Löwenheim-Skolem Theorems, Countable Approximations, and $L_{\infty\omega}$

David W. Kueker
(Lecture Notes, Fall 2007)

0. Introduction

In its simplest form the Löwenheim-Skolem Theorem for $L_{\omega_1\omega}$ states that if $\sigma \in L_{\omega_1\omega}$ and $M \models \sigma$ then $M_0 \models \sigma$ for some (in fact, 'many') countable $M_0 \subseteq M$. For sentences in $L_{\infty\omega}$ but not in $L_{\omega_1\omega}$ this property normally fails. But we will see that the $L_{\infty\omega}$ properties of arbitrary structures are determined by properties of their countable substructures. In particular we obtain a biconditional strengthening of the 'simple' Löwenheim-Skolem Theorem for $L_{\omega_1\omega}$ given above. We also obtain a characterization of $L_{\infty\omega}$-elementary equivalence of two structures in terms of isomorphisms between their countable substructures.

Central to this development are:

1) the concept of a *countable approximation* to structures and to formulas of $L_{\infty\omega}$,

2) a filter which defines a notion of *almost all* on countable approximations.

This work appeared in the author's paper [Ku3] (see also the announcement [Ku1]), which also discusses some other topics, including extensions to infinite quantifier logics. Some results from the paper and, more importantly, the methods, are used in our recent work on abstract elementary classes (see [Ku4]), which is the reason they are being covered in this course.

The usual formalization of $L_{\infty\omega}$ uses conjunctions and disjunctions of sets of formulas, thus if Φ is a set of formulas of $L_{\infty\omega}$ then $\bigwedge \Phi$ and $\bigvee \Phi$ are formulas. Many examples are more clearly written using indexed sets instead, so if $\varphi_i \in L_{\infty\omega}$ for all $i \in I$ then $\bigwedge_{i\in I}\varphi_i$ and $\bigvee_{i\in I}\varphi_i$ are formulas. We adopt the indexed sets version in these notes. Of course a set Φ can be treated as an indexed set with the index of φ being just φ so this is not a significant change.

1. Countable Approximations

A countable approximation to a structure M is just a countable substructure of M. A countable approximation to a formula $\varphi \in L_{\infty\omega}$ is the result of replacing all conjunctions and disjunctions over uncountable sets by conjunctions and disjunctions over countable subsets. Countable approximations are indexed by countable sets as in the following definition, which we state for arbitrary sets s to include the later generalization. For simplicity we assume throughout that all languages L are countable.

Definition. a) M^s is the substructure of M generated by $(M \cap s)$.

b) We define φ^s for $\varphi \in L_{\infty\omega}$ by recursion:

$\varphi^s = \varphi$ if φ is atomic
$(\neg\varphi)^s = \neg\varphi^s$
$(\forall x\varphi)^s = \forall x\varphi^s$

$(\exists x\varphi)^s = \exists x\varphi^s$
$(\bigwedge_{i\in I}\varphi_i)^s = \bigwedge_{i\in(I\cap s)}\varphi_i^s$
$(\bigvee_{i\in I}\varphi_i)^s = \bigvee_{i\in(I\cap s)}\varphi_i^s$

Some convention is needed for the case in which $(M \cap s) = \emptyset$ and L contains no constants. The easiest solution is to allow the empty structure, but the precise way in which this is done is irrelevant since $(M \cap s)$ will be non-empty for almost all countable s. M^s is a countable substructure of M for every countable s and $\varphi^s \in L_{\omega_1\omega}$ for every countable s. Note that in general a countable approximation to a formula is not a subformula of the formula and a formula neither implies nor is implied by its countable approximations, either individually or in concert. The reader should think about how these definitions must be modified to include the case of uncountable languages.

In any context we will be considering countable $s \subseteq C$ for some set C which is *large enough* to approximate whatever we are looking at — where C is *large enough* to approximate M if $M \subseteq C$ and C is *large enough* to approximate φ if $I \subseteq C$ for all I such that some subformula of φ is either a conjunction or disjunction indexed by I. Note that if C is large enough to approximate M and φ and $C \subseteq C'$ then C' is large enough to approximate M and φ and for every $s' \subseteq C'$ we have $M^{s'} = M^s$ and $\varphi^{s'} = \varphi^s$ where $s = (s' \cap C) \subseteq C$.

2. 'Almost All' Countable s

For any set C we define a filter on $\mathcal{P}_{\omega_1}(C)$, the set of countable subsets of C, and define 'for almost all countable $s \subseteq C$' to mean 'for all s in some set belonging to the filter'. We first introduce some terminology. Given C let $X \subseteq \mathcal{P}_{\omega_1}(C)$. Then:

X is *closed* iff X is closed under unions of countable chains;

X is *unbounded* iff for every $s_0 \in \mathcal{P}_{\omega_1}(C)$ there is some $s \in X$ such that $s_0 \subseteq s$.

These are the natural generalizations of closed and unbounded sets of ordinals to this context.

Definition. For any C, $D(C)$ is the set of all $X \subseteq \mathcal{P}_{\omega_1}(C)$ such that X contains some closed unbounded set.

The following Lemma gives the basic properties of these filters.

Lemma 1. Let $D = D(C)$.

a) D is a countably complete filter.

b) D is closed under *diagonalization*, that is, if $X_i \in D$ for every $i \in I$, where $I \subseteq C$, then $\bar{X} \in D$ where

$\bar{X} = \{s \in \mathcal{P}_{\omega_1}(C) : s \in X_i \text{ for every } i \in (s \cap I)\}$.

Proof: a) It suffices to show that the intersection of countably many closed unbounded sets is also closed unbounded. Let X_i be closed unbounded for all $i \in \omega$. It is clear that $\bigcap_{i\in\omega}X_i$ is closed. To show that it is unbounded we use an alternating chains argument. Partition ω into infinitely many infinite disjoint sets I_i for $i \in \omega$. Given countable s_0 we use the fact that each X_i is unbounded to define sets s_k for $k \in \omega$ such that $s_k \subseteq s_{k+1}$ and $s_{k+1} \in X_i$ whenever $(k+1) \in I_i$. Then $\bigcup_{k\in\omega}s_k \in \bigcap_{i\in\omega}X_i$.

b) We may assume that each X_i is closed unbounded. It is easy to see that $\bar{X}$ is closed. We show that $\bar{X}$ is unbounded. Let countable s_0 be given. By part a, $Y_0 = \bigcap\{X_i : i \in (s_0 \cap I)\}$ is closed unbounded, hence there is some $s_1 \in Y_0$ such that $s_0 \subseteq s_1$. We continue in this way, at stage n obtaining $s_{n+1} \in Y_n = \bigcap\{X_i : i \in (s_n \cap I)\}$ such that $s_n \subseteq s_{n+1}$. Let $s = \bigcup_{n \in \omega} s_n$. We claim $s \in \bar{X}$. Let $i \in (s \cap I)$. Then there is some n_0 such that $i \in (s_n \cap I)$ for all $n \geq n_0$, hence $s_n \in X_i$ for all $n > n_0$. Therefore $s = \bigcup_{n > n_0} s_n \in X_i$, since X_i is closed, and so $s \in \bar{X}$ as desired. $\dashv$

Since $X \in D$ is interpreted as '$(s \in X)$ almost everywhere (a.e.)', diagonalization can be interpreted as allowing us to interchange quantifiers in the following way:

if $(\forall i \in I)(s \in X_i$ a.e.$)$ then $[\forall i \in (I \cap s)(s \in X_i)]$ a.e.

This filter was also studied by T. Jech [Je1] who independently proved the preceding Lemma about the filter. He did not, however, obtain the game theoretic characterization which we now present.

Given $X \subseteq \mathcal{P}_{\omega_1}(C)$ we define the infinite two-person game $G(X)$ as follows: at stage n player I chooses some $a_n \in C$ and player II responds by choosing some $b_n \in C$. Player II *wins* iff $s = \{a_n : n \in \omega\} \cup \{b_n : n \in \omega\} \in X$.

Theorem 2. For any $X \subseteq \mathcal{P}_{\omega_1}(C)$, player II has a winning strategy in the game $G(X)$ iff $X \in D(C)$.

Proof: First, suppose that II has a winning strategy. This means that there are functions $f_n : C^{n+1} \to C$ such that whenever $a_n \in C$ for all $n \in \omega$ then $s = \{a_n : n \in \omega\} \cup \{f_n(a_0, \ldots, a_n) : n \in \omega\} \in X$. Now let X' be the set of all $s \in \mathcal{P}_{\omega_1}(C)$ such that s is closed under f_n for all $n \in \omega$. Clearly X' is closed unbounded. Now let $s \in X'$, say that $s = \{a_n : n \in \omega\}$. Then $s = \{a_n : n \in \omega\} \cup \{f_n(a_0, \ldots, a_n) : n \in \omega\} \in X$, since s is closed under all f_n and since they define II's winning strategy. Thus $X' \subseteq X$, hence $X \in D(C)$.

For the other direction suppose that $X \in D(C)$. We may assume that X is closed unbounded. We describe a winning strategy for II in the game $G(X)$. Suppose that I chooses a_0. II first picks some $s_0 \in X$ such that $\{a_0\} \subseteq s_0$ and will choose every element of s_0 in an infinite number of his succeeding moves. More formally, partition ω into an infinite number of infinite disjoint subsets X_i for $i \in \omega$ such that the smallest element of X_i is $\geq i$ for each $i \in \omega$. II will choose b_n's so that $s_0 = \{b_n : n \in X_0\}$. At the next move I chooses some a_1. Now II first picks some $s_1 \in X$ such that $s_0 \cup \{a_1\} \subseteq s_1$. II will choose b_n's so that $s_1 = s_0 \cup \{b_n : n \in X_1\}$. By continuing in this way II guarantees that $\{a_n ; n \in \omega\} \cup \{b_n : n \in \omega\} = \bigcup_{n \in \omega} s_n$ where s_n is a chain from X, hence its union is in X, as desired. $\dashv$

As an immediate consequence we have the following 'game-free' characterization of the filters.

Corollary 3. For $X \subseteq \mathcal{P}_{\omega_1}(C)$, $X \in D(C)$ iff there are functions f_n on C for $n \in \omega$, each with finitely many arguments, such that $s \in X$ whenever $s \in \mathcal{P}_{\omega_1}(C)$ and s is closed under f_n for every $n \in \omega$.

Theorem 2 is essential to our work on abstract elementary classes referred to earlier. We note that Corollary 3 has been used heavily in set theory. It is basic to Shelah's proper forcing (see [Je2], for example) and, especially in some of its uncountable generalizations (not considered in these notes, but see [Ku3]), is used in various combinatorial arguments (see [Do], for example).

3. Löwenheim-Skolem Theorems

A property of countable approximations of one or more models and/or formulas is said to hold *almost everywhere*, or *a.e.*, iff it holds for all $s \in X$ for some $X \in D(C)$ where C is large enough to approximate all of the models and formulas involved. Using the observation at the end of section 1 it is easy to see that this is independent of the choice of C. As an exercise, the reader should show that the universe of M^s is $(M \cap s)$ a.e.

For the proof of our next result we require the notion of *negation-normal form*. A formula φ is in *negation-normal form* iff whenever $\neg\alpha$ is a subformula of φ then α is atomic. The *canonical negation-normal form* φ^n of φ is defined by recursion:

$\varphi^n = \varphi$ if φ is atomic
$(\bigwedge_{i \in I} \varphi_i)^n = \bigwedge_{i \in I} (\varphi_i)^n$
$(\bigvee_{i \in I} \varphi_i)^n = \bigvee_{i \in I} (\varphi_i)^n$
$(\forall x \varphi)^n = \forall x \varphi^n$
$(\exists x \varphi)^n = \exists x \varphi^n$
$(\neg\varphi)^n$ is defined by cases on φ as follows:

$(\neg\varphi)^n = \neg\varphi$ if φ is atomic
$(\neg\neg\psi)^n = \psi^n$
$(\neg\bigwedge_{i \in I} \varphi_i)^n = \bigvee_{i \in I} (\neg\varphi_i)^n$
$(\neg\bigvee_{i \in I} \varphi_i)^n = \bigwedge_{i \in I} (\neg\varphi_i)^n$
$(\neg\forall x \psi)^n = \exists x (\neg\psi)^n$
$(\neg\exists x \psi)^n = \forall x (\neg\psi)^n$

It is easy to see that φ^n is in negation-normal form and is logically equivalent to φ. We need the additional easy fact: for every s, $(\varphi^n)^s = (\varphi^s)^n$.

The following is a Löwenheim-Skolem Theorem showing how the $L_{\infty\omega}$ properties of a structure are determined by properties of its countable substructures.

Theorem 4. For any M and any $\sigma \in L_{\infty\omega}$

$$M \models \sigma \text{ iff } M^s \models \sigma^s \text{ a.e.}$$

Proof: First note that it is enough to prove the implication from left to right, since if we know that direction and $M \not\models \sigma$ then $M \models \neg\sigma$ hence $M^s \models (\neg\sigma)^s$ a.e. and thus it can't happen that $M^s \models \sigma^s$ a.e. since $(\neg\sigma)^s = \neg\sigma^s$ for all s.

Secondly, because of the facts above about canonical negation-normal forms, it is sufficient to prove the implication for sentences in negation-normal form. We prove by induction that for all $\psi(\bar{x}) \in L_{\infty\omega}$ in negation-normal form and for all $\bar{a} \in M$, if $M \models \psi(\bar{a})$ then $M^s \models \psi^s(\bar{a})$ a.e.

Assume that $M \models \psi(\bar{a})$ and the implication is known for all proper subformulas of ψ. Let C be large enough to approximate both M and ψ (and hence all subformulas of ψ) and let $D = D(C)$. Let

$X(\psi, \bar{a}) = \{s \in \mathcal{P}_{\omega_1}(C) : M^s \models \psi^s(\bar{a})\}$.

We show that $X(\psi, \bar{a}) \in D$.

If ψ is atomic or negated atomic this is easy (why?).

If $\psi = \bigvee_{i \in I} \varphi_i$ then we must have $M \models \varphi_i(\bar{a})$ for some $i \in I$. By inductive hypothesis we know that $M^s \models \varphi_i^s(\bar{a})$ a.e. and so $M^s \models \psi^s(\bar{a})$ a.e., as desired (explain!).

The case $\psi = \exists y \varphi$ is similar to the preceeding since an existential quantification is just a disjunction, over the elements of M.

Suppose that $\psi = \bigwedge_{i \in I} \varphi_i$. For each $i \in I$ define $X_i = X(\varphi_i, \bar{a})$. By inductive hypothesis $X_i \in D$ for all $i \in I$. By diagonalization, $\bar{X} \in D$ where

$\bar{X} = \{s \in \mathcal{P}_{\omega_1}(C) : s \in X_i \text{ for all } i \in (s \cap I)\}$.

Translating this we see that

$\bar{X} = \{s \in \mathcal{P}_{\omega_1}(C) : M^s \models \varphi_i^s(\bar{a}) \text{ for all } i \in (s \cap I)\}$.

But this is exactly $X(\psi, \bar{a})$, so this case is finished.

Finally the case of a universal quantifier is just like the case of a conjunction, so the proof is complete. $\dashv$

As another exercise in diagonalization, the reader is invited to show that for every $\varphi(x_1, \ldots, x_n) \in L_{\infty\omega}$ and every M the following holds:

$\big(\text{for every } \bar{a} \in (M^s)^n [M \models \varphi(\bar{a}) \text{ iff } M^s \models \varphi^s(\bar{a})]\big)$ a.e.

We note a matched pair of consequences of the Theorem.

Corollary 5. a) If $\sigma \in L_{\omega_1\omega}$ then $M \models \sigma$ iff $M^s \models \sigma$ a.e. (also read, '$M_0 \models \sigma$ for almost all countable substructures M_0 of M').

b) If M is countable and $\sigma \in L_{\infty\omega}$, then $M \models \sigma$ iff $M \models \sigma^s$ a.e.

Certainly part a does not require our general theory, but part b seems to. It is easy to derive from part a that if M is countable and $M \equiv_{\omega_1\omega} N$ then $M \equiv_{\infty\omega} N$. It follows from Theorem 4 that if $\models \sigma^s$ a.e. then $\models \sigma$, but the converse fails in general. As an example where the converse does hold, the reader should show that if $\sigma, \varphi_i(x) \in L_{\omega_1\omega}$ for all $i \in I$ then

$\models (\sigma \rightarrow \exists x \bigwedge_{i \in I} \varphi_i)$ iff $\models (\sigma \rightarrow \exists x \bigwedge_{i \in I_0} \varphi_i)$ for all countable $I_0 \subseteq I$.

Using Theorem 4 we can prove the characterization of $L_{\infty\omega}$-elementary equivalence referred to in the Introduction. Note that part b of this result is not an immediate consequence of part a since the 'almost all' filter is not an ultrafilter.

Theorem 6. a) $M \equiv_{\infty\omega} N$ iff $M^s \cong N^s$ a.e.

b) $M \not\equiv_{\infty\omega} N$ iff $M^s \not\cong N^s$ a.e.

Proof: It suffices to show the left to right directions of each part since the reverse implications will then follow.

a) Assume that $M \equiv_{\infty\omega} N$. By the partial isomorphism characterization of $L_{\infty\omega}$-elementary equivalence there are functions

$f_n : (M^{n+1} \times N^n) \rightarrow N$ and $g_n : (M^n \times N^{n+1}) \rightarrow M$

such that whenever $a_0, \ldots, a_n \in M$, $b_0, \ldots, b_n \in N$ are such that

$(M, a_0, \ldots, a_n) \equiv_{\infty\omega} (N, b_0, \ldots, b_n)$

then for any $a_{n+1} \in M$ and for any $b_{n+1} \in N$ we have

$(M, a_0, \ldots, a_n, a_{n+1}) \equiv_{\infty\omega} (N, b_0, \ldots, b_n, f_n(a_0, \ldots, a_{n+1}, b_0, \ldots, b_n))$

and

$(M, a_0, \ldots, a_n, g_n(a_0, \ldots, a_n, b_0, \ldots, b_n, b_{n+1})) \equiv_{\infty\omega} (N, b_0, \ldots, b_n, b_{n+1})$.
(The functions f_n, g_n define the winning strategy for player II in the game version of the characterization). Let $C \supseteq (M \cup N)$ and define

$X = \{s \in \mathcal{P}_{\omega_1}(C) : s$ is closed under f_n, g_n for all $n \in \omega\}$.

Then, by the easy direction of Corollary 3, $X \in D(C)$. Finally, if $s \in X$ then you can use f_n, g_n to construct an isomorphism of M^s onto N^s. Thus, $M^s \cong N^s$ a.e.

b) If $M \not\equiv_{\infty\omega} N$ then there is some $\sigma \in L_{\infty\omega}$ such that $M \models \sigma$ and $N \models \neg\sigma$. It follows from Theorem 4 that $M^s \models \sigma^s$ a.e. and $N^s \models \neg\sigma^s$ a.e. We conclude that $M^s \not\cong N^s$ a.e. (Explain!). ⊣

We note the following:

Corollary 7. If M is countable, then $M \equiv_{\infty\omega} N$ iff $M \cong N^s$ a.e.

We also have the following application:

Corollary 8. A group G is $L_{\infty\omega}$-elementarily equivalent to a free group iff every countable subgroup of G is free.

Proof: If $G \equiv_{\infty\omega} H$ where H is free then $G^s \cong H^s$ a.e., hence G^s is free a.e. and thus all countable subgroups of G are free. Conversely if every countable subgroup of G is free and G is uncountable then G is not finitely generated, hence G^s is not finitely generated a.e. and so $G^s \cong H_\omega$ a.e. where H_ω is the free group on ω generators and thus $G \equiv_{\infty\omega} H_\omega$. ⊣

4. Uncountable Approximations

Let λ be an uncountable cardinal. Then λ-approximations are just M^s and φ^s for sets s of cardinality $\leq \lambda$. There is more than one way to define a corresponding filter to yield a notion of λ-a.e. This was done in [Ku3] so as to yield results about infinite quantifier logics, which are used in our recent work about abstract elementary classes with uncountable Löwenheim-Skolem number. Since in this course we will concentrate on the countable Löwenheim-Skolem number case, we present a simpler generalization which is adequate for $L_{\infty\omega}$. We omit all proofs.

Given C and $X \subseteq \mathcal{P}_{\lambda^+}(C)$ we say that X is *closed* iff X is closed under unions of chains of length $\leq \lambda$, and X is *unbounded* iff for every $s_0 \in \mathcal{P}_{\lambda^+}(C)$ there is some $s \in X$ such that $s_0 \subseteq s$.

Definition. For any C, $D_{\lambda^+}(C)$ is the set of all $X \subseteq \mathcal{P}_{\lambda^+}(C)$ such that X contains some closed unbounded set.

The following Lemma gives the basic properties of the filters.

Lemma 9. Let $D_{\lambda^+} = D_{\lambda^+}(C)$.

a) D_{λ^+} is a λ^+-complete filter.

b) D_{λ^+} is closed under diagonalization.

We use λ-a.e. to mean for all $s \in X$ for some $X \in D_{\lambda^+}(C)$ where C is large enough to approximate all of the models and formulas involved. The following two Löwenheim-Skolem results correspond to Theorems 4 and 6. (We omit the statements of the results corresponding to Corollary 5 and part b of Theorem 6).

Theorem 10. For any M and any $\sigma \in L_{\infty\omega}$

$$M \models \sigma \text{ iff } M^s \models \sigma^s \quad \lambda\text{-a.e.}$$

Theorem 11. $M \equiv_{\infty\omega} N$ iff $M^s \equiv_{\infty\omega} N^s$ λ-a.e.

Of course, Theorem 11 is less striking than Theorem 6 since $L_{\infty\omega}$-elementarily equivalent models of the same uncountable cardinality need not be isomorphic.

5. Other Löwenheim-Skolem Results

The Theorems in section 3 can be applied to yield Löwenheim-Skolem results about certain properties not given by sentences of $L_{\infty\omega}$. We give a few examples here.

Definition. a) M is *rigid* iff M has no proper automorphisms.
b) M is *definably rigid* iff every element of M is definable by a formula of $L_{\infty\omega}$.

Clearly, definably rigid structures are rigid, but the converse fails in general (there are rigid dense linear orders, for example). However the two notions coincide on countable structures.

Lemma 12. If M is countable then M is rigid iff it is definably rigid.

Proof: If M is not definably rigid then there are $a, b \in M$ with $a \neq b$ such that $(M, a) \equiv_{\infty\omega} (M, b)$. But then $(M, a) \cong (M, b)$ so M is not rigid. $\dashv$

We obtain the following Löwenheim-Skolem, or 'transfer', result about such models.

Theorem 13. M is definably rigid iff M^s is rigid a.e.

Proof: Assume first that M is definably rigid. Then there are $\varphi_a(x) \in L_{\infty\omega}$ for all $a \in M$ such that

$M \models \forall x \bigvee_{a \in M} [\varphi_a(x) \wedge \exists! x \varphi_a(x)]$.

By Theorem 4 this implies

$M^s \models \forall x \bigvee_{a \in (M \cap s)} [\varphi_a^s(x) \wedge \exists! x \varphi_a^s(x)]$ a.e.

But since $M \cap s$ is the universe of M^s a.e. we conclude that M^s is definably rigid a.e.

For the other direction suppose that M is not definably rigid. Then there are $a, b \in M$ such that $a \neq b$ but $(M, a) \equiv_{\infty\omega} (M, b)$. By Theorem 6 we conclude that $(M^s, a) \cong (M^s, b)$ a.e., so M^s is not rigid a.e. $\dashv$

As a consequence we obtain the following:

Corollary 14. Let $\sigma \in L_{\omega_1\omega}$.
a) If σ has a definably rigid model then σ has a countable rigid model.
b) If every countable model of σ is rigid then all models of σ are definably rigid.

We remark that the only property of $K = Mod(\sigma)$ needed for the Corollary is that $M \in K$ implies that $M^s \in K$ a.e.

We state without proof the analogous results for models with 'few' automorphisms.

Definition. a) M has *few automorphisms* iff there is some (finite) tuple $\bar{a} \in M$ such that $(M, \bar{a})$ is rigid.

b) M has *definably few automorphisms* iff there is some tuple $\bar{a} \in M$ such that $(M, \bar{a})$ is definably rigid.

Note that if M is infinite and has few automorphisms then it has $\leq |M|$ automorphisms. Once again, on countable M these notions coincide (see [Ku2], for example).

Lemma 15. If M is countable the following are equivalent:
i) M has few automorphisms
ii) M has just countably many automorphisms
iii) M has $< 2^\omega$ automorphisms
iv) M has definably few automorphisms

Theorem 16. M has definably few automorphisms iff M^s has few automorphisms a.e.

We omit the statement of the result corresponding to Corollary 14, but note the following consequence which is weaker but more likely to appeal to an algebraist (cf. [Hi]).

Corollary 17. Let $\sigma \in L_{\omega_1\omega}$. If every countable model of σ has few automorphisms then every model of σ has few automorphisms.

6. Logics with Game Quantification

The natural way to write down a sentence saying that almost all countable substructures have a certain property is to use a game quantifier. In this section we discuss one such logic, $L(\omega)$, studied by Keisler [Ke].

Definition. The set of formulas of $L(\omega)$ is defined as follows:
i) every atomic formula of L belongs to $L(\omega)$,
ii) if $\varphi \in L(\omega)$ then $\neg\varphi \in L(\omega)$,
iii) if $\Phi \subseteq L(\omega)$ then $\bigvee \Phi, \bigwedge \Phi \in L(\omega)$,
iv) if $\varphi \in L(\omega)$ then $\exists x\varphi, \forall x\varphi \in L(\omega)$ provided they have just finitely many free variables,
v) if $\varphi \in L(\omega)$ then $Q_0x_0 \ldots Q_nx_n \ldots \varphi \in L(\omega)$, where each Q_n is either $\forall$ or $\exists$, provided it has only finitely many free variables.

Theorem 18. Let K_0 be any class of countable structures. Then there is some $\sigma \in L(\omega)$ such that for every M we have $M \models \sigma$ iff $M^s \in K_0$ a.e.

Proof: For any $M \in K_0$ and any enumeration $\bar{a}$ of the elements of M, allowing repetitions, let $\theta_{M,\bar{a}}$ be

$$\bigwedge\{\alpha(x_0, \ldots, x_n) : \alpha \text{ is atomic or negated atomic and } M \models \alpha(a_0, \ldots, a_n)\}.$$

Let $\varphi(\bar{x})$ be the disjunction over all $M \in K_0$ and all enumerations $\bar{a}$ of M of the formulas $\theta_{M,\bar{a}}$. Finally let σ be

$$\forall x_0 \exists x_1 \ldots \forall x_{2n} \exists x_{2n+1} \ldots \varphi.$$

We leave the verification that σ is as desired to the reader. $\dashv$

For example, if K_0 is the class of all countable rigid models, then $Mod(\sigma)$ will be the class of all definably rigid models.

Certainly every formula of $L_{\infty\omega}$ belongs to $L(\omega)$ but the converse fails. In fact there are sentences of $L(\omega)$ which are not equivalent to any sentence of $L_{\infty\omega}$. The reader should give an example. But Keisler proved that every formula of $L(\omega)$ is preserved by $L_{\infty\omega}$-elementary equivalence.

Theorem 19. [Ke] Let $\varphi(\bar{x}) \in L(\omega)$. Given M and N, assume that $\bar{a} \subseteq M$ and $\bar{b} \subseteq N$, with $lh(\bar{x}) = lh(\bar{a}) = lh(\bar{b})$, are such that $(M, a_{i_0}, \ldots a_{i_n}) \equiv_{\infty\omega} (N, b_{i_0}, \ldots, b_{i_n})$ for all $i_0 < \ldots < i_n < lh(\bar{x})$. Then $M \models \varphi(\bar{a})$ iff $N \models \varphi(\bar{b})$.

Proof: By induction on φ using the partial isomorphism characterization of $L_{\infty\omega}$-elementary equivalence in the step involving the game quantifier. $\dashv$

Corollary 20. If $M \equiv_{\infty\omega} N$ then $M \equiv_{L(\omega)} N$.

References

[Do] N. Dobrinen. κ-stationary subsets of $\mathcal{P}_{\kappa^+}\lambda$, infinitary games, and distributive laws in boolean algebras, to appear.

[Hi] K. Hickin. Countable type local theorems in algebra, J. Algebra 27(1973) 523-537.

[Je1] T. Jech. Some combinatorial problems concerning uncountable cardinals, Annals of Math. Logic 5 (1973) 165-198.

[Je2] T. Jech. Stationary sets, to appear in the Handbook of Set Theory.

[Ke] H.J. Keisler. Formulas with Linearly Ordered Quantifiers, in: The Syntax and Semantics of Infinitary Languages, SLN 72 (1968) 96-130.

[Ku1] D.W. Kueker. Löwenheim-Skolem and interpolation theorems in infinitary languages, Bull. Amer. Math. Soc. 78 (1972) 211-215.

[Ku2] D.W. Kueker. Back-and-forth arguments and infinitary logics, in: Infinitary Logic: In Memoriam Carol Karp, SLN 492 (1975) 17-71.

[Ku3] D.W. Kueker. Countable approximations and Löwenheim-Skolem theorems, Annals of Math. Logic 11 (1977) 57-103.

[Ku4] D.W. Kueker. Abstract elementary classes and infinitary logics, in preparation.