Math 494: Topics in Algebra

Problem Set 1

Due Friday Feburary 14: Turn in 6 of the following problems. At the end of the problem set there is a list of useful MAPLE commands that may be useful in problems 1)–3).

1) Let $f(X) = X^5 + 2X^4 + 2X^3 + X^2 - X - 1$.

a) How many distict zeros does f have in $\mathbb{C}$? What are the multiplicities of the zeros?

b) We can also view f as a polynomial in $\mathbb{Z}_3[X]$. Suppose $K \supseteq \mathbb{Z}_3$ is an algebraically closed field. How many distict zeros does f have in K? What are the multiplicities of the zeros?

2) Let $f = X^3 + X + 3$ and $g = X^5 - X^4 - X + 6$.

a) Do f and g have a common zero in $\mathbb{C}$.

b) Suppose K is an algebraically closed field containing $\mathbb{Z}_p$. For which primes p do f and g have a common solution in K?

3) [The Euclidean Algorithm] Suppose F is a field and $f, g \in F[X]$ are nonzero. Define a sequence of polynomials $r_0, \ldots, r_n$, $q_0, \ldots, q_{n+1} \in F[X]$ such that deg $g >$ deg $r_0 > \ldots >$ deg r_n and

$$\begin{aligned}
f &= q_0 g + r_0 \\
g &= q_1 r_0 + r_1 \\
r_0 &= q_2 r_1 + r_2 \\
&\vdots \\
r_{n-2} &= q_n r_{n-1} + r_n \\
r_{n-1} &= q_{n+1} r_n.
\end{aligned}$$

a) Show that r_n divides f and g, if h divides f and g, then r_n divides h, and there are $s, t \in K[X]$ such that $sf + tg = h$. We call r_n a *greatest common factor* of f and g.[Hint: work backward by induction.]

b) Let $f(X) = X^4 + X^3 + 6X^2 + X + 5$ and $g(X) = X^4 - 2X^3 + 6X^2 - 2X + 5$. Use the Euclidean Algorithm to find h a greatest common factor of f and g and to find s, t such that $sf + tg = h$.

4) Prove that every algebraically closed field is infinite.

5) Suppose $f \in \mathbb{R}[X]$, $a, b \in \mathbb{R}$ and $f(a + bi) = 0$. Prove that $f(a - bi) = 0$. [Hint: Consider the polynomial $g(X) = X^2 - 2aX + a^2 + b^2$.]

6) Suppose $K \supseteq \mathbb{Z}_p$ is an algebraically closed field. Let $f(X) = X^{p^n} + X$.

a) Show that f has p^n distinct zeros in K.

b) Let $F = \{x \in K : f(x) = 0\}$. Prove that F is a field with p^n elements.

c) Suppose k is any field with p^n elements. Show that every element of k is a zero of f. [This is the key step in the proof that there is a unique field with p^n elements.]

7) Suppose K is a field, $f \in K[X_1, \ldots, X_n]$, $A \subseteq K$ is infinite and

$$f(a_1, \ldots, a_n) = 0$$

for all $a_1, \ldots, a_n \in A$. Prove that $f = 0$.[Hint: Use induction.]

Here are some MAPLE commands that may be useful in problems 1)–3).

`factor(f);` factors f in $\mathbb{Q}[X]$

`Factor(f) mod p;` factors f in $\mathbb{Z}_p[X]$

`resultant(f,g,X);` computes the resultant of f, g polynomials in the variable X

`rem(f,g,X);` computes the remainder when the polynomial f is divided by g in $\mathbb{Q}[X]$

`quo(f,g,X);` computes the quotient when the polynomial f is divided by g in $\mathbb{Q}[X]$