geq N$. Then $x_{i_n} \in I_n$ and $\alpha \in I_n$. Thus

$$|x_{i_n} - \alpha| \leq \frac{2M}{2^n} \leq \frac{2M}{2^N} < \epsilon$$

for all $n \geq N$ and $(x_{i_n})_{n=1}^{\infty} \to \alpha$.

2 Bounding and the Extreme Value Theorem

Theorem 5 (Bounding Theorem) *If $f : [a,b] \to \mathbb{R}$ is continous, then there is $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in [a,b]$.*

Proof For purposes of contradiction, suppose not. Then for any $n \in \mathbb{N}$ we can find $x_n \in [a,b]$ such that $|f(x_n)| > n$. By the Bolzano–Weierstrass Theorem, we can find a convergent subsequence $x_{i_1}, x_{i_2}, \dots$. Note that $|f(x_{i_n}| > i_n \geq n$. Thus, replacing $(x_n)_{n=1}^{\infty}$ by $(x_{i_n})_{n=1}^{\infty}$, we may, without loss of generality, assume that $(x_n)_{n=1}^{\infty}$ is convergent. Suppose $(x_n)_{n=1}^{\infty} \to x$. By Lemma 1, $x \in [a,b]$. By Lemma 2,

$$(f(x_n))_{n=1}^{\infty} \to f(x).$$

But the sequence $(f(x_n))_{n=1}^{\infty}$ is unbounded, and hence divergent, a contradication.

Theorem 6 (Extreme Value Theorem) *Suppose $a < b$. If $f : [a,b] \to \mathbb{R}$, then there are $c, d \in [a,b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a,b]$.*

Proof Let $A = \{f(x) : a \leq x \leq b\}$. Then $A \neq \emptyset$ and, by the Bounding Theorem, A is bounded above and below. Let $\alpha = \sup A$. We claim that there is $d \in [a,b]$ with $f(d) = \alpha$.

Since $\alpha = \sup A$, for each $n \in \mathbb{N}$, there is $x_n \in [a,b]$ with $\alpha - \frac{1}{n} < f(x_n) \leq \alpha$. Note that $(f(x_n))_{n=1}^{\infty}$ converges to α. By the Bolzano–Weierstrass Theorem, we can find a convergent subseqence. Replacing $(x_n)_{n=1}^{\infty}$ by a subsequence if necessary, we may assume $(x_n)_{n=1}^{\infty} \to d$ for some $d \in [a,b]$. Then $(f(x_n))_{n=1}^{\infty} \to f(d)$. Thus $f(d) = \alpha$. Note that $f(x) \leq \alpha = f(d)$ for all $x \in [a,b]$.

Similarly, we can find $c \in [a,b]$ with $f(c) = \beta = \inf A$ and $f(c) \leq f(x)$ for all $x \in [a,b]$.

3 Intermediate Value Theorem

Theorem 7 (Intermediate Value Theorem) *If $f : [a,b] \to \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then there is $a < c < b$ with $f(c) = 0$.*

Proof We start to build a sequence of intervals

$$I_0 \supseteq I_1 \supseteq I_2 \supseteq \dots$$

such that $I_n = [a_n, b_n]$, $f(a_n) < 0 < f(b_n)$ and $b_n - a_n = \frac{b-a}{/2^n}$. Let $a_0 = a, b_0 = b$ and $I_0 = [a_0, b_0]$. Then $f(a_0) < 0 < b_0$ and $b - a = (b-a)/2^0$.

Suppose we are given $I_n = [a_n, b_n]$ with $f(a_n) < 0 < f(b_n)$ and $b_n - a_n = b - a/over2^n$. Let $d = \frac{b_n - a_n}{2}$. If $f(d) = 0$, then we have found $a < d < b$ with $f(d) = 0$ and are done. If $f(d) > 0$, let $a_{n+1} = a_n$, $b_{n+1} = d$. If $f(d) < 0$, let $a_{n+1} = d$ and

Let $I_{n+1} = [b_{n+1}, a_{n+1}]$. Then $I_{n+1} \subset I_n$, $f(a_{n+1}) < 0 < f(b_{n+1})$ and $b_{n+1} - a_{n+1} = \frac{b-a}{2^n}$.

By the nested interval theorem, there is $c \in \bigcap_{n=0}^{\infty} I_n$. We claim that $f(c) = 0$.

Since $a_n, c \in I_n$, $|a_n - c| \leq \frac{b-a}{2^n}$ for all $n \in \mathbb{N}$. If $\epsilon > 0$, choose N such that $\frac{b-a}{2^N} < \epsilon$. Then $|a_n - c| < \epsilon$ for all $n \geq N$. Hence $(a_n)_{n=1}^{\infty}$ converges to c. Thus, by Lemma 2, $(f(a_n))_{n=1}^{\infty}$ converges to $f(c)$. Since $f(a_n) \leq 0$ for all n, we must have $f(c) \leq 0$.

Similarly, $(b_n)_{n=1}^{\infty} \to c$ and $(f(b_n))_{n=1}^{\infty} \to f(c)$. But each $f(b_n) > 0$, thus $f(c) \geq 0$. Hence $f(c) = 0$.

Thus there is $a < c < b$ with $f(c) = 0$.