Analyticity of Power Series

Differentiability of Power Series

Consider a power series with radius of convergence $R > 0$).

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then $f(z)$ is differentiable at $z = 0$ and $f'(0) = a_1$:

$$\begin{aligned} f(z) - f(0) &= \sum_{n=1}^{\infty} a_n z^n \\ &= a_1 z + \sum_{n=2}^{\infty} a_n z^n \\ &= a_1 z + o\left(z\right). \end{aligned}$$

If $|z_0| < R$, and z is near[1] z_0, the power series for $f(z)$ converges absolutely so that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n z^n \\ &= \sum_{n=0}^{\infty} a_n \left((z - z_0) + z_0\right)^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} \left(z - z_0\right)^k z_0^{n-k} \\ &= \sum_{k=0}^{\infty} \left(z - z_0\right)^k \sum_{n=k}^{\infty} \binom{n}{k} a_n z_0^{n-k} \\ &= \sum_{k=0}^{\infty} d_k \left(z - z_0\right)^k, \end{aligned}$$

where

$$\begin{aligned} d_k &= \sum_{n=k}^{\infty} \binom{n}{k} a_n z_0^{n-k} \\ &= \sum_{n=0}^{\infty} \binom{n+k}{k} a_{n+k} z_0^n \end{aligned}$$

[1] More precisely $|z_0| + |z - z_0| < R$.

The argument for $z = 0$ shows that $f(z)$ is differentiable at $z = z_0$, with $f'(z_0) = d_1$, so that

$$\begin{aligned} f'(z_0) &= \sum_{n=0}^{\infty} \binom{n}{1} a_{n+1} z_0^n \\ &= \sum_{n=1}^{\infty} n a_{n+1} z_0^n \end{aligned} .$$

We have shown

- If $f(z)$ is represented by a convergent power series for $|z| < R$, then $f(z)$ is an analytic function in the region $|z| < R$ and its derivative is represented by the convergent series $\sum_{n=1}^{\infty} n a_n z^{n-1}$, $|z| < R$.

All Analytic Functions Can Be Represented by Power Series

With a great deal more work, we will show that every analytic function can be represented locally as a convergent power series:

- If $f(z)$ is an analytic function in the region $|z| < R$, then $f(z)$ is represented by a convergent power series for $|z| < R$. Moreover, the derivatives of all orders exist and can be represented by the formally differentiated series for $|z| < R$.

An Exercise Using Absolute Convergence of Power Series

The power series for the exponential function is

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, |z| < \infty.$$

Use absolute convergence of the series and changing the order of summation to show that

$$\begin{aligned} \exp(z_1 + z_2) &= \sum_{n=0}^{\infty} \frac{1}{n!} (z_1 + z_2)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \cdots \\ &\vdots \\ &= \exp(z_1) \cdot \exp(z_2). \end{aligned}$$