

The Laurent Expansion

Theorem. Let $f(z)$ be analytic in the region $\{z \mid 0 < |z| < R\}$. Then for $0 < |z| < R$,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

$$a_n = \frac{1}{2\pi i} \oint_{C_r} f(\zeta) \zeta^{-n-1} d\zeta.$$

Here r is any number such that $0 < r < R$.

The series

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n$$

is analytic in $\{z \mid |z| < R\}$,

The series

$$f_2(z) = \sum_{n=-\infty}^{-1} a_n z^n$$

is analytic in $\{z \mid 0 < |z| < R\}$.

Note that

- If $f(z)$ be analytic in the region $\{z \mid |z| < R\}$, then

$$a_n = \frac{1}{2\pi i} \oint_{C_r} f(\zeta) \zeta^{-n-1} d\zeta = 0, n = -1, -2, \dots$$

- If $f(z)$ be analytic in the region $\{z \mid 0 < |z| < R\}$, then

$$a_{-1} = \frac{1}{2\pi i} \oint_{C_r} f(\zeta) d\zeta$$

is called the *residue* of $f(z)$ at $z = 0$.

Isolated Singularities

For the moment, we shall consider a function $f(z)$ analytic in the *punctured disk*

$$\dot{D}_R = \{z \mid 0 < |z| \leq R\}.$$

Thus the possible singularity of $f(z)$ at $z = 0$ is *isolated*.

Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

$$a_n = \frac{1}{2\pi i} \oint_{C_r} f(\zeta) \zeta^{-n-1} d\zeta.$$

- The coefficient of z^{-1} is called the *residue* of $f(z)$ at $z = 0$, and is written

$$\text{Res}(f, z = 0) = \text{Res} f(z)|_{z=0} = \frac{1}{2\pi i} \oint_{C_r} f(\zeta) d\zeta.$$

- If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $f(z)$ may be extended by defining $f(0) = a_0$, and the resulting function is analytic in $|z| \leq R$. In this case the singularity is *removable*.

- If $f(z) = \sum_{n=N}^{\infty} a_n z^n$, $N \geq 0$, $a_N \neq 0$, $f(z)$ is said to have a *zero of order N* at $z = 0$. Near $z = 0$,

$$f(z) = z^N \cdot g(z)$$

, where $g(z)$ is analytic in $|z| \leq R$, $g(0) \neq 0$.

- If $f(z) = \sum_{n=-M}^{\infty} a_n z^n$, $M \geq 0$, $a_{-M} \neq 0$, $f(z)$ is said to have a *pole of order M* at $z = 0$. Near $z = 0$,

$$f(z) = z^{-M} \cdot g(z)$$

, where $g(z)$ is analytic in $|z| \leq R$, $g(0) \neq 0$. The function is also *meromorphic* in \dot{D}_R .

- If $f(z) = O(|z|^{-M})$, $M \geq 0$, the preceding exercises show that $a_{-M-1} = 0$, $a_{-M-2} = 0$, ... Thus $f(z)$ at $z = 0$ has a pole of order *at most* M .
- At $z = 0$, $f(z)$ has a pole of order M iff there are positive constants c_1 and c_2 such that

$$\frac{c_1}{|z|^M} \leq |f(z)| \leq \frac{c_2}{|z|^M}.$$

Isolated Essential Singularities

Definition. If $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, $a_n \neq 0$ for infinitely many negative n , then $f(z)$ is said to have an *essential singularity* at $z = 0$.

Analytic functions which have isolated essential singularities behave very badly near the essential singularity.

Theorem (Little Picard). Suppose that $f(z)$ has an essential singularity at $z = 0$. Then for any complex number w_0 , in any neighborhood of $z = 0$, $f(z)$ gets arbitrarily close to w_0 .

Proof of the Little Picard Theorem: The proof is by contradiction. If there is a neighborhood $\dot{D}_r = \{z | 0 < |z| < r\}$ in which $f(z) - w_0$ is bounded away from 0, then

$$g(z) = \frac{1}{f(z) - w_0}$$

is analytic and bounded in \dot{D}_r . Thus $g(z)$ has a removable singularity at $z = 0$ and a zero of order N , $N \geq 0$. Thus $g(z) = z^N \cdot h(z)$, $h(z)$ analytic near $z = 0$ and $h(0) \neq 0$. Possibly shrinking r , we may assume that $h(z) \neq 0$ in $D_r = \{z | |z| < r\}$. Then

$$f(z) - w_0 = z^{-N} \cdot \frac{1}{h(z)}.$$

It follows that $f(z)$ has a pole of order at most N at $z = 0$.