Exam 2 Math 310 Anton Leykin Friday 14 Nov 2003

1. **[15 points]** The linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ is defined as

$$L(\mathbf{x}) = \begin{pmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 + 3x_2 + 4x_3 \\ 3x_1 + 4x_2 + 5x_3 \end{pmatrix}$$

(a) What is the matrix A that represents L with respect to the standard basis?

solution

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

(b) Find the kernel of L.

solution

The reduced echelon form of A is

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $\alpha = x_3$, the free variable, then

$$\ker L = N(A) = \begin{pmatrix} \alpha \\ -2\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

2. **[15 points]** The subspace V of $\mathbb{R}^3$ is spanned by

$$\mathbf{u}_1 = (1, -2, -2)^T, \ \mathbf{u}_2 = (1, 2, 3)^T, \ \mathbf{u}_3 = (4, 0, 2)^T.$$

(a) Find a basis for its orthogonal complement $V^\perp$.

solution

A vector $\mathbf{x} \in \mathbb{R}^3$ belongs to $V^\perp$ if and only if it is orthogonal to $\mathbf{u}_1$,$\mathbf{u}_2$ and $\mathbf{u}_3$. This condition boils down to the system of linear equations for the coordinates of $\mathbf{x}$ represented by the following augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 1 & 2 & 3 & 0 \\ 4 & 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{5}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

therefore, $\mathbf{x} = \alpha(-\frac{1}{2}, -\frac{5}{4}, 1)^T$. Hence, $\{(-\frac{1}{2}, -\frac{5}{4}, 1)^T\}$ is a basis for $V^\perp$.

(b) What are the dimensions of V and $V^\perp$? Explain.

solution

Since the basis above consists of one element, $\dim V^\perp = 1$.
Also, $\mathbb{R}^3 = V \oplus V^\perp$, therefore, $\dim \mathbb{R}^3 = \dim V + \dim V^\perp$ and $\dim V = 2$.

3. **[15 points]** Let $L : P_3 \to P_3$ be the linear operator defined by

$$L(p) = p'' + p' + p$$

Find the matrix A representing L in the standard basis $[1, x, x^2]$.

solution

$$\begin{array}{lclclclcl} L(1) & = & 1 & = & \underline{1} \cdot 1 & + & \underline{0} \cdot x & + & \underline{0} \cdot x^2 \\ L(x) & = & 1+x & = & \underline{1} \cdot 1 & + & \underline{1} \cdot x & + & \underline{0} \cdot x^2 \\ L(x^2) & = & 2+2x+x^2 & = & \underline{2} \cdot 1 & + & \underline{2} \cdot x & + & \underline{1} \cdot x^2 \end{array}$$

The columns of the matrix are the coordinates of $L(1), L(x), L(x^2)$ in the basis $[1, x, x^2]$:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. **[20 points]** Find the equation of the line $y = c_0 + c_1 x$ which is a least squares best fit to the points $(1, 2), (2, 3), (3, 5), (4, 4)$.

solution

To find c_0 and c_1 we need to solve the matrix equation

$$A^T A \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = A^T \mathbf{b},$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 4 \end{pmatrix}.$$

This leads to the system represented by the augmented matrix

$$\left(\begin{array}{cc|c} 4 & 10 & 14 \\ 10 & 30 & 39 \end{array} \right) \to \left(\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & \frac{4}{5} \end{array} \right)$$

Answer: $y(x) = \frac{3}{2} + \frac{4}{5} x$.

5. **[20 points]** The functions $\cos x, \sin x, \cos 2x$ form an orthonormal basis of their span in $C[-\pi, \pi]$ equipped with the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f g \; dx.$$

(a) Compute

$$\int_{-\pi}^{\pi} (\cos x + 2 \sin x + 3 \cos 2x)(\cos x + \cos 2x) \; dx$$

(b) Compute $||\cos x + 2\sin x + 3\cos 2x||$ and $||\cos x + \cos 2x||$.

(c) What is the cosine of the angle between $f = \cos x + 2\sin x + 3\cos 2x$ and $g = \cos x + \cos 2x$

solution

The functions $f = \cos x + 2\sin x + 3\cos 2x$ and $g = \cos x + \cos 2x$ are represented by coordinate vectors $(1,2,3)^T$ and $(1,0,1)^T$ with respect to the basis $[\cos x, \sin x, \cos 2x]$.

(a) Since

$$\langle f, g\rangle = (1,2,3)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 4,$$

$$\int_{-\pi}^{\pi} (\cos x + 2\sin x + 3\cos 2x)(\cos x + \cos 2x)\ dx = \pi\langle f, g\rangle = 4\pi.$$

(b)

$$||f|| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$
$$||g|| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

(c) The cosine of the angle α between f and g is determined by

$$\cos\alpha = \frac{\langle f, g\rangle}{||f|| \cdot ||g||} = \frac{4}{\sqrt{28}} = \frac{2}{\sqrt{7}}.$$

6. **[15 points]** Let

$$A = \begin{pmatrix} 4 & 7 \\ 3 & -1 \\ 0 & 5 \end{pmatrix}$$

Use the Gram-Schmidt method to give an orthonormal basis for the column space $R(A)$.

solution

Step 1.

$$\mathbf{u}_1 = \frac{1}{\sqrt{4^2+3^2}}\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{3}{5} \\ 0 \end{pmatrix}.$$

Step 2.

$$\mathbf{p}_1 = \langle \begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix}, \mathbf{u}_1\rangle \mathbf{u}_1 = (\frac{28}{25} - \frac{3}{25})\mathbf{u}_1 = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 7 \\ -1 \\ 5 \end{pmatrix} - \mathbf{p}_1 = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix},$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{3^2+(-4)^2+5^2}}\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} = \frac{1}{\sqrt{50}}\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} = \frac{1}{5\sqrt{2}}\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}.$$

Answer: $[(\frac{4}{5}, \frac{3}{5}, 0)^T,\ \frac{1}{5\sqrt{2}}(3, -4, 5)^T]$