Finite Character and $L_{\infty\omega}$

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We assume throughout these notes that $(\mathcal{K}, \prec_{\mathcal{K}})$ is an a.e.c. in a countable vocabulary with $LS(\mathcal{K}) = \omega$. We prove the following result.

Theorem 1. Assume that $(\mathcal{K}, \prec_{\mathcal{K}})$ has finite character. Let $M \in \mathcal{K}$ and assume that $M \equiv_{\infty\omega} N$. Then $N \in \mathcal{K}$.

For completeness we give the definition of finite character which we use.

Definition. $(\mathcal{K}, \prec_{\mathcal{K}})$ has *finite character* iff for $M, N \in \mathcal{K}$ we have $M \prec_{\mathcal{K}} N$ iff $M \subseteq N$ and for every (finite) $\bar{a} \in M$ there is some $\mathcal{K}$-embedding of M into N fixing $\bar{a}$.

Shelah [Sh:88] showed that the closure properties for chains given by the a.e.c. axioms also hold for directed families. In particular he established the following.

Lemma 2. Let $\mathcal{S}$ be a family of countable structures in $\mathcal{K}$ which is directed under $\prec_{\mathcal{K}}$. Then $\bigcup \mathcal{S} \in \mathcal{K}$.

We use the methods of countable approximations (see our earlier Lecture Notes). We first note the following.

Lemma 3. a) If $M \in \mathcal{K}$ then $M^s \prec_{\mathcal{K}} M$ a.e.

b) If $M \in \mathcal{K}$ and $M_0 \prec_{\mathcal{K}} M$ is countable then $M_0 \prec_{\mathcal{K}} M^s$ a.e.

Proof: a) $\{s \in \mathcal{P}_{\omega_1}(M) : M^s \prec_{\mathcal{K}} M\}$ is unbounded, since $LS(\mathcal{K}) = \omega$. It is closed by the coherence and chains axioms.

b) Since $M_0 \subseteq M^s$ a.e. this follows from part a and the coherence axiom. $\dashv$

From now on, $\bar{x}, \bar{a}$ etc are used exclusively for ω-sequences, and $ran(\bar{a}) = \{a_k : k \in \omega\}$. We define $(M, \bar{a}) \equiv_{\infty\omega} (N, \bar{b})$ to hold iff $\bar{a}$ and $\bar{b}$ satisfy the same formulas of $L_{\infty\omega}$. Since formulas of $L_{\infty\omega}$ have just finitely many free variables, this happens iff $(M, (a_i)_{i<n}) \equiv_{\infty\omega} (N, (b_i)_{i<n})$ for all $n \in \omega$. The serious work towards proving Theorem 1 begins with the following Lemma.

Lemma 4. Let $M \in \mathcal{K}$, let $M_0 \prec_{\mathcal{K}} M$ be countable, let $n \in \omega$, and let $a_0, \ldots, a_{n-1} \in M_0$. Let $N \in \mathcal{K}$ be arbitrary, let $b_0, \ldots, b_{n-1} \in N$, and assume $(M, (a_i)_{i<n}) \equiv_{\infty\omega} (N, (b_i)_{i<n})$. Then

$\big($there is a $\mathcal{K}$-embedding h of M_0 into N^s such that $h(a_i) = b_i$ for all $i < n\big)$ a.e.

Proof: We abbreviate the assertion within parentheses which we wish to show holds a.e. by $\mathcal{E}(N, (b_i)_{i<n}, s)$. Define

$Y = \{s \in \mathcal{P}_{\omega_1}(N) : \mathcal{E}(N, (b_i)_{i<n}, s)\}$.

We will show that $Y \in D_{\omega_1}(N)$ by showing that player II_Y has a winning strategy in the game $G(Y)$.

By Lemma 3, $M_0 \prec_{\mathcal{K}} M^s$ a.e., so

$X = \{s \in \mathcal{P}_{\omega_1}(M) : M_0 \prec_{\mathcal{K}} M^s, M^s = s\} \in D_{\omega_1}(M)$

and thus player II_X has a winning strategy in the game $G(X)$.

The winning strategy for II_Y is obtained as follows. Say I_Y chooses $d_0 \in N$. Pick some $c_0 \in M$ such that $(M, (a_i)_{i<n}, c_0) \equiv_{\infty\omega} (N, (b_i)_{i<n}, d_0)$. In the game $G(X)$ have player I_X choose c_0. Using his winning strategy player II_X responds by choosing c_1. Pick some d_1 such that $(M, (a_i)_{i<n}, c_0, c_1) \equiv_{\infty\omega} (N, (b_i)_{i<n}, d_0, d_1)$. Player II_Y now chooses this d_1 as his response to I_Y's move.

Continuing in this way we obtain sequences $\bar{c} = (c_i)_{i\in\omega}$ from M and $\bar{d} = (d_i)_{i\in\omega}$ from N such that

$(\star)$ $(M, (a_i)_{i<n}, \bar{c}) \equiv_{\infty\omega} (N, (b_i)_{i<n}, \bar{d})$.

Since II_X plays using his winning strategy we know that $ran(\bar{c}) = s_0 \in X$, so $M_0 \prec_{\mathcal{K}} M^{s_0}$ and $M^{s_0} = s_0$.

Let $s_1 = ran(\bar{d})$. Define $g : M^{s_0} \to N$ by $g(c_i) = d_i$ for all $i \in \omega$. Then g is an isomorphism of M^{s_0} onto N^{s_1} and $N^{s_1} = s_1$. Let $N_0 = g[M_0]$. Then $N_0 \prec_{\mathcal{K}} N^{s_1}$, since $\prec_{\mathcal{K}}$ is preserved by the isomorphism g. Necessarily $g(a_i) = b_i$ for all $i < n$, by $(\star)$. Therefore $h = g|M_0$ is a $\mathcal{K}$-embedding of M_0 into N^s such that $h(a_i) = h(b_i)$ for all $i < n$, and so $s_1 \in Y$. $\dashv$

Lemma 5. Assume that $(\mathcal{K}, \prec_{\mathcal{K}})$ has finite character. Let $M \in \mathcal{K}$, $M_0 \prec_{\mathcal{K}} M$ countable, and let $\bar{a}$ be an ω-sequence with $ran(\bar{a}) = M_0$. Let N be arbitrary, let $\bar{b}$ be an ω-sequence from N, and assume that $(M, \bar{a}) \equiv_{\infty\omega} (N, \bar{b})$. Then $ran(\bar{b}) = N_0$ where $N_0 \prec_{\mathcal{K}} N^S$ a.e.

Proof: By Lemma 4 we know that $\mathcal{E}(N, (b_i)_{i<n}, s)$ holds a.e., for each $n \in \omega$. Note that g defined by $g(b_i) = a_i$ for all $i \in \omega$ is an isomorphism of N_0 onto M_0. Thus $\mathcal{E}(N, (b_i)_{i<n}, s)$ implies

$\mathcal{E}^*(N, (b_i)_{i<n}, s)$: there is a $\mathcal{K}$-embedding of N_0 into N^s fixing b_i for all $i < n$,

so for each $n \in \omega$, $\mathcal{E}^*(N, (b_i)_{i<n}, s)$ holds a.e. But the 'almost all' filter is countably complete, so in fact

$\big($for every $n \in \omega$ $\mathcal{E}^*(N, (b_i)_{i<n}, s)\big)$ holds a.e.

By finite character we conclude that $N_0 \prec_{\mathcal{K}} N^s$ a.e., as desired. $\dashv$

Lemma 6. Assume that $(\mathcal{K}, \prec_{\mathcal{K}})$ has finite character. Let $M \in \mathcal{K}$ and assume that $M \equiv_{\infty\omega} N$. Then for every countable $B_0 \subseteq N$ there is some countable $N_0 \subseteq N$ such that $B_0 \subseteq N_0$ and $N_0 \prec_{\mathcal{K}} N^s$ a.e.

Proof: We show how to find a countable ω-sequence $\bar{a}$ from M such that $ran(\bar{a}) = M_0$ where $M_0 \prec_{\mathcal{K}} M$ and an ω-sequence $\bar{b}$ from N such that $(M, \bar{a}) \equiv_{\infty\omega} (N, \bar{b})$ and $B_0 \subseteq ran(\bar{b})$. It then follows from Lemma 5 that $N_0 = ran(\bar{b})$ is as desired.

Since $M^s \prec_{\mathcal{K}} M$ a.e. by Lemma 3, we know that

$X = \{s \in \mathcal{P}_{\omega_1}(M) : M^s \prec_{\mathcal{K}} M, M^s = s\} \in D_{\omega_1}(M)$

and so player II has a winning strategy in the game $G(X)$.

Enumerate B_0 as $\{b_{2k} : k \in \omega\}$. Pick $a_0 \in M$ such that $(M, a_0) \equiv_{\infty\omega} (N, b_0)$ and have player I choose a_0 in the game $G(X)$. Using the winning strategy, player II chooses $a_1 \in M$. Now find $b_1 \in N$ such that $(M, a_0, a_1) \equiv_{\infty\omega} (M, b_0, b_1)$.

Continuing in this way, $\{a_k : k \in \omega\} = s \in X$, so $M_0 = M^s \prec_{\mathcal{K}} M$ and $ran(\bar{a}) = M_0$. Since $(M, \bar{a}) \equiv_{\infty\omega} (N, \bar{b})$ and $B_0 \subseteq ran(\bar{b})$ by construction, we are done. $\dashv$

Note that this argument actually establishes the stronger conclusion that ($N_0 \prec_{\mathcal{K}} N^s$ a.e.) holds for almost all countable $N_0 \subseteq N$.

We now easily obtain the Theorem.

Proof of Theorem 1: Define $\mathcal{S}$ to be

$\{N_0 \subseteq N : N_0$ is countable, $N_0 \prec_{\mathcal{K}} N^s$ a.e.$\}$.

We first note that if $N_0, N_1 \in \mathcal{S}$ and $N_0 \subseteq N_1$ then $N_0 \prec_{\mathcal{K}} N_1$ by coherence, since there will be some $N' \subseteq N$ such that both $N_0 \prec_{\mathcal{K}} N'$ and $N_1 \prec_{\mathcal{K}} N'$.

Secondly, by Lemma 6, $N = \bigcup \mathcal{S}$ and $\mathcal{S}$ is directed under $\subseteq$. But by our first remark, $\mathcal{S}$ will then be directed under $\prec_{\mathcal{K}}$, and so $N \in \mathcal{K}$ by Lemma 2. $\dashv$

In fact, Theorem 1 is a consequence of the following stronger result.

Theorem 7. Assume that $(\mathcal{K}, \prec_{\mathcal{K}})$ has finite character. Then $\mathcal{K} = Mod(\theta)$ for some $\theta \in L(\omega)$.

We outline, without proof, what needs to be done to obtain this stronger Theorem. The first, and most important, step is to show that the property in the conclusion of Lemma 4 is $L(\omega)$-definable.

Lemma 8. Let $M_0 \in K$ be countable, let $n \in \omega$, and let $a_0, \ldots, a_{n-1} \in M_0$. Then there is $\varphi^{(M_0,(a_i)_{i<n})}(x_0, \ldots, x_{n-1}) \in L(\omega)$ such that for all N and $b_0, \ldots, b_{n-1} \in N$,

$N \models \varphi^{(M_0,(a_i)_{i<n})}(b_0, \ldots, b_{n-1})$ iff $\mathcal{E}(N, (b_i)_{i<n}, s)$ holds a.e.

For any countable $M_0 \in \mathcal{K}$ and any ω-sequence $\bar{a}$ such that $ran(\bar{a}) = M_0$ we define $\varphi^{M_0,\bar{a}} = \bigwedge_{n\in\omega} \varphi^{(M_0,(a_i)_{i<n})}$. Note that $\varphi^{M_0,\bar{a}} \in L(\omega)$ even though it has infinitely many free variables. The next Lemma follows using the proof of Lemma 5.

Lemma 9. Assume $(\mathcal{K}, \prec_{\mathcal{K}})$ has finite character. Let $M_0 \in \mathcal{K}$ be countable and let $\bar{a}$ be such that $ran(\bar{a}) = M_0$. Then for any N and ω-sequence $\bar{b}$ from N, $N \models \varphi^{M_0,\bar{a}}(\bar{b})$ iff the mapping g defined by $g(a_k) = b_k$ for all $k \in \omega$ defines an isomorphism of M_0 onto some N_0 such that $N_0 \prec_{\mathcal{K}} N^s$ a.e.

Next, define $\varphi(\bar{x}) = \bigvee\{\varphi^{M_0,\bar{a}} : M_0 \in \mathcal{K}$ is countable and $ran(\bar{a}) = M_0\}$. The following is clear.

Lemma 10. Assume that $(\mathcal{K}, \prec_{\mathcal{K}})$ has finite character. For any N and ω-sequence $\bar{b}$ from N, $N \models \varphi(\bar{b})$ iff $ran(\bar{b}) = N_0$ where $N_0 \prec_{\mathcal{K}} N^s$ a.e.

Finally we define $\theta = (\forall x_{2n} \exists x_{2n+1})_{n\in\omega} \varphi(\bar{x})$. Then $\theta \in L(\omega)$ since it has no free variables and $\varphi \in L(\omega)$. If $M \in \mathcal{K}$ then $M \models \theta$ by Lemmas 3 and 10. The proof that $M \models \theta$ implies $M \in \mathcal{K}$ is just like the proof of Theorem 1, using the hypothesis that $M \models \theta$ in place of Lemma 6.