An Invitation to Mathematical Logic

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Preface

My goal was to write a text for a one semester graduate level introduction to mathematical logic, one that I would have liked to learn from when I was a student, and one I would like to teach from as a professor Two thirds of the book evolved from lecture notes for courses given over 30 years teaching introductory logic at the University of Illinois Chicago.

The heroes of most introductory logic texts are Gödel and Turing.[1] Certainly the Gödel's Completeness and Incompleteness Theorems and Turing's formalization of computability, universal machines and undecidable problems must be at the center of any course in mathematical logic, but I think focusing only on Gödel and Turing gives an unbalanced view. Gödel's results on incompleteness and undecidability in arithmetic become even more interesting when contrasted with Tarski's tameness results for the real and complex fields. One of my goals is to raise Tarski to the podium alongside Gödel and Turing.

I titled this book an **invitation** to mathematical logic as I hope that it will excite readers and make them hungry for further study in logic. One can easily get bogged down at the beginning of a logic course when confronted with a myriad of new definitions, formalism and syntax. I have tried to streamline this part of the text in order to arrive as quickly as possible to the meat. From my experience in the classroom this seems to work well. This book definitely contains more material than could possibly covered in one semester. My hope is that a student would be excited by what they have seen and will be motivated to work through some of the more advanced material in Chapters 8, 12, 14, 15 and 16. Another goal is to put in one place some topics that often fall through the gaps in one's logic education such as cut elimination and models of arithmetic.

Mathematical logic grew out of the study of questions on the foundations of mathematics. Foundational questions are the focus of Part's I and IV and Part III develops the foundations of computability. I have also tried, particularly in Chapters 5, 7, 8, 14 and 16, to illustrate the interplay between logic and other areas of mathematics, notably algebra, number theory and combinatorics. To me this is one of the most fascinating aspects of modern logic.

One difficult decision was to exclude set theory–except for Appendix A which contains a brief introduction to some useful fundamentals. While I view set

[1] Gödel and Turing are even embedded in popular culture in books like *Gödel, Escher Bach* and films like *The Immitation Game* and *Oppenheimer.* Indeed, "Pharma bro" Martin Shkreli named two of his companies after Turing and Gödel.

theory as a central, highly important part of logic, I think it belongs in a separate companion course. Including the material I would want for a proper introduction to set theory would have probably added another 200 pages (and several years of writing) to this volume, defeating the purpose of a one semester introductory text. While perhaps there is a niche for a future *An Invitation to Set Theory*, for now there are excellent introductory texts by Kunen [57], Schindler [86] and Jech [38].

Another decision that some will disagree with is postponing the Incompleteness Theorem until after an introduction to computability theory. Historically, the Incompleteness Theorem preceded and inspired the development of computability and it is both possible and reasonable to take a faster path toward these results. I, however, believe that incompleteness phenomena are best understood having first encountered computability, particularly, the arithmetic hierarchy and computably inseparable computably enumerable sets.

Detailed Overview

In part I we begin by introducing the basic concepts of logic: structures, truth, proofs and Gödel's Completeness Theorem. Chapter 1 begins with the basic concepts of logic–the syntactic notions of languages, terms, formulas and theories and the semantic notions of structures, truth in a structure, logical consequences and definability. It is easy to get bogged down in some of the technical formalism so I try to go though this material as quickly as possible. To this end, I cut some corners on issues such as unique readability of formulas; for completeness, I return to these issues in Appendix B. Induction on the complexity of formulas is a basic proof technique in logic that does not have a counterpart in other areas of mathematics. I include a number of results on equivalent normal forms in this chapter, in part to give the reader more examples of this method.

A fundamental lesson of category theory is that when one studies any type of mathematical objects it is imperative to study structure preserving maps between them. Chapter 2 is devoted to the study of embeddings of structures, isomorphisms of structures and how the truth of formulas is preserved under mappings. This chapter provides the foundation for Part II Elements of Model Theory, and will also be needed in Chapters 13 and 16 of Part IV. Chapter 2 concludes with the Tarski–Vaught characterization of elementary submodels and the Downward Löwenheim–Skolem Theorem. This chapter also has the pedagogical goal of reinforcing the student's proficiency with proofs by induction on complexity of formulas.

Chapters 3 and 4 culminate in Gödel's Completeness Theorem, one of the intellectual gems of logic. I still find it remarkable that the semantic notion of logical consequence, which *a priori* requires quantification over all models of a theory, is completely captured by a finitistic syntactic notion of proof. Chapter 3 introduces and studies a system of formal proof. I have chosen a variant of sequent calculus. There are simpler systems, but this one has the advantage that it is relatively easy to formalize proofs in this framework. Chapter 4 is devoted to a proof, in the style of Henkin, of the Completeness Theorem. We need to show that every consistent theory has a model. How do we build a structure from scratch? In a surprising twist, the syntactic elements of the language can be used to build the desired structure.

Part II is an introduction to model theory and algebraic applications. Chapter 5 begins with the Compactness Theorem. Though the Compactness Theorem is a simple consequence of the Completeness Theorem, it has many surprising and intriguing consequences. I introduce the notions of complete and κ-categorical theories, and Vaught's test for completeness, and use these to show that we can completely axiomatize the theory of the field of complex numbers by saying it is an algebraically closed field of characteristic zero. This is applied in Ax's astonishing proof that an injective polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ must be surjective, where he reduces the problem to polynomial maps on finite fields. I also introduce the back-and-forth method to prove several countable categoricity results. One interesting combinatorial consequence is the 0-1 law for random graphs.

Chapter 6 introduces ultraproducts, a model theoretic method for averaging structures and building rich elementary extensions. Ultraproducts can be used to give an alternative proof of the Compactness Theorem.[2] In my earlier book [63], I chose to deemphasize ultraproducts, a decision I came to regret. Over the last two decades ultraproducts have come to play an important role in continuous model theory and in connections between model theory and combinatorics. The usefulness of ultraproducts in set theory is also a strong argument for inclusion in any first course in logic.

Chapters 7 and 8 are devoted to Tarski's results that, in contrast to the Gödel phenomena of incompleteness and undecidability in arithmetic, the theories of the complex and real field are quite tame. Chapter 7 develops a model theoretic test for quantifier elimination. As a warm up we apply these results to divisible abelian groups and ordered divisible abelian groups. We then prove quantifier elimination for algebraically closed fields, apply this to characterize the definable sets as the constructible sets of algebraic geometry, and give a model theoretic proof of Hilbert's Nullstellensatz.

In Chapter 8 we begin be "reviewing" some of the basic algebra of ordered fields. As I expect most students will not have seen this before, I include a detailed survey of real algebra with proofs in Appendix C. We then prove quantifier elimination for the theory of real closed fields. Some of the applications we include are a characterization of definable sets as the semialgebraic sets, discussion of tameness of definable functions and sets, and Abraham Robinson's model theoretic version of Artin's solution to Hilbert's 17th Problem.[3] The chapter concludes with a brief survey of more recent results on o-minimal expansions of the real field and exponentiation.

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[2]Indeed, if one wanted to get to the model theoretic material as quickly as possible, one could do Chapter 6 directly after Chapter 2 and then cover Chapters 5, 6 and 7, postponing formal proofs to a later part of the course.

[3]I have adopted the style of referring to most people only by their last name–this is impossible with Abraham, Julia and Raphael Robinson.

Part III is devoted to computability. Chapter 9 begins by introducing register machines as a simple model of a programmable computing device. While Turing machines are even simpler, register machines are much easier to program. Although they are simple, the Church–Turing thesis asserts that they capture completely our intuitive notion of potentially computable by algorithm. As examples of things computable by register machines we introduce the primitive and partial recursive functions of Gödel and Herband. We first show that they are all register machine computable and that indeed they describe the same class of computable functions. As evidence for the Church–Turing thesis we introduce random access machines and show that even though they at first seem more powerful, anything they can do can be done on a register machine. For completeness, we conclude with a description of Turing machines and provide some examples.

Chapter 10 discusses Turing's celebrated results that there are universal machines and that the halting problem is undecidable. We use this to give several other examples of undecidability, including Church's theorem on the undecidability of validity in first order logic and Rice's theorem on index sets. We conclude with Kleene's Recursion Theorem, one of the more intriguing consequences of the existence of universal machines.

Chapter 11 introduces the computably enumerable sets and the arithmetic sets and discusses some of their properties.[4] This material will be crucial in our approach to the First Incompleteness Theorem in Chapter 13. As an example of an interesting Π^0_1-set, we study the Kolmogorov random numbers and arising incompleteness phenomena.

Chapter 12 is a brief introduction to some further topics in computability theory. We start by introducing Turing reducibility, the Turing jump, and giving Post's characterization of the arithmetic hierarchy. I then have chosen to survey some of my favorite results in computability theory: the Kleene–Post construction of incomparable degrees, Spector's construction of a minimal degree, a finite injury priority argument to prove the existence of incomplete noncomputable computably enumerable sets, and the Jockusch–Soare Low Basis Theorem. These may seem like random choices but I've included these results because a reasonable number of logic students will never take a more advanced course in computability, and I think it is important that they see these results. In set theory, the Kleene–Post construction is a precursor to Cohen forcing, and the minimal degree construction led to Sacks forcing. I think it is important for a student in set theory to see these results in their original context. The Low Basis Theorem gives new insight on problems like finding the completion of a theory. Finally, every student of logic should understand at least one basic priority argument.

Part IV is devoted to various manifestations of the incompleteness and undecidability of arithmetic. Chapter 13 is devoted to Gödel's Incompleteness Theorems and refinements. We first prove the first Incompleteness Theorem by

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[4] Computably enumerable sets used to be called *recursively enumerable*. While I have adopted the new terminology, I still can not bring myself to use the common abbreviation *c.e.* sets.

showing that we can define the graph of every primitive recursive function in $\mathbb{N}$, then concluding that the sets definable in $\mathbb{N}$ are exactly the arithmetic sets. The results of Part III can then be used to conclude, in a very strong way, that $\mathrm{Th}(\mathbb{N})$ is undecidable and can not be recursively axiomatized. We then turn to Gödel's original proof, where we code formulas by numbers and use diagonalization to produce a sentence asserting its own unprovability. A sketch is given of a proof of the Second Incompleteness Theorem, that PA does not prove its own consistency, though we leave out some tedious, but necessary, details showing that some of the basic properties of proof systems can be formalized in PA. Finally, we sketch the proof by Hilbert and Bernays that the Completeness Theorem can be formalized in PA and Kreisel's model theoretic proof of the Second Incompleteness Theorem.

Chapters 14–16 explore different aspects of incompleteness phenomena. One of the most astonishing manifestations of the incompleteness phenomena is the undecidability of Hilbert's 10th problem on the solvability of Diophantine equations. In Chapter 14 we discuss some aspects of the proof. In particular, following the early work of Davis, Putnam and Julia Robinson [14], we will show that if we assume that $y = 2^x$ is Diophantine definable, then the Diophantine definable sets are exactly the computably enumerable sets. We then discuss Pell equations, one of the key ideas in Matiyasevich's proof that $y = 2^x$ is Diophantine. I will not say much about the rest of the proof. It is long, clever and detailed, relying mostly on very elementary number theory but there are no further logical aspects. As I feel I have nothing new to add, I refer the reader to clear, elegant published treatments such as Murty and Fodden's book [72].

Goodstein found a surprising number theoretic statement whose proof makes essential use of ordinals below ϵ_0. Chapter 15 begins with Goodstein's proof, and then proves the independence of Goodstein's result by showing that the use of ϵ_0 is essential and beyond Peano Arithmetic. The bulk of the chapter is devoted to a theorem of Wainer, building on work of Gentzen and Kreisel, calibrating the growth rates of computable functions provably total in PA. My treatment of this material follows closely unpublished notes of Henry Towsner [100]. and I thank him for letting me adapt his presentation.

Chapter 16 centers on a model theoretic proof due to Paris and Harrington of the independence from PA of a combinatorial statement that is a minor variant of the fundamental result of Ramsey Theory. As a warm up we give model theoretic proofs of two results in the spirit of Chapter 15 characterizing the growth rates of computable functions provably total in weak fragments of Peano Arithmetic. Having introduced the study of nonstandard models of PA as a tool in independence results, we turn to studying these models as interesting objects in their own right and prove several fundamental results about extensions and embeddings of models of PA.

Using this book as a text

This book contains significantly more material than could be covered in a one semester course. This was done intentionally. First, I wanted to give instructors some flexibility in shaping the content of their course. At the end of this introduction I have included a graph of the essential dependencies between chapters. Secondly, I wanted to provide additional material with the hope that interested students would be tempted to explore material beyond what can be covered in one semester.

At a bare minimum I think a course should cover:

- Chapter 1: omitting the material on normal forms;
- Chapter 2: through Proposition 2.16;
- Chapters 3 and 4: all;
- Chapter 5: through the Upward Löwenheim–Skolem Theorem;
- Chapter 9: through the proof of the equivalence of register machine computable and general recursive;
- Chapter 10: omitting index sets and the Recursion Theorem;
- Chapter 11: through the analysis of the arithmetic hierarchy;
- Chapter 13: through the Second Incompleteness Theorem.[5]

When I taught the course at the University of Illinois Chicago, I would usually also cover: the rest of Chapters 2 and 5, all of Chapter 6, the quantifier elimination tests with applications to algebraically closed fields in Chapter 7, quantifier elimination for real closed fields and some consequences in Chapter 8, Kolmogorov randomness from Chapter 11, Turing reducibility, Post's theorem, the existence of incomparable degrees and the finite injury priority argument from Chapter 12 and, if time permitted, a briefly discuss Goodstein's Theorem from Chapter 15–though the exact content changed from year to year.

Chapter 14, 15 and 16 will probably not be covered in most one semester courses. They, like some of the material in Chapters 8 and 12, would make excellent subject matter for a reading course, a graduate student working group, or self-study by motivated students.

Each chapter ends with a section of exercises. The exercises range from quite easy to quite challenging. Some of the exercises develop important ideas that I would have included in a longer text. I have left some important results as exercises because I think students will benefit by working them out. Some exercises are embedded in the chapters. These tend to be ones where I think

[5]If one needs a quicker path to the Incompleteness Theorem, it would be possible to do Chapter 13 immediately after proving the undecidability of the Halting Problem in Chapter 10, adapting slightly some of the arguments in Chapter 13.

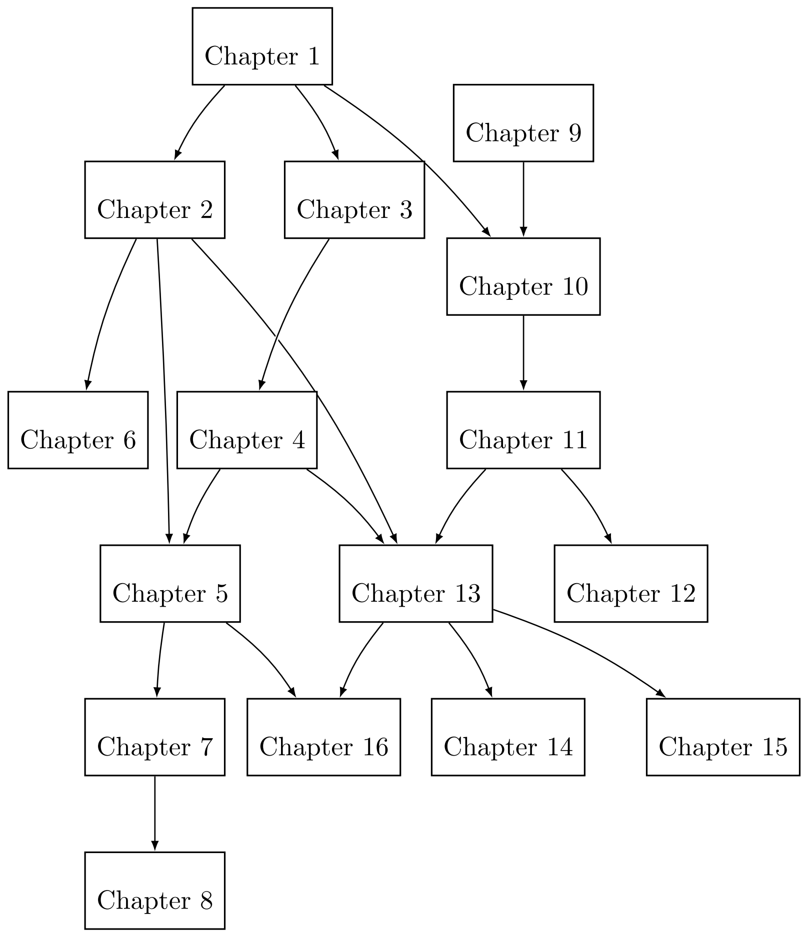
it would be good for the student to work on the exercise immediately to fully understand what is going on. Some exercises will require more comfort with algebra, computability, or set theory than I assume in the rest of the book. I mark those exercises with a dagger †.

Prerequisites

For most of the text the only prerequisite is “mathematical maturity”. It should be suitable for first year graduate students or advanced undergraduates in mathematics, philosophy graduate students with a solid math background or students in computer science who want a mathematical introduction to logic. While some prior exposure to logic would be helpful, it is not assumed. We assume some familiarity with basic set theory–countability, cardinality and Zorn’s Lemma–more or less at the same level one would expect for a first graduate analysis course. Appendix A covers most of this material. In Chapter 15 we assume some familiarity with ordinals. This is also covered in Appendix A.

In Chapters 5, 7 and 8 we assume some familiarity with algebra, particularly algebraically closed fields. A student simultaneously taking a graduate algebra course should be well prepared. Chapter 8 uses algebraic results on ordered fields. While this material is included in many graduate algebra texts–Lang’s *Algebra* [58] is a good reference for the material we need–I suspect most students will not have seen it, so I develop the necessary results in Appendix C.

Chapter dependencies



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