To Louis Kauffman
who knows better...
Francisco

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A CALCULUS FOR SELF-REFERENCE

FRANCISCO J. VARELA G.

Department of Anatomy, University of Colorado Medical Center, Denver, Colorado, U.S.A.

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An extension of the calculus of indications (of G. Spencer Brown) is presented to encompass all occurrences of self-referential situations. This is done through the introduction of a third state in the form of indication, a state seen to arise autonomously by self-indication. The new extended calculus is fully developed, and some of its consequences for systems, logic and epistemology are discussed.

INDEX TERMS Self-reference, self-referential systems, calculus of indications, paradoxes, autonomy.

"Was wir liefern, sind eigentlich Bemerkungen zur Naturgeschichte des Menshen: aber keine kuriose Beiträge, sondern Fetstellungen von Fakten, an denen niemand gezweifelt hat, und die dem Bemerktwerden nur entgehen, weil sie sich ständig vor unsern Augen herumtreiben."

(What we are supplying are really remarks on the natural history of man: not curiosities, however, but rather observations on facts which no one has doubted and which have only gone unremarked because they are always before our eyes.)

Wittgenstein.

ONE: THE DOMAIN

1. *Presence.*

Self-reference is awkward: one may find the axioms in the explanation, the brain writing its own theory, a cell computing its own computer, the observer in the observed, the snake eating its own tail in a ceaseless generative process.

Stubbornly, these occurrences appear as outstanding in our experience. Particularly obvious is the case of living systems, where the self-producing nature of their entire dynamic is easy to observe, and it is this very fact that can be taken as a characterization for the organization of living systems.[1,2] Similarly, the physiological and cognitive organization of a self-conscious system may be understood as arising from a circular and recursive neuronal network, containing its own description as a source of further descriptions.[3,4,5] We have mentioned some of the few reports which address themselves directly to the self-referential nature of such systems, normally avoided as undesirable difficulty (or circulus vitiotus).

The difficulties in dealing with self-referential systems are rooted in language. Antinomies are to be expected when the self-referential capacity of language is used upon itself, as known for long in the capsular form of the epiminidean paradox, later to become, in mathematical language, Cantor's diagonal argument.[6] This singularity of self-referential antinomies, where a proposition is equivalent to its own negation, has been used for the establishment of internal limitations on certain formalisms.[7,8]

2. *Mechanism.*

Whether in dealing with the organization of systems or with the structure of languages, hardships with self-referential situations have the same root: the distinction between actor or operand, and that which is acted or operated upon, collapses. There seems to be an irreducible duality between the act of expression and the content to which this act addresses itself; self-referential occurrences blend these two immiscible components of our cognitive behavior and engender a dual nature which, apparently, succeeds in escaping this universal behavior and thus seems peculiar in our knowledge. Their peculiarity lies in being self-indicative in a given domain, in standing out of a background by their own means, in being *autonomous* as the strict meaning of the word enounces.

3. *Antecedents.*

It is ordinarily assumed that self-reference leads inevitably to contradictions even in ordinary discourse, let alone in formal languages, and hence, as said, is carefully avoided. Yet, true as this may be, language *is* self-referential, and, if we are not prepared to avail ourselves fully of self-referential notions, it is not possible to deal either with this aspect of discourse or with the many systems where self-reference is a central feature of their organization.

Moreover, consistent analysis of several classes of systems *has* been done with such concepts.[1,2,3,4,5,9] And in the mathematical domain it is too often forgotten that even classic works, such as Gödel's,[8] are based on the non-contradictory use of self-reference. More recently, Günther[10] and Löfgren[11] have addressed the problem more directly from the logical and set-theoretic point of view, and firmly prove that, although not without consequence, self-reference need not lead into contradictions.

I believe, however, that these antecedents are not enough to do justice to the central and recurrent nature of self-reference. We have no unified way of dealing with it in several domains, from biological to mathematical.

4. *The Calculus of Indications.*

I also believe that new possibilities opened, in this and other domains, after the formulation of the calculus of indications by G. Spencer Brown.[12] By succeeding in going deeper than truth, to indication and the laws of its form, he has provided an account of the common ground in which both logic and the structure of any universe are cradled, thus providing a foundation for a genuine theory of general systems.

By pointing out the need to rehabilitate what he calls Boolean equations of higher degree, he has also indicated a way of constructing a unified formalism for self-reference. In higher degrees, an expression is allowed to re-enter its own indicative space, and thus allowed to be self-indicative or self-informed. Self-reference, in this calculus, can be identified with the notion of *re-entry*, and in this way its basic form is recovered at this deep level, from which all its manifestations can be contemplated, whether in logic and formal language, or in the organization of certain systems.

By this revealing shift in perspective, the whole problem of self-reference is seen in a much more tractable light, as I hope to show here. It follows that hereinafter extensive use of the calculus of indications will be made and I shall assume the reader to be familiar with it. For those readers not yet acquainted with this calculus, Spencer Brown's book is irreplaceable; however, the Appendix contains an index of results.

5. *The Calculus and Higher Degree.*

It would be of interest in itself to explore further the description of self-referential notions with the tools of higher degree equations. Yet, as Spencer Brown says, he has only indicated a direction for work, and not provided a firmly constructed theory of re-entering expressions[12] (page xv). In fact, with a closer look, the departure from the calculus of indications proper, into re-entering forms, is not without its difficulties which render the treatment of higher degree equations, as it now stands, in need of revision. Spencer Brown claims that "it is evident that (the two algebraic initials) hold for all equations whatever their degree"[12] (page 57). However, let us consider the simple second degree equation

$$f = \overline{f}| \,, \qquad (1)$$

in relation to the first algebraic initial, J1,

$$\overline{\overline{p}|\quad p|}| = \quad , \qquad \text{J1}$$

which we may choose to write as

$$\overline{p}|\ p = \overline{\ \ }|$$

and hence, if Spencer Brown is right, we have

$$\begin{aligned} \overline{\ \ }| &= \overline{f}|\, f && \text{J1} \\ &= f f && (1) \\ &= f && \text{CI C3}\dagger \end{aligned}$$

But this is clearly untrue, since replacing in (1) we obtain

$$\overline{\overline{\ \ }|}| = \overline{\ \ }|$$

contrary to J1 itself.

Thus by allowing re-entry we lose connection not only with the arithmetic, as Spencer Brown saw,

† Any labeling preceded by CI indicates a corresponding result of the calculus of indications, which can be located in the Appendix.

but *also* with the algebra which cannot be used freely without running into contradictions. Spencer Brown himself hinted somewhat at this, by noting that certain transformations must be avoided if the memory properties of second degree equations are to be preserved[12] (page 62). The trouble runs deeper than these partial restrictions. The imaginary value, required to interpret re-entering expressions, is not rooted enough in the calculus, and, as we have seen above, is unable to come to grips with the pending antinomies in a formal sense.

6. *Intention.*

Since, for our purpose, we would like to keep both the calculus of indications and the presence in it of higher degree equations, rather than drop either of them, it seems worth an attempt to reconcile them both in a context different from the one originally envisioned by Spencer Brown. The new or third value introduced by re-entry as an imaginary state in the form, may be taken as a value in an extended arithmetic to arrive at a calculus capable of containing re-entering expressions.

This defines the principal intention of this work: to construct an extended calculus of indications fully compatible with equations of higher degree, and thus capable of handling self-referential forms at a deep enough level. The starting point is viewing the basic form of self-reference as belonging intrinsically to the arithmetical domain, that is, to take its self-indicatory or autonomous value as a state in the form. To be sure, there are many implications to confront by taking autonomy at face value: implications of departing from the primary arithmetic and enlarging its domain. These I propose to explore in the following calculus.

TWO: THE CALCULUS

1. *Context.*

Let the calculus of indications, and the context from which it is seen to arise, be valid, except for the modifications introduced hereinafter.

2. *Definition* (Third State).

Let there be a third state, distinguishable in the form, distinct from the marked and unmarked states. Let this state arise autonomously, that is, by self-indication. Call this third state appearing in a distinction, the *autonomous* state.

3. *Notation.*

Let the autonomous state be marked with the mark □, and let this mark be taken for the operation of an autonomous state, and be itself called self-cross to indicate its operation.

4. *Definition* (Arrangement).

Call the form of a number of tokens ⌉, , □, considered with respect to one another an arrangement. Call any arrangement intended as an indicator an expression. Call a state indicated by an expression the value of the expression.

5. *Notation.*

Let v stand for any one of the marks of the states distinguished or self-distinguished: ⌉, , □. Call v a marker.

6. *Definition* (Simple expressions).

Note that the arrangements ⌉, , □ are, by definition, expressions. Call a marker a simple expression. Let there be no other simple expressions.

ARITHMETIC

7. *Initials.*

Let the following initials be valid, and be used to determine a calculus out of them. Call this calculus the *Extended Calculus of Indications.*

Initial 1: Dominance

⌉ v = ⌉ I1

Initial 2: Order

⌉⌉ = I2

Initial 3: Constancy

□⌉ = □ I3

Initial 4: Number

□ □ = □ I4

8. *Theorem 1* (CI T1).

The value indicated by an expression consisting of a finite number of crosses and self-crosses can be taken to be the value of a simple expression, that is, any expression can be simplified to a simple expression.

Proof:

Let α be any expression, and let s be its indicative space. Being finite, α must have a reachable space which is the deepest in it. Call it s_d. s_d is either (1.1) contained in a cross, or (1.2) not contained in a cross.

1.2) If s_d is not contained in a cross, then s_d either contains a finite number of self-crosses or it does not. In either case it is already simple, since the self-crosses can be condensed by I4.

1.1) If s_d is in a cross c_d, then c_d is either empty or contains a finite number of self-crosses, otherwise s_d would not be deepest.

Now, c_d either stands alone in s (2.1), or (2.2) does not stand alone in s.

2.1) If c_d stands alone in s, then α is already simple, since it is either a cross or a self-cross, according to I3, I4.

2.2) If c_d does not stand alone in s, the c_d must stand either (2.2.1) in a space together with a marker (otherwise s_d would not be deepest) or (2.2.2) alone in the space under another cross.

In either case the initials apply and two markers are eliminated from α, and the expression reduced in one depth.

There will be a time when α has been simplified to a marker.

9. *Theorem 2* (CI T2).

If any space pervades an empty cross, the value indicated by the space is the marked state.

Proof:

Let α be any expression consisting of a part p and a cross.

We must show

$$\urcorner\, p = \urcorner$$

By the first theorem, p simplifies to a simple expression v.

Thus, in any case, after simplification we can write

$$p\urcorner = v\urcorner$$

which, according to I1,

$$v\urcorner = \urcorner$$

This completes the proof.

10. *Rule of Dominance* (CI Canon 6).

Let m stand for any number, larger than zero, of expressions indicating the marked state. Let a stand, similarly, for any number of expressions indicating the autonomous state. Let n stand for any number of expressions indicating the unmarked state.

We have

$$m\,m = m = \urcorner \qquad \text{I1}$$

$$a\,a = a = \square \qquad \text{I4}$$

$$n\,n = n =$$

also by I1

$$m\,n = m$$

$$m\,a = m$$

and

$$n\,a = a$$

Call m a dominant value, a a mixed value, n a recessive value. Then we obtain the following rule:

If an expression α in a space s shows a dominant value in s, then the value of α is the marked state. Otherwise the value of α either shows a mixed value in s, and then the value of α is the autonomous state, or it does not, and then the value of α is the unmarked state.

Also

$$\overline{m}\rceil = n$$

$$\overline{n}\rceil = m$$

but

$$\overline{a}\rceil = a.$$

11. *Theorem 3* (CI T3)

The simplification of an expression is unique.

Proof:

Let α be any expression in a space s. Find the deepest space s_d. By hypothesis the crosses covering s_d are either empty or contain a self-cross (perhaps after condensation via I4), and they are the contents of s_{d-1} together, perhaps, with self-crosses. Mark m outside of each empty cross in s_{d-1}, mark an a outside a cross covering a self-cross, and an a next to every self-cross in s_{d-1}. We know that

$$\urcorner \to \urcorner\, m = \urcorner\,\urcorner = \urcorner$$

$$\square \to \square\, a = \square\,\square = \square$$

$$\overline{\square|} \to \overline{\square|}\, a = \overline{\square|}\,\square = \square\,\square = \square = \overline{\square|}$$

Thus no value in s_{d-1} is changed.

Consider next the markers in s_{d-2}. Mark every self-cross with an a. Any cross in s_{d-2} either is empty or covers some marker, already marked with m or a. If it is empty, mark it with m. If it covers a mark m, mark it with n; if it covers no m but an a, mark it with a. We know

$$n =$$

$$\overline{a|} \to \overline{a|}\; a = \overline{\square|}\,\square = \square = \overline{\square|} = \overline{a|}$$

so that no value in s_{d-2} is changed.

Continue the procedure to subsequent spaces up to $s_o = s$. By the procedure each marker is uniquely marked with m, n, or a. Therefore by the Rule a unique value of α is determined. But the procedure leaves α unchanged, and the rules of the procedure are taken from the initials. Therefore, the value of α uniquely determined by the procedure is the same as the value determined by simplification. Thus the simplification of an expression is unique.

12. *Corollary* (CI T4).

The value of an expression constructed by taking steps from a given simple expression is distinct from the value of an expression constructed from a different simple expression.

Proof:

Every step in the construction is reversible by simplification. But the simplification is unique according to the preceding theorem. Thus the corollary follows.

13. *Commentary* (Consistency).

The preceding results show that the three values of the calculus are not confused, that is, the calculus is consistent. Indeed its consistency is seen, by the form of the proofs, to follow closely that of the calculus of indications. By this consistency the following rules are seen to be evident consequences (CI T5, T6, T7).

14. *Rules of Consistency.*

Rules of Identity: In every case of an expression p, $p = p$.

Rules of Value: In every case where p, q, express the same value, $p = q$.

Rules of Consequence: Expressions equivalent to an identical expression are equivalent to one another.

15. *Theorem 3* (CI C4).

Let p, q be of any expressions. Then in any case

$$\overline{\overline{p|}\quad q|}\; p = p$$

Proof:

Let $p = \urcorner$. Then

$\overline{\overline{p|}\ q|}\; p = \overline{\overline{\urcorner|}\ \ q|}\ \urcorner$	substitution (S)
$= \urcorner$	theorem 2(T2)
$= p.$	S

Let $p =$. Then

$\overline{\overline{p|}\ q|}\; p = \overline{\urcorner\, q|}$	S
$= \overline{\urcorner|}$	T2
$=$	I2
$= p.$	S

Let $p = \square$. Then

$\overline{\overline{p|}\ q|}\; p = \overline{\overline{\square|}\ q|}\ \square$	S
$= \overline{\square\, q|}\ \square.$	I3

Take $q = \rceil$,

$$\overline{\square\, q|}\,\square = \overline{\square\, \rceil|}\,\square \qquad \text{S}$$

$$= \square \qquad \text{I1, I3}$$

$$= p; \qquad \text{S}$$

take $q =$,

$$\overline{\square\, q|}\,\square = \overline{\square|}\,\square \qquad \text{S}$$

$$= \square \qquad \text{I3, I4}$$

$$= p; \qquad \text{S}$$

take $q = \square$,

$$\overline{\square\, q|}\,\square = \overline{\square\,\square|}\,\square \qquad \text{S}$$

$$= \square \qquad \text{I4, I3, I4}$$

$$= p. \qquad \text{S}$$

There is no other case of q. There is no other case of p. Thus the theorem follows.

16. *Theorem 4.*

Let p be any expression. Then in every case

$$\overline{p\,\square|}\;p = p\,\square$$

Proof:

Let $p = \rceil$. Then

$$\overline{p\,\square|}\;p = \overline{\rceil\,\square|}\;\rceil \qquad \text{S}$$

$$= \rceil, \qquad \text{T2}$$

$$p\,\square = \rceil\,\square \qquad \text{S}$$

$$= \rceil. \qquad \text{I1}$$

Let $p =$. Then

$$\overline{p\,\square|}\;p = \overline{\square|} \qquad \text{S}$$

$$= \square, \qquad \text{I3}$$

$$p\,\square = \square. \qquad \text{S}$$

Let $p = \square$. Then

$$\overline{p\,\square|}\;p = \overline{\square\,\square|}\,\square \qquad \text{S}$$

$$= \overline{\square|}\,\square \qquad \text{I4}$$

$$= \square, \qquad \text{I3, I4}$$

$$p\,\square = \square\,\square \qquad \text{S}$$

$$= \square. \qquad \text{I4}$$

There is no other case of p. Thus the theorem follows.

17. *Theorem 5* (CI T9).

Let p, q, r, be any expressions. Then in any case

$$\overline{\overline{p\,r|}\;\overline{q\,r|}|} = \overline{\overline{p|}\;\overline{q|}|}\;r$$

Proof:

Let $r = \rceil$. Then

$$\overline{\overline{p\,r|}\;\overline{q\,r|}|} = \overline{\overline{p\,\rceil|}\;\overline{q\,\rceil|}|} \qquad \text{S}$$

$$= \overline{\overline{\rceil|}\;\overline{\rceil|}|} \qquad \text{T2}$$

$$= \rceil, \qquad \text{I2}$$

and

$$\overline{\overline{p|}\;\overline{q|}|}\;r = \overline{\overline{p|}\;\overline{q|}|}\;\rceil \qquad \text{S}$$

$$= \rceil. \qquad \text{T2}$$

Let $r =$. Then

$$\overline{\overline{p\,r|}\;\overline{q\,r|}|} = \overline{\overline{p|}\;\overline{q|}|} \qquad \text{S}$$

and

$$\overline{\overline{p|}\;\overline{q|}|}\;r = \overline{\overline{p|}\;\overline{q|}|}\,. \qquad \text{S}$$

Let $r = \square$. Then

$$\overline{\overline{p\,r|}\;\overline{q\,r|}|} = \overline{\overline{p\,\square|}\;\overline{q\,\square|}|}\,, \qquad \text{S}$$

$$\overline{\overline{p|}\;\overline{q|}|}\;r = \overline{\overline{p|}\;\overline{q|}|}\;\square, \qquad \text{S}$$

thus we must show

$$\overline{\overline{p\,\square|}\ \overline{q\,\square|}|} = \overline{\overline{p|}\ \overline{q|}|}\ \square.$$

Take $p = q = \urcorner$: in this case

$$\overline{\overline{p\,\square|}\ \overline{q\,\square|}|} = \overline{\overline{\urcorner\,\square|}\ \overline{\urcorner\,\square|}|} \qquad \text{S}$$

$$= \overline{\overline{\urcorner|}\ \overline{\urcorner|}|} \qquad \text{T2}$$

$$= \urcorner, \qquad \text{I2}$$

$$\overline{\overline{p|}\ \overline{q|}|}\ \square = \overline{\overline{\urcorner|}\ \overline{\urcorner|}|}\ \square \qquad \text{S}$$

$$= \urcorner\,\square \qquad \text{I2}$$

$$= \urcorner, \qquad \text{T2}$$

Take $p = q =$: in this case

$$\overline{\overline{p\,\square|}\ \overline{q\,\square|}|} = \overline{\overline{\square|}\ \overline{\square|}|} \qquad \text{S}$$

$$= \overline{\square\ \square|} \qquad \text{I3}$$

$$= \overline{\square|} \qquad \text{I4}$$

$$= \square \qquad \text{I3}$$

$$\overline{\overline{p|}\ \overline{q|}|}\ \square = \overline{\urcorner\ \urcorner|}\ \square \qquad \text{S}$$

$$= \overline{\urcorner|}\ \square \qquad \text{I1}$$

$$= \square. \qquad \text{I2}$$

Take $p = \urcorner, q =$; in this case

$$\overline{\overline{p\,\square|}\ \overline{q\,\square|}|} = \overline{\overline{\urcorner\,\square|}\ \overline{\square|}|} \qquad \text{S}$$

$$= \overline{\overline{\urcorner|}\ \square|} \qquad \text{I1, I3}$$

$$= \square, \qquad \text{I2, I3}$$

$$\overline{\overline{p|}\ \overline{q|}|}\ \square = \overline{\overline{\urcorner|}\ \urcorner|}\ \square \qquad \text{S}$$

$$= \square. \qquad \text{I2}$$

Take $p = \urcorner, q = \square$; in this case

$$\overline{\overline{p\,\square|}\ \overline{q\,\square|}|} = \overline{\overline{\urcorner\,\square|}\ \overline{\square\,\square|}|} \qquad \text{S}$$

$$= \overline{\overline{\urcorner|}\ \overline{\square|}|} \qquad \text{I1, I4}$$

$$= \square, \qquad \text{I2, I3}$$

$$\overline{\overline{p|}\ \overline{q|}|}\ \square = \overline{\overline{\urcorner|}\ \overline{\square|}|}\ \square \qquad \text{S}$$

$$= \square. \qquad \text{I2, I3, I4}$$

Take $p =$, $q = \square$; in this case

$$\overline{\overline{p\,\square|}\ \overline{q\,\square|}|} = \overline{\overline{\square|}\ \overline{\square\ \square|}|} \qquad \text{S}$$

$$= \square, \qquad \text{I4, I3}$$

$$\overline{\overline{p|}\ \overline{q|}|}\ \square = \overline{\urcorner\ \overline{\square|}|}\ \square \qquad \text{S}$$

$$= \square. \qquad \text{I2, I3, I4}$$

There is no other distinct way of substituting p, q. There is no other case of r. Therefore the theorem follows.

ALGEBRA

18. *Initials.*

Let the results of the three preceding theorems be taken as initials to determine a new calculus. Call this calculus the *Extended Algebra*.

Initial 1: Occultation

$$\overline{\overline{p|}\ q|}\ p = p \qquad \text{A1}$$

Initial 2: Transposition

$$\overline{\overline{p\,r|}\ \overline{q\,r|}|} = \overline{\overline{p|}\ \overline{q|}|}\ r \qquad \text{A2}$$

Initial 3: Autonomy

$$\overline{p\,\square}|\;p = p\,\square \qquad \text{A3}$$

19. *Proposition 1* (CI C1).

$$p = \overline{\overline{p}|}| \qquad \text{P1}$$

Demonstration:

We first note that by A1

$$\overline{\overline{p}|}|\;p = p,$$

and

$$\overline{\rceil\;q}| = \;.$$

Now,

$$p = \overline{\overline{p}|}|\;p \qquad \text{A1}$$

$$= \overline{\overline{p}|}|\;\rceil\;\overline{\rceil}|\;p \qquad \text{A1}$$

$$= \overline{\overline{p}|}|\;\overline{\overline{p}|\;\overline{p}|}| \qquad \text{A2}$$

$$= \overline{\overline{\overline{\overline{p}|}|\;p}|\;\overline{\overline{\overline{p}|}|\;p}|}| \qquad \text{A2}$$

$$= \overline{\overline{p}|\;\overline{\overline{\overline{p}|}|\;p}|}| \qquad \text{A1}$$

$$= \overline{\overline{p}|}|\;. \qquad \text{A1}$$

20. *Proposition 2* (CI C5).

$$p\,p = p \qquad \text{P2}$$

Demonstration:

We first find by P1

$$\overline{\rceil}| = \;.$$

Now,

$$p = \overline{\overline{p}|}|\;p \qquad \text{A1}$$

$$= p\,p. \qquad \text{P1}$$

21. *Proposition 3* (CI C3).

$$p\,\rceil = \rceil \qquad \text{P3}$$

Demonstration:

$$\rceil = \overline{p\,\overline{\rceil}|}| \qquad \text{A1}$$

$$= p\,\rceil. \qquad \text{P1}$$

22. *Proposition 4* (CI C7).

$$\overline{\overline{\overline{p}|\;q}|\;r}| = \overline{p\,r}|\;\overline{\overline{q}|\;r}| \qquad \text{P4}$$

Demonstration:

$$\overline{\overline{\overline{p}|\;q}|\;r}| = \overline{\overline{\overline{p}|\;\overline{\overline{q}|}|}|\;r}| \qquad \text{P1}$$

$$= \overline{\overline{\overline{\overline{p\,r}|\;\overline{\overline{q}|\;r}|}|}|}| \qquad \text{A2}$$

$$= \overline{p\,r}|\;\overline{\overline{q}|\;r}| \qquad \text{P1}$$

23. *Proposition 5* (CI C8).

$$\overline{\overline{p}|\;\overline{q\,r}|\;\overline{s\,r}|}| = \overline{\overline{p}|\;\overline{q}|\;\overline{s}|}|\;\overline{\overline{p}|\;\overline{r}|}| \qquad \text{P5}$$

Demonstration:

$$\overline{\overline{p}|\;\overline{q\,r}|\;\overline{s\,r}|}| = \overline{\overline{p}|\;\overline{\overline{\overline{q\,r}|\;\overline{s\,r}|}|}|}| \qquad \text{P1}$$

$$= \overline{\overline{p}|\;\overline{\overline{\overline{q}|\;\overline{s}|}|\;r}|}| \qquad \text{A2}$$

$$= \overline{\overline{p}|\;\overline{q}|\;\overline{s}|}|\;\overline{\overline{p}|\;\overline{r}|}|. \qquad \text{P4}$$

24. *Proposition 6.*

$$\square = \overline{\overline{p}|\;p}|\;\square \qquad \text{P6}$$

Demonstration:

We first note that by A3

$$\overline{\square}| = \square.$$

Now,

$$\overline{\overline{p}\ p}\ \square = \overline{\overline{p}\ \overline{\overline{p}}}\ \square \qquad \text{P1}$$

$$= \overline{\overline{p\ \square}\ \overline{\overline{p}\ \square}} \qquad \text{A2}$$

$$= \overline{\overline{\overline{p\ \square}\ p}\ \overline{\overline{p\ \square}\ \square}} \qquad \text{A2}$$

$$= \overline{\overline{p\ \square}\ p}\ \square \qquad \text{P1, A1}$$

$$= \overline{p\ \square}\ \square \qquad \text{A3}$$

$$= \square. \qquad \text{A1}$$

25. *Proposition 7.*

$$\overline{\overline{p}\ p}\ \overline{p\ \square} = \overline{p\ \square} \qquad \text{P7}$$

Demonstration:

$$\overline{p\ \square}\ \overline{\overline{p}\ p} = \overline{\overline{\square}\ p}\ p \qquad \text{P4}$$

$$= \overline{p\ \square}\ p \qquad \text{A3}$$

$$= \overline{p\ \square}. \qquad \text{A3}$$

26. *Proposition 8.*

$$\overline{p\ \overline{r}}\ \overline{q\ \overline{r}}\ \square = \overline{\overline{p}\ \overline{r}\ \overline{q}\ r\ \overline{\overline{r}\ r}}\ \square \qquad \text{P8}$$

Demonstration:

$$\square\ \overline{\overline{p}\ \overline{r}\ \overline{q}\ r\ \overline{\overline{r}\ r}}$$

$$= \overline{\overline{\overline{\overline{p}\ \overline{r}}\ \overline{\overline{q}\ r}}\ \overline{\overline{r}\ r}}\ \square \qquad \text{P1}$$

$$= \overline{\overline{\overline{\overline{p}\ \overline{r}}\ \overline{\overline{q}\ r}}\ \square}\ \overline{\overline{r}\ \square\ r} \qquad \text{A2}$$

$$= \overline{\overline{\overline{p}\ \overline{r}\ \square}\ \overline{\overline{q}\ r\ \square}\ \overline{\overline{r}\ \square\ r}} \qquad \text{A2}$$

$$= \overline{\overline{\overline{\overline{p}\ \overline{r}}\ \square}\ \overline{\overline{\overline{q}\ r}\ \square}\ \overline{\overline{r}\ r\ \square}} \qquad \text{P1, A3, P2}$$

$$= \overline{\overline{\overline{p}\ \overline{r}}\ \overline{\overline{r}\ r}\ \square\ \overline{\overline{r}\ r}}$$

(continued)

$$\overline{\overline{\overline{\overline{q}\ r}\ \overline{\overline{r}\ r}\ \square\ \overline{\overline{r}\ r}}} \qquad \text{A2}$$

$$= \overline{\overline{p\ \overline{\overline{r}\ \ r\ \square}\ r\ \overline{\overline{r}\ r\ \square}}\ \overline{\square\ \overline{r}\ r\ \square}}$$

(cont.) $$\overline{\overline{r}\ \overline{\overline{r}\ r\ \square}\ q\ \overline{\overline{r}\ r\ \square}}$$

(cont.) $$\overline{\overline{\overline{r}\ r\ \square}\ \square} \qquad \text{A2}$$

$$= \overline{\square\ \overline{r}\ \overline{\overline{\overline{r}\ r\ \square}\ \overline{p}}}$$

(cont.) $$\overline{r\ \square\ \overline{\overline{r}\ r\ \square}\ q} \qquad \text{P1, A1}$$

$$= \overline{\overline{\overline{r}\ r\ \square}\ \overline{\overline{\overline{p}\ \overline{r}\ \square}}\ \overline{r\ \square\ \overline{r\ q}}}$$

(cont.) $$\overline{\overline{\overline{r\ \square\ \overline{q}}}} \qquad \text{A2, P4, P1}$$

$$= \overline{\overline{r}\ p\ \overline{r}\ \square\ \overline{r\ q}\ r\ \square}$$

(cont.) $$\overline{r\ \overline{q}\ \square} \qquad \text{A2, A1}$$

$$= \overline{\overline{r}\ p\ \overline{\overline{r}\ \square}\ \overline{\overline{rq}\ r\ \square}} \qquad \text{P1, A2}$$

$= \overline{\overline{\overline{r}|\;p}|\;\overline{r}|\;\overline{\overline{q\,r}|\;r}|}|\;\square$ A2

$= \overline{p\,\overline{r}|\;\overline{\overline{rq}|\;r}|}|\;\overline{\overline{r}|\;\overline{\overline{rq}|\;r}|}|\;\square$ P4

$= \overline{\overline{r}|\;p}|\;\overline{\overline{\overline{qr}|\;r}|\;r}|\;\square$ A1, P1

$= \overline{p\,\overline{r}|}|\;\overline{qr}|\;\overline{\overline{r}|\;r}|\;\square$ P4

$= \overline{p\,\overline{r}|}|\;\overline{qr}|\;\square$. P6

27. *Commentary* (Primary and Extended Algebra).

It is interesting to note how some of the results valid in the primary algebra, are also valid in this algebra. In fact, only the following are found to be invalid

$\overline{\overline{p}|\;p}|\;=$ CI J1

$\overline{a\,b}|\;b = \overline{a}|\;b$ CI C2

$\overline{\overline{a}|\;\overline{b}|}|\;\overline{\overline{a}|\;b}|\;= a$ CI C6

$\overline{\overline{\overline{b}|\;\overline{r}|}|\;\overline{\overline{a}|\;\overline{r}|}|\;\overline{\overline{x}|\;r}|\;\overline{\overline{y}|\;r}|}|$

$= \overline{\overline{r}|\;a\,b}|\;\overline{r\,x\,y}|$. CI C9

For example, in

$\overline{a\,b}|\;b = \overline{a}|\;b$

take $a =$, $b = \square$, then

$\overline{a\,b}|\;b = \overline{\square}|\,\square$ S

$= \square$, A3, P2

and

$\overline{a}|\;b = \urcorner\;\square$ S

$= \urcorner$. P3

These consequences of the primary algebra have a direct dependence on the validity of CI J1, which is exactly the key difference, as is reflected at the algebraic level, between the two calculi (cf. Part Three).

The propositions proved also show, in regard to the primary algebra, that CI C4 (A1) and CI J2 (A2) generate a set of consequences compatible with a three-valued Boolean arithmetic. This fact may also be taken as an indirect proof that CI C4 and CI J2 cannot form a complete set of initials for the primary algebra (as it is the case with CI C5 and CI C6). For, if they were, then in the extended algebra, CI C2, for example, would be demonstrable, and thus an expression could be simplified to more than a simple expression, contrary to T1.

28. *Theorem 6* (CI T14).

For any given expression, an equivalent expression not more than two crosses deep can be derived.

Proof:

Let α be any expression. Since α is finite, it will have a finite number of deepest spaces. According to P4 we can reduce the depth, so as to produce $\alpha_1 = \alpha$ again with a finite number of deepest spaces, of one depth less than α. By repeating this procedure we finally come to an expression $\alpha_{d-(d-2)}$ of depth 2 not reducible any further by P4. Thus the theorem follows.

29. *Theorem 7* (CI T15).

From any given expression an equivalent expression can be derived so as to contain not more than two appearances of any given variable.

Proof:

The proof is done constructively, based on the preceding theorem.

Let α be any expression, and let p be any variable in it. By T6, α will be of the form

$$\alpha = \ldots\;\overline{\overline{p\,a}|\;b}|\;\overline{\overline{p\,c}|\;d}|\;h\;\;\overline{p\,x}|\;\overline{p\,y}|\;\ldots$$

in which $a, b, c, h, x, y, \ldots$, are expressions appropriate to α. Now we have

$$\alpha = \ldots\;\overline{\overline{p}|\;a}|\;\overline{\overline{a}|\;b}|\;\overline{\overline{p}|\;d}|\;\overline{\overline{c}|\;d}|$$

(cont.) $h\;\;\overline{p\,x}|\;\overline{p\,y}|\;\ldots$ P1, A2

$$= \dots \overline{\overline{p|}\ a|}\ \overline{\overline{p|}\ d|}\ g\ \ \overline{p\ x|}\ \overline{p\ y|}\dots;$$

after calling

$$g = h\ \overline{\overline{a|}\ b|}\ \overline{\overline{c|}\ d|}\ \dots,$$

$$\alpha = \dots \overline{\overline{\overline{\overline{a|}\ d|}\ \overline{p|}|}\ \overline{\overline{x|}\ \overline{y|}\ \dots|}\ p|}\ g. \quad \text{P1, A2}$$

So that any variable p in a given expression α can be taken to appear in the form

$$\alpha = \overline{\alpha_1\ \overline{p|}|}\ \overline{\alpha_2 p|}\ \alpha_3.$$

This completes the proof.

30. *Commentary*.

If the algebra is to be of real interest with respect to the arithmetic, it must be shown to be complete, that is, we must be convinced that every valid arithmetic form must be demonstrable in the algebra. This is shown in the next theorem.

31. *Theorem 8* (CI T17).

The extended algebra is complete.

Proof:

We must show that if $\alpha = \beta$ can be proved true in the arithmetic, it is also algebraically demonstrable. The proof is done by induction on the number of variables of the equation $\alpha = \beta$.

Suppose the theorem true for $\alpha = \beta$ containing an aggregate of less than n variables. Let now $\alpha = \beta$ contain n variables. By the preceding theorem, let us write α and β in their canonical forms with respect to a variable p,

$$\alpha = \overline{\alpha_1\ \overline{p|}|}\ \overline{\alpha_2 p|}\ \alpha_3 \qquad (1)$$

$$\beta = \overline{\beta_1\ \overline{p|}|}\ \overline{\beta_2 p|}\ \beta_3 \qquad (2)$$

as these identities are demonstrable since the theorems 6 and 7 were proved without use of the arithmetic. We now have by hypothesis

$$\overline{\alpha_1\ \overline{p|}|}\ \overline{\alpha_2 p|}\ \alpha_3 = \overline{\beta_1 \overline{p|}|}\ \overline{\beta_2 p|}\ \beta_3.$$

Substituting values for p we find

$$\overline{\alpha_1|}\ \alpha_3 = \overline{\beta_1|}\ \beta_3 \qquad (3)$$

$$\overline{\alpha_2|}\ \alpha_3 = \overline{\beta_2|}\ \beta_3 \qquad (4)$$

$$\overline{\alpha_1\ \overline{\square|}|}\ \overline{\alpha_2\ \overline{\square|}|}\ \alpha_3 = \overline{\beta_1\ \overline{\square|}|}\ \overline{\beta_2\ \overline{\square|}|}\ \beta_3 \qquad (5)$$

having at most $n-1$ variables and therefore demonstrable. By (5),

$$\overline{\alpha_1\ \overline{\square|}|}\ \overline{\alpha_2\ \overline{\square|}|}\ \alpha_3\ \square$$

$$= \overline{\beta_1\ \overline{\square|}|}\ \overline{\beta_2\ \overline{\square|}|}\ \beta_3 \qquad (6)$$

is also demonstrable by substitution. Thus

$$\alpha_3 \square = \overline{\alpha_1\ \overline{\square|}|}\ \overline{\alpha_2\ \overline{\square|}|}\ \alpha_3\ \square \qquad \text{A3}$$

$$= \overline{\beta_1\ \overline{\square|}|}\ \overline{\beta_2\ \overline{\square|}|}\ \beta_3\ \square \qquad (6)$$

$$= \beta_3\ \square, \qquad \text{A3}$$

so that

$$\alpha_3\ \square = \beta_3\ \square \qquad (7)$$

is demonstrable.

Now

$$\alpha\ \square = \overline{\alpha_1\ \overline{p|}|}\ \overline{\alpha_2 p|}\ \alpha_3\ \square \qquad (1)$$

$$= \overline{\overline{\overline{p|}\ \overline{\alpha_1|}|}\ \overline{p\ \ \overline{\alpha_2|}|}\ \overline{\overline{p|}\ p|}|}\ \alpha_3\ \square \quad \text{P8}$$

$$= \overline{\overline{\overline{\overline{p|}\ \overline{\alpha_1|}|}\ \overline{\overline{\alpha_2|}\ p|}|}\ \overline{\overline{p|}\ p|}|}\ \alpha_3\ \square\ \text{P1}$$

$$= \overline{\overline{\overline{\overline{p|}\ \overline{\alpha_1|}|}\ \overline{\overline{\alpha_2|}\ p|}|}\ \square\ \alpha_3|}$$

(cont.) $\overline{\overline{\overline{p|}\ p\ \alpha_3\ \overline{\square|}|}|}$ A2

$$= \overline{\overline{\overline{p}\rceil\ \overline{\alpha_1}\rceil\ \alpha_3 \square}\rceil\ \overline{\overline{\alpha_2}\rceil\ \alpha_3 \square p}\rceil}\rceil$$

(cont.) $\overline{\overline{\square \overline{p}\rceil\ p\,\alpha_3}\rceil}\rceil$ A2

$$= \overline{\overline{\overline{p}\rceil\ \overline{\beta_1}\rceil\ \beta_3 \square}\rceil\ \overline{\overline{\beta_2}\rceil\ \beta_3\, p \square}\rceil}\rceil$$

(cont.) $\overline{\overline{\beta_3 \square \overline{p}\rceil\ p}\rceil}\rceil$ (3), (4), (7)

$= \overline{\overline{\overline{p}\rceil\ \overline{\beta_1}\rceil}\rceil\ \overline{p\ \overline{\beta_2}\rceil}\rceil\ \overline{\overline{p}\rceil\ p}\rceil}\rceil\ \beta_3 \square$ A2, P1

$= \overline{\overline{p}\rceil\ \beta_1}\rceil\ \overline{p\,\beta_2}\rceil\ \beta_3 \square$ P8

$= \beta \square$ (2)

showing that

$\alpha \square = \beta \square$ (8)

is demonstrable. Since by hypothesis $\alpha = \beta$ is true, although perhaps not demonstrable, it is also true, although perhaps not demonstrable, that $\overline{\alpha}\rceil = \overline{\beta}\rceil$, by substitution. An exactly similar reasoning to the preceding one about this new identity will show that

$\overline{\alpha}\rceil \square = \overline{\beta}\rceil \square$ (9)

is demonstrable.

Now,

$\alpha = \overline{\overline{\alpha}\rceil \square}\rceil\ \alpha$ A1

$= \overline{\overline{\beta}\rceil \square}\rceil\ \alpha$ (9)

$= \overline{\overline{\alpha\beta}\rceil\ \overline{\alpha \square}\rceil}\rceil$ A2

$= \overline{\overline{\alpha\beta}\rceil\ \overline{\beta \square}\rceil}\rceil$ (8)

$= \overline{\overline{\alpha}\rceil \square}\rceil\ \beta$ A2

$= \overline{\overline{\beta}\rceil \square}\rceil\ \beta$ (9)

$= \beta.$ A1

Thus $\alpha = \beta$ is demonstrable with n variables if it is demonstrable with less than n variables. We now prove the theorem to be true for $n = 1$, that is, for expressions with no variables. By T1–T3 it is enough to show that the initials of the arithmetic are demonstrable. In P3 let p be any marker v,

$$\overline{\ }\rceil v = \overline{\ }\rceil$$

which is I1.

In P1 let $p =$,

$$\overline{\overline{\ }\rceil}\rceil =$$

which is I2.

In A3, let $p =$,

$$\overline{\square}\rceil = \square$$

which is I3.

In P2 let $p = \square$,

$$\square \square = \square$$

which is I4.

This completes the proof.

EQUATIONS OF HIGHER DEGREE

32. *Context.*

Let any expression in the calculus be permitted to re-enter its own indicative space at an odd or an even depth.

33. *Commentary* (Indeterminancy).

Consider the expression

$$f = \overline{\overline{f}\rceil\ f}\rceil \qquad (1)$$

where f re-enters its own space at an odd and an even depth. In this case the value of f cannot be obtained by fixing the values of the variables which appear in the expression.

For example, let $f =$, then

$= \overline{\overline{f}\rceil\ f}\rceil$ S, (1)

$= \overline{\overline{\ }\rceil}\rceil$ S

$=$, P1

and let now $f = \square$,

$$\square = \overline{\overline{f|}\ f|} \qquad \text{S, (1)}$$

$$= \overline{\overline{\square|}\ \square|} \qquad \text{S}$$

$$= \square. \qquad \text{A3, P2}$$

By allowing re-entry we have introduced a degree of indeterminancy which we must try to classify.

34. *Definition* (Degree).

Let the deepest space in which re-entry occurs in an expression determine a way to classify such expression. Call an expression with no re-entry, of first degree: those expressions with deepest re-entry in the next most shallow space of second degree, and so on.†

Thus

$$f = \overline{f\,p|} \qquad (2)$$

is of second degree, while

$$f = \overline{\overline{f\,p|}\ q|} \qquad (3)$$

is of third degree.

35. *Notation.*

Where re-entry takes place as part of a larger expression it is necessary to indicate clearly the part reinserted and where re-entry takes place. We shall indicate this by direct connection[12] (page 64). Thus we can re-write the preceding expressions

$$f = \boxed{\overline{\ \ |}\ |} \qquad (1)$$

$$f = \boxed{p|} \qquad (2)$$

$$f = \boxed{\overline{p|}\ q|}\ . \qquad (3)$$

Let the notion of a marker be extended so as to include any such non-literal parts of an expression.

† Note the difference from the definition in Spencer Brown[12] (page 57), where the degree is given by the number of re-entering expressions, not by their depth. This difference will be seen to be very superficial by the next theorems.

36. *First Rule of Lexicographical Consistency.*

Any of the re-entries of a marker may be replaced by writing, in the place of re-insertion, an expression equivalent to the marker.

Thus we may write

$$f = \boxed{\overline{\ \ |}\ |} = \overline{\overline{f|}\ f|}\ . \qquad (1)$$

In the case of a larger expression, for example,

$$g = \overline{\boxed{\overline{p|}\ q|}\ \overline{r|}\ s|}$$

by the rule we can now write

$$f = \boxed{\overline{p|}\ q|}$$

and furthermore

$$f = \overline{\overline{f\,p|}\ q|}\ .$$

Thus we chose to view a marker as always being a token for re-entering variables.

37. *Second Rule of Lexicographical Consistency.*

Consider expression (2) and take $p =$. Thus

$$f = \overline{f|}\ ,$$

or, by the preceding rule and notation

$$f = \square.$$

To escape any ambiguity in writing it is therefore necessary to adopt the following rule:

> Any variable whose value is the autonomous state can be taken to be a second degree expression.

Thus by the rule, if $p = \square$, then this equation is of second degree, and by the preceding rule we have also

$$p = \overline{p|}\ .$$

Alternatively, any self-cross represents a re-entering expression because we may write

$$\square = p$$

and thence

$$p = \overline{p|}\ .$$

In this way we may look at a self-cross alternatively as a value in the arithmetic or as a basic form of a higher degree equation, and thus they provide the connection between the arithmetic and re-entering expressions. The following theorems show this fact clearly.

38. *Theorem 9.*

For a given expression of any degree an equivalent expression can be found of degree at most 3 and containing a number of additional variables equal to the number of higher degree markers other than self-crosses.

Proof:

We shall prove this theorem by induction on the degree k of any expression α. The proof is trivial, in view of the preceding rules, for an α of $k \leqq 3$. So let us assume the theorem to be true for any expression of degree $k-1$, and consider an expression α of degree k.

It will contain a finite number of parts g_k^i, which re-enter at a depth k. By the first rule we may write a variable g_k^i equal to the marker in all places of re-entry, and proceed to reduce the depth of the resulting expression according to theorem 6. Since by the hypothesis all other expressions within g_k^i have degree at most 3, we obtain an expression equivalent to g_k^i which is also of degree at most 3, since it will re-enter at depth at most 2. Moreover, by repeating this procedure in all of the g_k^i we introduce exactly i new variables. Thus the theorem follows.

39. *Commentary* (Example).

To illustrate the procedure used in the proof, consider

$$\alpha = \overline{\underline{\overline{\underline{\overline{\overline{p}|\ q}|}\ \overline{r}|\ s}|}\ t}|$$

so that we can write

$$g_2^1 = g_2 = \underline{\overline{\overline{p}|\ q}|}$$

and

$$g_3^1 = g_3 = \overline{\overline{\overline{g_3\, g_2\, p}|\ q}|\ \overline{r\, g_3}|\ s}|$$

$$= \overline{g_3\, g_2\, p\, s\ \overline{r\, g_3}|}|\,\overline{\overline{q}|\ \overline{r\, g_3}|\ s}|$$

so that g_2 is of degree 3, and g_3 also re-enters at a depth 2, so that α can be now written as third degree equation with two new variables,

$$\alpha = \overline{\overline{g_3}|\ t}|$$

$$g_3 = \overline{g_2\, g_3\, p\, s\ \overline{r\, g_3}|\ \overline{q}|\ \overline{r\, g_3}|\ s}| \; .$$

40. *Commentary* (Confusion).

An expression consisting of variables derived from markers can be seen by this theorem to confuse the richness that the markers convey to a point that is impossible to follow. By approaching the algebra with an expression of higher degree, the structure is lost, although not its sense, which we can keep by recursive records of what the variables actually indicate at successive depths. Yet this same confusion also reveals a connection between the variety of re-entering expressions and more simple forms in the calculus.

41. *Definition* (Solution).

Let α be an expression of any degree. Call a solution of α any simple expression, when it exists, to which α can be shown to be equivalent.

42. *Commentary.*

According to the definition, any first degree expression will have one and only one solution. For higher degree we have seen that more than one solution is possible (Cf. 31). But we have no assurance that any such solution exists in all cases of re-entering expressions.

43. *Theorem* 10.

Every expression has at least one solution in the extended calculus.

Proof:

By theorem 9, we only need to prove the result for expressions re-entering at a depth of at most two. Three possible such forms exist: of second degree

$$f = \overline{p f}| \qquad (1)$$

and of third degree,

$$f = \overline{\overline{p f}|\ q}| \qquad (2)$$

$$f = \overline{\overline{p f}|\ q f}| \; . \qquad (3)$$

Consider (1). Let p = , then

$$f = \overline{f}\,|$$

$$= □;$$

let p = ⌉ , then

$$f = \overline{f \; ⌉}\,|$$

$$= \quad ;$$

let p = □ , then

$$f = \overline{f \; □}\,|$$

so that it must be f = □. Thus every expression of second degree is equivalent to either a self-cross or a blank. Consider (2). Let p and q take all possible values, and let us record the value of f as entries in the following table

p \ q	⌉	□
⌉	⌉	□
	[⌉ □]	[□]
□	[⌉ □]	□

Consider (3), and similarly let us draw a table

p \ q	⌉	□
⌉	□	□
	[□]	[□]
□	□	□

Thus every expression of third degree is equivalent to at least one simple expression. This completes the proof.

44. *Commentary* (Classification of Indeterminancy).

The preceding proof provides a way of classifying the indeterminancies in re-entering expressions, as proposed in 31. In fact, by inspecting the two tables and the results on second degree equations, we see that there is a total of six basic indeterminate forms. Since every higher degree equation can be reduced to some of these basic forms presented in the proof, we conclude also that these are the only six possible indeterminancies that arise in the calculus by allowing re-entry.

THREE: THE INTERPRETATION

1. *Recapitulation.*

I have endeavored to present an extension of the calculus of indications to encompass all occurrences of self-referential situations, through the introduction of a third state in the form of indication, seen to arise autonomously by self-indication.

The principal idea behind this work can be stated thus: we choose to view the form of indication and the world arising from it as containing the two obvious dual domains of indicated and void states, and a third, not so obvious but distinct domain, of a self-referential autonomous state which other laws govern and which *cannot* be reduced by the laws of the dual domains. If we do not incorporate this third domain explicitly in our field of view, we force ourselves to find ways to avoid it (as has been traditional) and to confront it, when it appears, in paradoxic forms.

We have shown that a third value can be introduced in a Boolean arithmetic preserving consistency, and even more, providing a complete algebra to represent every arithmetic form. When departing from the calculus to re-entering expressions, these new forms are seen to fit without contradiction in the calculus, and thus it indeed serves as a basis for a rigorous foundation of higher degree equations, as was our intention (One: 6). In this sense, we have arrived at a satisfactory result to what we were looking for, and it is necessary to stop and consider several possible interpretations to which the calculus can be subjected.

2. *Autonomy as a Paradigm for Self-Reference.*

A key to the basic form of self-reference is how self-reference finds its way into language. As mentioned (One: 1), antinomies appear when language is used onto itself, that is, a proposition equivalent to its own negation. This antinomic form is paradigmatic of self-referential situations not only in language[3,9,2,4], and is in fact just the consequence of the circular interlocking of operator and operand in any self-referential situation we choose to look at (One: 2). If, for a moment, we interpret the calculus of indications for logic[12] (Appendix 2), and consider a cross to be the negation of its content, we see that indeed a self-cross is equivalent to its own negation. Alternatively, if a variable is autonomous, then it is necessarily equivalent to its negation (Two: 34). Thus a self-cross embodies the basic paradigm of self-reference. By putting autonomy as a third value in the form we therefore render self-referential instances as singular with respect to indication, singular because they are unmodified by indication (or, in logic, by negation).

3. *Autonomy as the Basis for Any Self-Referential Form.*

Although a self-cross represents the paradigm for self-reference, it is the re-entry of an expression into its own indicative space that is the way to recover all the forms of circularity, linguistic or otherwise. The results proven, however, show that, as is clear to the intuition, all the variety of re-entering expressions can be made equivalent to the basic values of the arithmetic (Two: 40). The connection of these expressions with the calculus hinges critically on the autonomous value, in itself simultaneously a state in the form and a re-entering expression. Many such re-entering expressions can be shown to be equivalent to a self-cross, that is, shown to behave essentially as the basic paradigm of self-reference; however, as seen in the proof of Theorem 10, not all re-entering expressions take an autonomous value, as some of them are equivalent to a mark or a blank. Thus although some re-entering expressions may appear to be self-referring, in fact, at a closer inspection, they are not. The calculus shows, not only that indeed all self-referential situations can be treated on an equal footing as belonging essentially to one class, but also shows a way to decide when an apparently self-referring situation is truly such.

When restricted to the calculus itself we can contemplate the behavior of self-reference; when allowed re-entry we can contemplate the unity in the diversity of self-referring situations. By moving farther from the arithmetic to free re-entry we permit diversity to appear; by confining to the calculus we simplify back to the basic forms and regain uniqueness.

4. *Self-Reference and Time.*

When Spencer Brown introduces re-entry and arrives at an expression equivalent to its content, $f = \overline{f}|$, what we call a self-cross, he notes its disconnection with his arithmetic and thus chooses to interpret it as an *imaginary* state in the form seen in *time* as alternation of the two states of the form. This interpretation is, in my opinion, one of his most outstanding contributions. He succeeds in linking time and description in a most natural fashion.

However, we have seen that this interpretation was not sufficient to hold consistently to the equations of higher degree; we took the alternative path of introducing a third value. What for the calculus of indications is contradictory with the arithmetic, here is a constitutive part of it, and we do not need any other interpretation of a self-cross other than as an embodiment of self-reference or autonomy. But we should pay attention to the fact that the double nature of self-reference, its blending of operand and operator, cannot be conceived of outside of time as a process in which two states alternate, and thus retrieving Spencer Brown's interpretation. True as it is that a cell is both the producer and the produced which embodies the producer, this duality can be pictured only when we represent for ourselves a sequence of processes of a circular nature in time. Apparently our cognition cannot hold both ends of a closing circle simultaneously; it must travel through the circle ceaselessly. Therefore we find a peculiar equivalence of self-reference and time, insofar as self-reference cannot be conceived outside time, and time comes in whenever self-reference is allowed.

It is worthwhile to note in this connection that a re-entering expression, since it can be substituted an indefinite number of times in itself, can engender an *infinite* expression, something that we have not explicitly (but at this light implicitly) allowed in the calculus. An excursion to infinity is precisely the way in which Spencer Brown introduces re-entry in his context. We should not be surprised by the connection between infinity and time since the nature of a re-entering expression is precisely that of an infinite recursion in time of a closed system. Thus in a cell we deal with productions of productions of productions ...; in self-consciousness with descriptions of descriptions of descriptions ..., and so on. By writing an infinite expression we only expand in a different fashion the basic form of self-reference.

We may interpret a self-cross, a value in the extended arithmetic, as an alternation of the other values in time. Conversely we may take the states, marked and unmarked, as timeless constituents of a self-cross occurring as an oscillation in time. Either point of view reattaches time directly to our dealing with self-referential forms. We may note in this connection that, by considering a self-cross as an oscillation in time, we may also consider other re-entering expressions as modulations of a basic frequency. This is one of the applications Spencer Brown finds for higher degree expressions[12] (page 67). To what extent a re-entering expression can be characterized by a certain frequency, remains to be investigated.

5. *The Extended Calculus and Logic.*

The extended calculus can be interpreted for logic in much the same manner as the primary calculus[12] (Appendix 2) and we need not repeat it here. In fact the key difference between the two calculi, in this interpretation, is the same as between a two and a three valued logic. The adoption of a third value leads necessarily to the abandonment of the law of excluded middle (tertium non datur), which, in the primary calculus, takes the form of C1 J1

$$\overline{\overline{p}\rceil\ p\rceil} = \quad .$$

This form is not valid in the extended calculus, and it can be shown to be the source of contradictions when re-entering expressions are allowed in the primary calculus (One: 3). We find a similar but not identical form in the extended calculus in P6

$$\overline{\overline{p}\rceil\ p\rceil}\,\square = \square\ .$$

There are, of course, several consequences of the abandonment of such a classical principle, but these are not so serious as one might expect. Ackerman and Fitch[13,14] have presented consistent contradiction-free logical systems leaving out tertium non datur, and have been able to show that the richness of such logic is ample enough to permit the construction of most of the classical mathematics. In this sense, a three-valued logic, although it forces us to abandon logical principles which appear so basic to our common discourse, can nevertheless be reconstructed so as to yield enough richness to deal in some other way with the common forms of discourse (and thus with basic mathematics). For the extended algebra, being interpretable as one of these logics, similar conclusions are valid.

To introduce more than two values in a calculus or a logical system has been a current field of investigation since Lukasiewicz.[15] Such additional values are usually interpreted in terms of probability or necessity.[14] Günther[10] has been alone in pointing out that another possible interpretation of many-valued logics as a basis for a cybernetic ontology, that is, for systems capable of self-reference, and precisely one such additional value, he claims, must be taken as time. I follow here Günther's suggestion that a third value might be taken as time. But I have shown that this third value can be seen at a level deeper than logic, in the calculus of indications, where the *form* of self-reference is taken as a third value in itself, and in fact confused with time as a necessary component for its contemplation. In the extended calculus, self-reference, time, and re-entry are seen as aspects of the same third value arising autonomously in the form of distinction.

This logical interpretation has bearings on classical meta-mathematical results of the internal limitations of formalisms,[6,7,8,13,14] as Spencer Brown saw for the primary calculus,[12] (page xv), which indicate the need for a review of the real import of these results, a review which cannot be undertaken here, and I shall restrict myself to some remarks. The present calculus, when interpreted for logic, is clearly non-Gödelian by the presence of a third value; it can deal with self-referential situations which are the basis of the Gödelian limitations. If we are prepared to avail ourselves of re-entry in the present form, we can see the classic paradoxes (such as Russell's) in a new light, as being a domain distinguishable precisely because of their antinomic behavior. Instead of finding ad-hoc means of *avoiding* their appearance (as in Russell's theory of types) we let them appear *freely* by taking their apparent anomaly as a characteristic, namely, autonomy, which we find in so many of our descriptions that it seems futile to avoid rather than confront it. Thus the epiminidean is a liar precisely because it is *not* a liar, that is, the epiminidean sentence is, in the extended calculus, autonomous not anomalous.

6. *The Extended Calculus and General Systems.*

I have already stated my view that the calculus of indications is a sound basis for a theory of general systems, insofar as it provides a grounding for every description of any universe. I also believe the present calculus to have a similar bearing on those systems which are self-referential in nature. In fact, I have undertaken the present work urged by the need of tools to deal adequately with the organization of living systems.[1,2] Lacking the actual presentation of results, I can only say here on the basis of my unpublished work, that this approach is, to say the least, very fertile.

7. *The Imaginary State and the Intercrossing of Domains.*

In this calculus antinomic forms are allowed to appear without restrictions and thus we have found a way to *construct* from an antinomic situation, which, formerly, we might have avoided rather than face. By not doing so, we have found a new, wider domain where all the preceding forms can be lodged. A similar case, at the numerical level is to be seen in the construction of the complex number,[12] starting from the antinomic form of $x^2 = -1$, not solvable in the real domain because it needs a number which is both positive and negative. This antinomy is solved by admitting this behavior within a larger arithmetic containing a new value $i = \sqrt{-1}$, and thus extending the real domain to the complex domain. In analogy, we have presented a similar construction at the Boolean level. By allowing an antinomic form (from the point of view of logic) we have constructed a new larger domain akin to the complex plane, where new forms can be lodged, including those of the preceding primary domain found to be in conflict by the introduction of re-entering expressions. Again, rather than avoid the antinomy, by confronting it, a new domain emerges.

This intercrossing of domains at the point of self-referring, hence, antinomic, situations in a given domain, repeats itself. The most impressive instance being the appearance of living systems when a set of chemical productions closes onto itself to become a self-productive and self-constructive unity. Later on, when in a living system cognitive structures become capable of self-description, again a significant new domain emerges, that of self-consciousness. By uniting two constituents of a domain, producer and produced, description and describer, into a third state which blends the two preceding ones through circular closing, we see the appearance of a much more inclusive domain. It appears as if different, successively larger levels are connected and intercross at the point where the constituents of the new lower level refer to themselves, where antinomic forms appear, and time sets in. We recognize this fact in ordinary speech.[16] When trying to convey a description of a new domain we often construct an apparent antinomy to induce the listener's cognition in a way such as to compel his imagination towards the construction of a larger domain where the apparent opposites can exist in unity. (A moral example: once you lose everything, you have everything; a philosophical one: a being is when it ceases to be).

Thus self-reference is the hinge upon which levels of serial inclusiveness intercross. Rather than recording any particular such instances (as in some of the above example) the extended calculus provides a record of the general form of this situation, and can serve, therefore, as the paradigm for all of them.

8. *Conclusion.*

The starting point of this calculus, following the key line of the calculus of indications, is the act of indication. In this primordial act we separate forms which appear to us as the world itself. From this starting point, we thus assert the primacy of the role of the observer who draws distinctions wherever he pleases. Thus the distinctions made which engender our world reveal precisely that: the distinctions we make—and these distinctions pertain more to a revelation of where the observer stands than to an intrinsic constitution of the world which appears, by this very mechanism of separation between observer and observed, always elusive. In finding the world as we do, we forget all we did to find it as such, and when we are reminded of it in retracing our steps back to indication, we find little more than a mirror-to-mirror image of ourselves and the world. In contrast with what is commonly assumed, a description, when carefully inspected, reveals the properties of the observer. We, observers, distinguish ourselves precisely by distinguishing what we apparently are not, the world.

We then see that we stand in relation to the world by mutual negation, and that the union *of us two* has therefore an autonomous structure whereby the negation engenders a distinction which leads to its own negation in a ceaseless circular process which is, in fact, the symbol which tradition has chosen to represent the creation of everything since time inmemorial.

Autonomy is seen in this light to *engender* the two stages of the form when this ceaseless process is broken into its constituents. By the introduction of a third autonomous state in the form, we do nothing but restore to our field of view that which was there at the beginning, and which we can only see now reflected as segments of the world or in language itself. Conversely, by taking self-reference and time as our filum ariadnis through a succession of levels, we dwell upon the re-union of the constituents of these levels up to our own union with the world, and thus we find a way to retrieve the unity originally lost.

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Appendix

Index of Forms From the Calculus of Indications[12] (page 138).

Definition:

Distinction is perfect continence.

Axioms:

1 The value of a call made again is the value of the call.

2 The value of a crossing made again is not the value of the crossing.

Arithmetic Initials:

I1	$\rceil\ \rceil = \rceil$	Number
I2	$\overline{\rceil}| =$	Order

Algebraic Initials:

J1	$\overline{\overline{p}\,\ p} =$	Position
J2	$\overline{\overline{pr}\ \overline{qr}} = \overline{\overline{p}\ \overline{q}}\ r$	Transposition

Theorems:

T1 The form of any finite number of crosses can be taken as the form of an expression.

T2 If any space pervades an empty cross, the value indicated in the space is the marked state.

T3 The simplification of an expression is unique.

T4 The value of any expression constructed by taking steps from a given simple expression is distinct from the value of any expression constructed by taking steps from a different simple expression.

T5 Identical expressions express the same value.

T6 Expressions of the same value can be identified.

T7 Expressions equivalent to an identical expression are equivalent to one another.

T8 Invariance: $\overline{\overline{p}\rceil\; p}\rceil \;=$

T9 Variance: $\overline{\overline{pr}\rceil\;\overline{qr}\rceil}\rceil = \overline{\overline{p}\rceil\;\overline{q}\rceil}\rceil\; r$

T10 The scope of J2 can be extended to any number of divisions of the space s_{n+2}.

T11 The scope of C8 can be extended as in T10.

T12 The scope of C9 can be extended as in T10.

T13 The generative process in C2 can be extended to any space not shallower than that in which the generated variable first appears.

T14 From any given expression, an equivalent expression not more than two crosses deep can be derived.

T15 From any given expression, an equivalent expression can be derived so as to contain not more than two appearances of any given variable.

T16 If expressions are equivalent in every case of one variable, they are equivalent.

T17 The primary algebra is complete.

T18 The initials of the primary algebra are independent.

Consequences:

C1 Reflexion $\overline{\overline{a}\rceil}\rceil = a$

C2 Generation $\overline{a\,b}\rceil\; b = \overline{a}\rceil\; b$

C3 Integration $\overline{\ }\rceil\, a = \overline{\ }\rceil$

C4 Occultation $\overline{\overline{a}\rceil\; b}\rceil\; a = a$

C5 Iteration $a\,a = a$

C6 Extension $\overline{\overline{a}\rceil\;\overline{b}\rceil}\rceil\;\;\overline{\overline{a}\rceil\; b}\rceil = a$

C7 Echelon $\overline{\overline{\overline{a}\rceil\; b}\rceil\; c}\rceil = \overline{ac}\rceil\;\overline{\overline{b}\rceil\; c}\rceil$

C8 Modified transposition

$$\overline{\overline{a}\rceil\;\overline{br}\rceil\;\overline{cr}\rceil}\rceil = \overline{\overline{a}\rceil\;\overline{b}\rceil\;\overline{c}\rceil}\rceil\;\;\overline{\overline{a}\rceil\;\overline{r}\rceil}\rceil$$

C9 Crosstransposition

$$\overline{\overline{\overline{b}\rceil\;\overline{r}\rceil}\rceil\;\;\overline{\overline{a}\rceil\;\overline{r}\rceil}\rceil\;\;\overline{\overline{x}\rceil\; r}\rceil\;\;\overline{\overline{y}\rceil\; r}\rceil}\rceil$$

$$= \overline{\overline{r}\rceil\; ab}\rceil\;\overline{rxy}\rceil$$

FRANCISCO JAVIER VARELA GARCIA was born in Talcahuano, Chile, on September 7, 1946. He received his B.A. from the University of Chile, and his Ph.D. from Harvard University in 1970.

From 1970 to 1973 he was Associate Professor of Biology at the Faculty of Sciences, University of Chile, Santiago. By the end of 1973 he was Visiting Professor at the Universidad Nacional, Heredia, Costa Rica. He is currently associated with the Department of Anatomy at the University of Colorado Medical Center, in Denver.

His scientific activity is centered around the problems of bio-computers, both molecular and cognitive, and has contributed two books and several articles to these problems.