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A TUTTE POLYNOMIAL FOR SIGNED GRAPHS*

Louis H. KAUFFMAN
Department of Mathematics, Statistics and Computer Science, The University of Illinois at Chicago, Chicago, IL 60680, USA

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1. Introduction

This paper introduces a generalization of the Tutte polynomial [14] that is defined for *signed graphs*. A signed graph is a graph whose edges are each labelled with a sign (+1 or −1). The generalized polynomial will be denoted $Q[G]=Q[G](A,B,d)$. Here G is the signed graph, and the letters A, B, d denote three independent polynomial variables. The polynomial $Q[G]$ can be specialized to the Tutte polynomial, and it satisfies a spanning tree expansion analogous to the spanning tree expansion for the original Tutte polynomial.

Planar signed graphs are, by a medial construction, in one-to-one correspondence with diagrams for knots and links. By this correspondence, the polynomial $Q[G]$ specializes to the Kauffman bracket polynomial [5-8] and hence (with a normalization) to the Jones polynomial invariant [3] for knots and links. The Jones polynomial is an important invariant in knot theory. One purpose of this paper is to provide a link between knot theory and graph theory, and to explore a context embracing both subjects.

Since the relationship with knots and knot diagrams is the primary motivation for our polynomial, we will explain this connection early in the paper. The first two sections provide graph theoretic and topological background. The reader may wish to begin reading directly in Section 4 and then turn to Section 2 and Section 3 for this background. On the other hand, a direct reading of the sections in order will give an account of the genesis of the polynomial $Q[G]$.

Section 2 discusses chromatic, dichromatic and Tutte polynomials. Section 3 explains the medial graph construction and the relation to the bracket polynomial for unoriented link diagrams. Section 3 also contains a result of independent interest: a reformulation of the definitions of activities in maximal trees (if the graph is disconnected, one should properly refer to maximal *forests* to denote disjoint collections of trees; we shall speak of trees and ask the reader to read forest for tree when the graphs are disconnected) of a planar graph in terms of properties of Euler trails

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on the corresponding medial graph. This reformulation gives new insight into the structure of the original Tutte polynomial for planar graphs, and it is useful topologically, and for the generalizations. In Section 4 we introduce the polynomial $Q[G]$ and delineate its relationships with Tutte and bracket polynomials. Section 5 discusses the spanning tree expansion for $Q[G]$. Section 6 summarizes our results, and indicates other directions and connections with topology and combinatorics.

2. Chromatic, dichromatic and Tutte polynomials

Recall first the *chromatic polynomial*, $C[G]=C[G](q)$. This polynomial enumerates (for q a fixed positive integer) the number of proper colorings of a graph G using q colors. In a proper coloring, vertices that are connected by an edge receive different colors.

In this discussion, and throughout the rest of the paper, all graphs are finite—with loops and multiple edges allowed. A *loop* is an edge having a single vertex.

Given a graph G and an edge e of G, let G' be the graph obtained by deleting e from G (retain the endpoints of e) and let G'' be the graph obtained from G by contracting e, that is, collapsing e to a point. We then have the deletion/contraction formula

$$C[G]=C[G']-C[G'']$$

(see [15]). It follows easily from the definition of proper coloring: If the edge e is a loop (equal endpoints), then $C[G]=0$ and $C[G']=C[G'']$. If e has distinct endpoints a and b, then colorings of G' with the same colors at a and b are in one-to-one correspondence with colorings of G''. Hence the difference on the right-hand side of the formula enumerates proper colorings of G.

The deletion/contraction formula is augmented with the following two formulas for disjoint unions, and for the graph consisting in a single vertex:

$$C[G \sqcup H]=C[G]C[H], \qquad C[\bullet]=q.$$

Coloring numbers multiply under disjoint unions, and a single vertex has q colorings. The deletion/contraction formula together with these formulas is sufficient to calculate $C[G](q)$ recursively (since the graphs G' and G'' have fewer edges than G). It follows from this recursive form of calculation that $C[G](q)$ is a polynomial in q, and that $C[G](q)$ is well defined as a polynomial in the abstract variable q.

A successor to the chromatic polynomial is the *dichromatic polynomial*, $Z[G](q, v)$. Here an extra polynomial variable v has been added—replacing the -1 in the deletion/contraction formula for $C[G]$ by v. That is, $Z[G](q,-1)=C[G]$. The dichromatic polynomial is determined by the recursive formulas:

$$Z[G]=Z[G']+vZ[G''],$$

$$Z[G \sqcup H]=Z[G]Z[H],$$

$$Z[\bullet]=q.$$

Note that the dichromatic polynomial is not necessarily trivial on graphs with loops. For example,

$Z[$  $] = Z[\bullet] + vZ[\bullet] = (1+v)q.$

While the chromatic polynomial is defined in terms of proper colorings, it is not so obvious that the dichromatic polynomial can be defined in an analogous manner. However, for q a positive integer, the dichromatic polynomial is in fact related to all possible vertex colorings of the graph G. To see this, let $E=E(G)$ denote the set of edges of G, and $V=V(G)$ the set of vertices of G. A *coloring* of G (with q colors) is a mapping $c: V(G) \to S(q)$ where $S(q)$ is a set of q distinct colors. Let $\mathrm{Col}(G)$ denote the set of colorings of G, and let

$$D : E(G) \times \mathrm{Col}(G) \to \{0,1\}$$

be defined by the formula $D(e,c)=1$ only if the coloring c assigns the same color to both endpoints of e. Then

$$Z[G] = \sum_{c \in \mathrm{Col}(G)} \prod_{e \in E(G)} (1 + vD(e,c)).$$

It is easy to see that this formula satisfies the deletion/contraction property for the dichromatic polynomial. Hence it provides a model and proves the existence of the dichromatic polynomial.

A short-cut in calculating either the chromatic or dichromatic polynomials is obtained by noting that it is easy to give a specific formula for $Z[G]$ when the edges of the graph G are all isthmuses or loops. An edge in a connected component of a graph G is said to be an *isthmus* if deletion of this edge disconnects the component.

Tutte [14] provided an elegant reformulation of the dichromatic polynomial. Tutte's polynomial will be denoted $T[G](x,y)$ where x and y are independent commuting algebraic variables.

The following properties determine the Tutte polynomial:

(1) Let G, G', G'' be a deletion/contraction triple obtained for an edge that is neither an isthmus nor a loop. Then

$$T[G] = T[G'] + T[G''].$$

(2) Suppose that G consists entirely of isthmuses and loops with i isthmuses and l loops. Then

$$T[G] = x^i y^l.$$

The dichromatic polynomial is related to the Tutte polynomial by the formula:

$$Z[G](q,v) = q^k v^{N-k} T[G](1 + qv^{-1}, 1+v),$$

where N denotes the number of vertices of G and k is the number of connected components of G. In this sense *the dichromatic polynomial and the Tutte polynomial* *are equivalent*, and *the existence of the dichromatic polynomial implies the existence of the Tutte polynomial.*

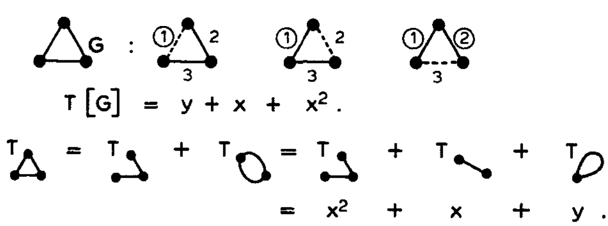
Tutte proved a remarkable theorem, showing that his polynomial could be computed from weightings assigned to the maximal trees of the graph G. While this weighting depends upon an ordering of the edges of G, the resulting polynomial is indepenent of the particular choice of ordering.

Definition 2.1. Let G be a connected graph whose edges have been labelled $1,2,3,\ldots,n$. Let $H \subset G$ be a maximal tree in G. Let i in $\{1,2,\ldots,n\}$ denote an edge of H. Let Hi denote $H-$(the ith edge). Since H is a tree, Hi has two components. One says that i is *internally active* if $i<j$ for every edge j in $G-H$ with endpoints in both components of Hi. Let i be an external edge (external to the tree H). One says that i is *externally active* if $i<j$ for all edges j on the unique path in H extending from one end of i to the other.

Theorem 2.2 (Tutte). *Let $\mathscr{T}$ denote the collection of maximal trees in a connected graph G. Let $i(H)$ denote the number of internally active edges in G (with respect to the tree H), and let $e(H)$ denote the number of externally active edges. Then the Tutte polynomial is given by the formula:*

$$T[G](x,y)=\sum_{H\in\mathscr{T}} x^{i(H)}y^{e(H)}.$$

Example.



The next section explains the passage to link diagrams via the medial graph construction.

3. The medial graph and bracket polynomial

It may seem from the definitions of external and internal activity (in Section 2) that they are somewhat different. Actually, there is a symmetry of definition for planar graphs. It is a very pleasant thing to see this symmetry, and thus obtain a more intuitive feeling for the spanning tree expansion of the Tutte polynomial. In order to see it we need to discuss the *medial graph construction* that associates a 4-valent planar graph $M(G)$ to any graph G embedded in the plane.

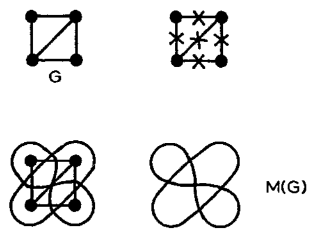


Fig. 1.

Thus, our first motivation for the medial construction is a clarification of the Tutte polynomial for planar graphs. The second motivation is that this construction provides the transition to ideas and structures related to knots and links.

3.1. The medial construction

Let G be a planar graph (meaning a graph G together with an embedding of this graph in the plane). The *medial graph*, $M(G)$, is obtained as shown in Fig. 1: Each edge of G is marked by a transversal crossing ().

These crossings are connected to each other by tracing from a given crossing, parallel to an edge of G, past a vertex, and connecting with the next available crossing. Each crossing becomes a vertex in the medial graph.

The resulting medial graph, $M(G)$, is 4-valent in the sense that it has four edges locally present at each vertex. Some of the edges of $M(G)$ may be loops. A graph of this type, embedded in the plane, will be called a *universe* (see [4]).

The medial construction has an inverse. To each connected universe U in the plane we can associate a planar graph $G(U)$ such that $M(G(U)) = U$. The inverse process is illustrated in Fig. 2. First the universe is checker-board shaded so that the unbounded region is unshaded, and two regions sharing an edge have opposite shading. Then a graph $G(U)$ is formed with vertices in one-to-one correspondence with the shaded regions of U. Two vertices are joined by an edge in $G(U)$ whenever the corresponding regions share a crossing.

Note that if U is a Jordan curve, then $G(U)$ is an isolated vertex. Therefore we

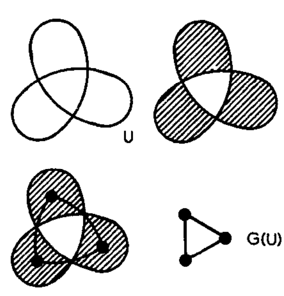


Fig. 2.

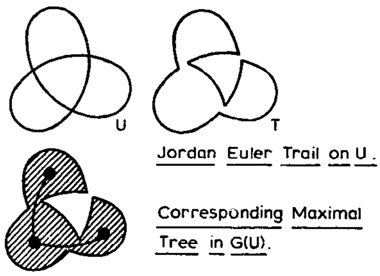


Fig. 3.

let the medial graph of an isolated vertex be an isolated Jordan curve surrounding that vertex.

The upshot of these remarks is the following.

Proposition 3.1. *The set of connected planar graphs is in one-to-one correspondence with the set of connected universes.*

Furthermore, structures in the category of planar graphs may correspond to perspicuous structures in the category of universes. In particular, call an Euler trail on a universe U a *Jordan-Euler trail* if it never crosses at a crossing, and hence, after a slight perturbation, can be drawn as a Jordan curve in the plane. See Fig. 3. Then *the collection of Jordan-Euler trails on a universe U is in one-to-one correspondence with the collection of maximal trees on $G(U)$.* For a proof of this fact see [4]. The process is illustrated in Fig. 3: In the Jordan-Euler trail, each crossing ($\asymp$) is replaced by a pair of cusps in the forms

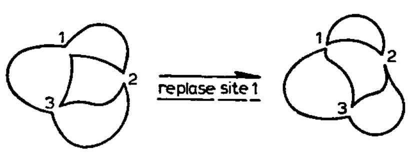
$$(>\,<) \quad \text{or} \quad (\genfrac{}{}{0pt}{}{\vee}{\wedge}).$$

Call each such pair of cusps a *site*. Regard the cusps as the sides of a doorway between regions ($> - <$), and connect all vertices in shaded regions that stand joined by open doors. This procudes the tree.

In defining activities for trees we labelled all the edges of G from an ordered set $\{1,2,\ldots,n\}$. This means that the crossings in the universe U are labelled from this same set. Thus the sites of any trail T (henceforth *trail* denotes Jordan-Euler trail) are so labelled.

Now note that if, in a trail, we replace a given site by its opposite as in

$$\text{“}\genfrac{}{}{0pt}{}{\vee}{\wedge}\text{”} \longrightarrow \text{“}>\,<\text{”}$$



then the trail breaks up into two components. These components *interact* at a subset of sites that includes the site at which the replacement was made. Thus, in the example above, the two components interact at 1 and at 2. An *interaction* is a site consisting of one cusp from each component.

Let T be a trail, and i a site of T, to be exact, the site labelled by i. Let Ti be the set of two curves resulting from replacement at i. Call i *active* if all the other interaction sites in Ti have label greater than i. With this we have a single definition of activity.

Call a site on a trail T *internal* if its two cusps point to the inside (bounded side) of the Jordan curve, and *external* if its cusps point to the outside of the Jordan curve. Thus, in the example above, sites 1 and 3 are internal, while site 2 is external.

Finally, a site is *internally active* if it is internal and active. A site is *externally active* if it is external and active.

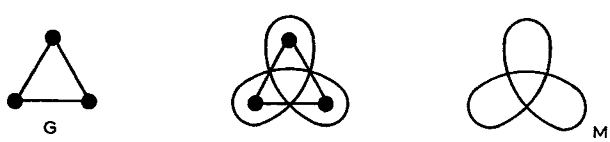
These definitions of activity for sites of a trail T on a universe U correspond exactly to the discriminations of activity for the edges of G(U) with respect to the maximal tree determined by T. I leave the verification of this statement to the reader.

In this reformulation, Tutte's theorem becomes

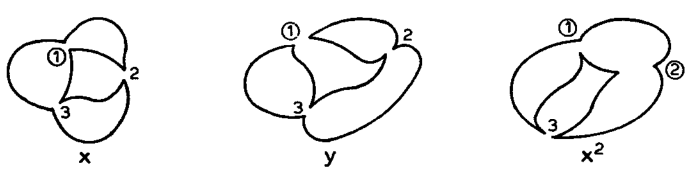
$$T[G](x,y) = \sum_{T \in \mathrm{Trails}(M(G))} x^{i(T)} y^{e(T)},$$

where the sum is taken over all Jordan–Euler trails T on $M(G)$, and $i(T)$, $e(T)$ denote respectively the number of internally active and the number of externally active sites relative to the trail T.

Example. The medial M of the triangle graph G is the trefoil universe.



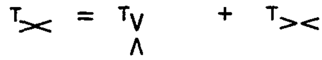
The Jordan–Euler trails on M are shown below. Active sites are circled, and the corresponding contribution to the Tutte polynomial is shown.



T  = x + y + x^2

This form of activity calculation gives the Tutte polynomial for the triangle graph as $T[G]=x^2+x+y$. This tallies with the recursive calculation.

We can also rewrite the formalism of the Tutte deletion/contraction algorithm in terms of the medial graph. Then $T[G]=T[G']+T[G'']$ becomes



where each replacement creates a connected universe. Thus

This reformulation contains the seed of the connection with knots and links.

3.2. Link diagrams

Any knot or link (an embedding of two or more circles) in three-dimensional space can, by appropriate projection, be represented by a diagram whose structure is that of a 4-valent planar graph—with extra structure at the vertices to indicate how the corresponding space curve crosses over or under at the vertex. The vertex with this extra structure is called a *crossing*, and is depicted as shown in Figs. 4–6.

A *link diagram* is a universe with extra structure. Each crossing is supplied with a break (or undercrossing) structure as shown in Fig. 4. The result of such a choice (there are 2^N link diagrams corresponding to a universe with N crossings) is a standard schematic for a knot or link in three-dimensional space. The broken line is, in this interpretation, regarded as crossing underneath the unbroken (overcrossing) line.

In order to use these diagrams to investigate topological problems about knots and links it is necessary to understand how certain diagrammatic changes (the Reidemeister moves) correspond to topological deformations of the links in three-dimensional space. These moves are shown in Fig. 5. See [7, 8] for more information about the topology.

The fundamental result about the Reidemeister moves is that *two diagrams represent ambient isotopic knots or links if and only if the diagrams can be transformed to one another by a finite sequence of Reidemeister moves.* Two knots or links are said to be ambient isotopic if there is a continuous family of embeddings in three-

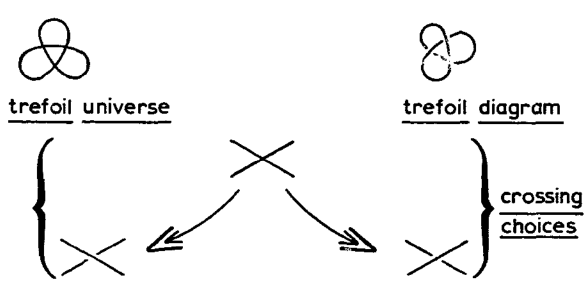


Fig. 4.

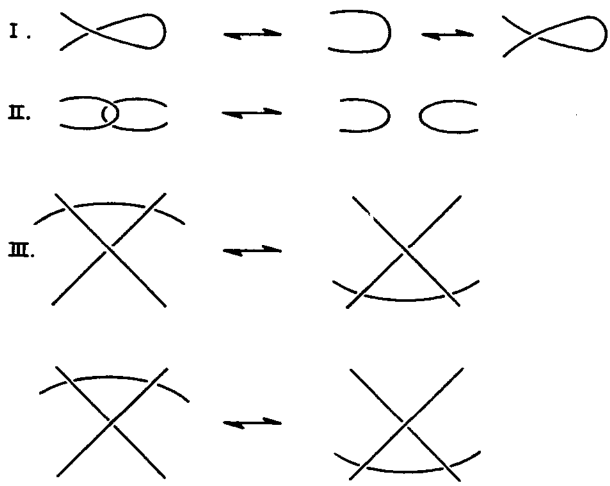


Fig. 5. Reidemeister moves.

space taking one to the other. Thus ambient isotopy captures the physical idea of deforming a knotted rope without tearing or breaking it. The theorem about the Reidemeister moves tells us that questions about ambient isotopy are equivalent to purely combinatorial questions about diagrams and moves.

The 4-valent planar graph underlying a diagram will be called its *universe*. We do not distinguish diagrams whose underlying universes are carried one to another under a homeomorphism of the plane, preserving the extra crossing structure.

The fundamental combinatorial result regarding link diagrams is:

Proposition 3.2. *The collection of connected planar link diagrams is in one-to-one correspondence with the collection of connected signed planar graphs.*

Proof. Associate crossings and signed edges as shown in Fig. 6. This association assigns a unique crossing to each signed edge, and then by the medial construction gives a link diagram $K(G)$ associated with each signed graph G. The inverse process proceeds just as in the medial construction. □

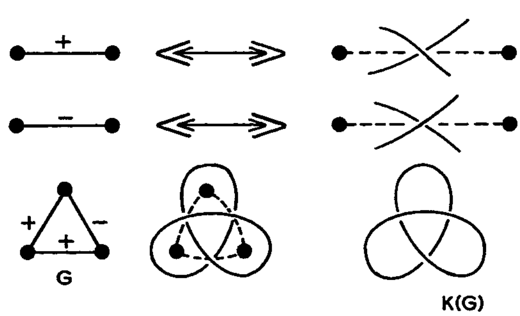


Fig. 6. Link diagram associated with signed graph.

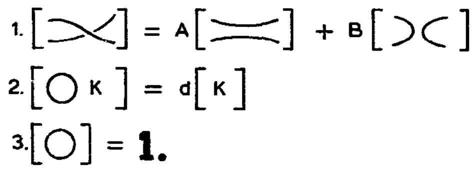
Remark. The signed graph G has all signs of the same type, if and only if the link diagram $K(G)$ is *alternating*. A link diagram is alternating if the weave alternates between over and under as one traverses the strands (a strand is traversed by choosing a starting point, and walking so as to cross under or over at each crossing that presents itself).

3.3. *The bracket polynomial*

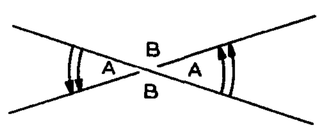
In analogy to the deletion/contraction algorithm for graphs, there is a general polynomial of three variables A, B, d that is associated with link diagrams. For a link diagram K, we denote this polynomial by

$$[K] = [K](A, B, d)$$

and refer to it as the *bracket polynomial* (see [5–8]). The bracket is defined by the formulas:



Here it is understood that the three small diagrams are parts of otherwise identical larger diagrams. Note also that the association of the variables A and B to the two modes of splicing a crossing are well defined:



A given crossing can be seen to distinguish two out of its four logal regions by *rotating its overcrossing line counterclockwise to sweep out two regions*. See the diagram above.

Finally, in the second equation, the extra circle denotes any disjoint component that is a Jordan curve in the plane. Thus if K consists solely of N disjoint Jordan curves, then $[K] = d^{N-1}$.

Remark. It follows from these axioms that the bracket satisfies the following formula for disjoint unions of diagrams:

$$[K \sqcup L] = d[K][L].$$

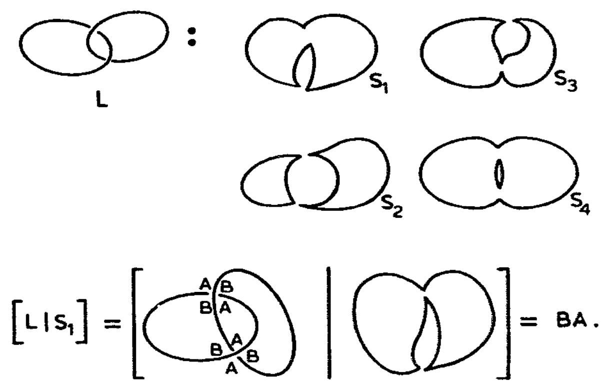
It is easy to see that the bracket is well defined. Just note that the recursive calculation is independent of the order in which the vertices are spliced. In fact, there is a direct formula for the bracket as a sum over *states* of the diagram. A *state*

S of a link diagram K consists in a choice of splitting at each crossing of K. As indicated in the axioms for the bracket, there are two possibilities for splitting a crossing—type A and type B. In this sense, each crossing in the diagram contributes an A or a B to the state S. Let $[K \mid S]$ denote the product of A's and B's contributed by the crossings of the diagram to the state S. Let the *norm* of the state S, denoted $|S|$, be the *number of Jordan curves* obtained by splitting each crossing according to the state's choice at that crossing. Then the bracket is expressed by the formula:

$$[K] = \sum_S [K \mid S] d^{|S|-1}.$$

This formula can be taken as a definition of the bracket.

Example.

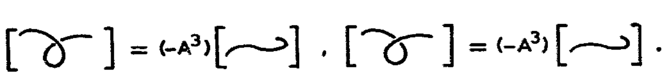


$$[L] = BA + A^2d + AB + B^2d.$$

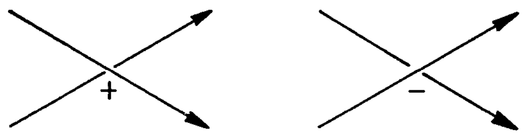
The bracket polynomial is very significant for the theory of knots and links because one can determine conditions on A, B, d for which $[K]$ is invariant under Reidemeister moves (Fig. 5). In particular, one finds that if

$$B = A^{-1}, \qquad d = -(A^2 + A^{-2}),$$

then $[K]$ is invariant under the Reidemeister moves of type II and type III. This specialized bracket is not invariant under the type I move, but it behaves as follows:

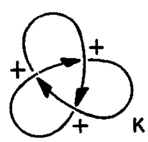


An appropriate normalization then yields an invariant of all three moves. This normalization is most easily explained for *oriented* links. An oriented link has a directionality associated to each of its components—indicated by arrow-heads placed on these components. Each crossing in an oriented link has a *sign* (plus or minus one) associated with it. In a positive crossing, the overcrossing segment will coincide in direction with the undercrossing segment after a counterclockwise rotation. In a negative crossing this coincidence uses a clockwise rotation. See the figure below:



Define the *write*, $w(K)$, of an oriented diagram K to be the *sum of the signs of the crossings of K*.

Example.



$$w(K) = +1+1+1 = 3.$$

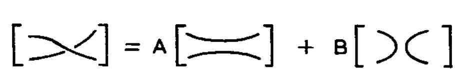
Then we can define a Laurent polynomial $f[K]$ by the formula:

$$f[K] = (-A^3)^{-w(K)}[K].$$

This polynomial $f[K]$ is invariant under all three Reidemeister moves. Hence it is an invariant of ambient isotopy for knots and links in three-dimensional space. In fact, $f[K]$ is a state summation model (after a change of variables) for the original Jones polynomial [3]. The bracket yields an elementary construction of the Jones polynomial. It also specializes to give the partition function for the Potts model in statistical mechanics (see [6]). It is worth remarking that the Jones polynomial was originally discovered by a somewhat different route—a new representation of the Artin braid group into Von Neumann algebras. These matters are also very closely tied with this combinatorics (see [5, 6]).

3.4. Calculating the bracket

In calculating the bracket it is not necessary to go all the way down to collections of disjoint Jordan curves. In fact, we can parallel our translation of the Tutte polynomial to the medial graphs and restrict the recursion



to those cases where the universes underlying these three diagrams remain connected.

It is not hard to see that a connected universe that is disconnected by one or the other of the splits at each of its crossings is the medial graph of a connected planar graph whose edges are each either loops or isthmuses. Call such a universe *irreduci-*

ble and say that a link diagram is *irreducible* if its underlying universe is a disjoint union of irreducible universes.

It is easy to see that if K had a connected, irreducible universe, then the bracket for K is given by the formula:

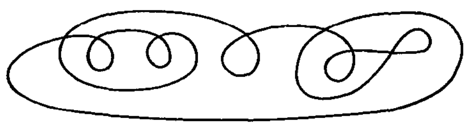
$$[K] = X^{n(K)} Y^{p(K)},$$

where

$$X = A + Bd, \qquad Y = Ad + B,$$

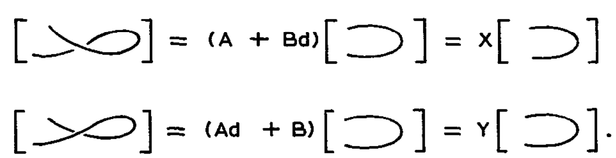
and $p(K)$ denotes the number of positive crossings in K, and $n(K)$ denotes the number of negative crossings in K. (K necessarily is a knot if it is connected and irreducible. The sign of the crossings of a knot diagram (one component in the space curve) is independent of the assignment of orientation to the diagram.)

The connected, irreducible universes then have the form



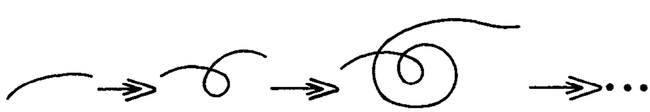
That is, they are locally composed of curls.

Note that each curl contributes via



Call a curl *positive* if it produces the factor Y, and *negative* if it produces the factor X. Note that a positive curl can correspond to a positive loop in the associated graph (or to a negative isthmus); see Fig. 7.

A connected, irreducible universe consists entirely of curls in the sense that it is built out of them. Curls may be constructed on curls as in:



With this formulation, the bracket definition closely parallels the recursive definition of the Tutte polynomial.

In fact, the bracket has the analog of a spanning tree expansion. It can be evaluated from the site-weightings associated with Jordan–Euler trails on the universe underlying K.

Theorem 3.3. *Let K be a link diagram, and T a (Jordan–Euler) trail on U, the universe underlying K. Assume that the crossings of $K(U)$ are labelled from the set*

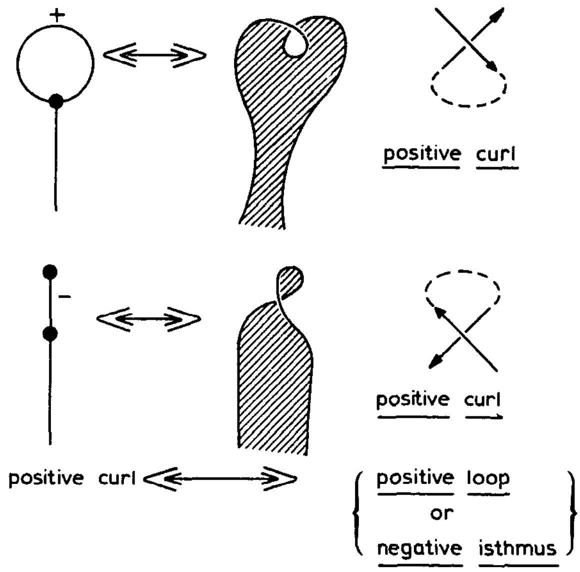
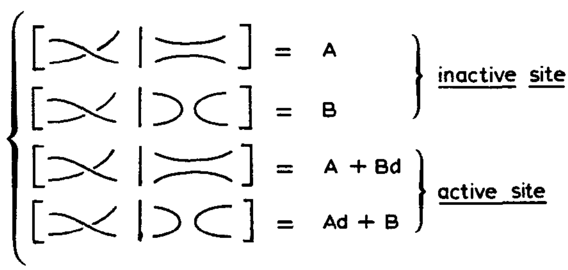


Fig. 7.

1, 2, …, n so that sites of T may be labelled active and inactive according to the definitions given in this section. Define local site contributions from T according to the following scheme:



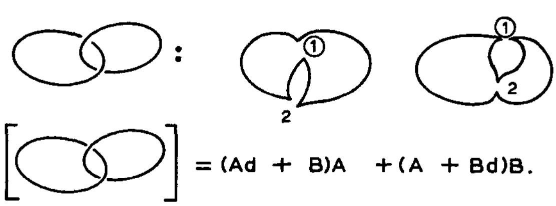
Here the crossing indicates the crossing type in K, in comparison with the corresponding site in T.

Let $[K \mid T]$ *denote the product of the contributions from each site of T.*

Then the bracket $[K]$ *is the sum of these contributions from each trail*:

$$[K] = \sum_{T \in \text{Trails}} [K \mid T].$$

Example.



The proof of Theorem 3.3 is deferred to Section 5. [*K*] has the Tutte polynomial for planar graphs (via medial translation) as a special case. See Section 4 for this specialization. It is remarkable that Tutte's expansion generalizes in this way. Thistlethwaite [13] was the first person to notice this possibility (in the case of the Jones polynomial).

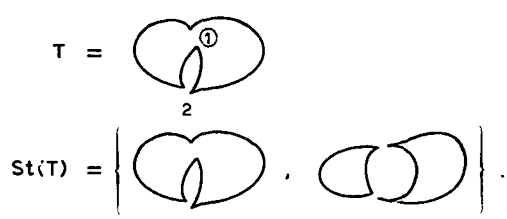
We are now in a position to translate the bracket into the category of signed planar graphs, and thereby obtain a further generalization. That is the subject of the next section.

Remark. It is interesting to note how the expansion of the bracket via trails is related to the "state" expansion that I explained just after introducing the bracket. Each trail is a state, but states (being the result of arbitrary splittings of the crossings), may be disconnected. Note that in the trail expansion of the bracket each trail has a set of active sites. *Associate to each trail the collection of states obtained by switching*

$$(> < \twoheadrightarrow \genfrac{}{}{0pt}{}{\vee}{\wedge})$$

a subset of the active sites. The trail itself is one such state. Call the collection of states associated with a trail T the "states of T", denoted St(*T*). Then it is not hard to see that the sum of the contributions of the states in St(*T*) (via the bracket state expansion) is equal to the contribution of the trail *T* in the trail expansion. These same remarks apply to our generalization of the Tutte polynomial for signed graphs, and will be the subject of another paper.

Example.



4. A polynomial for signed graphs

Let G be a signed graph. Let e be an edge of G, and let sign(e) denote the sign of this edge (+1 or −1).

The edge e may be an isthmus or a loop. Let $i_+ = i_+(G)$ denote the number of positive isthmuses, $i_- = i_-(G)$ denote the number of negative isthmuses in G. Similarly, $l_+ = l_+(G)$ and $l_- = l_-(G)$ denote the number of positive and negative loops in G.

We shall define a polynomial

$$Q[G] = Q[G](A, B, d)$$

for signed graphs via deletion/contraction and evaluation formulas.

The following abbreviations are of use:

$$X = A + Bd, \qquad Y = Ad + B.$$

The defining formulas for $Q[G]$ are:

(1) Let G, G', G'' be a deletion/contraction triple for an edge e that is neither an isthmus nor a loop in G. Then

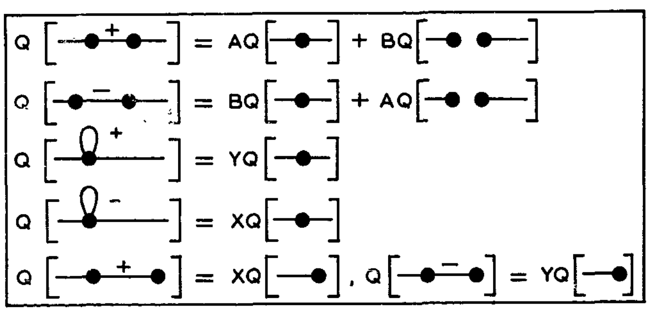
$$Q[G] = AQ[G'] + BQ[G''] \quad \text{if } \operatorname{sign}(e) < 0,$$

$$Q[G] = BQ[G'] + AQ[G''] \quad \text{if } \operatorname{sign}(e) > 0.$$

(2) If every edge of G is either an isthmus or a loop and G is connected, then

$$Q[G] = X^{i_+ + l_-} Y^{i_- + l_+}.$$

(3) If G is the disjoint union of graphs $G1$ and $G2$, then $Q[G] = dQ[G1]Q[G2]$.



In order to establish the existence of $Q[G]$ for all signed graphs, a spanning tree expansion for $Q[G]$ will be established in Section 5 of this paper. Our next two results show that $Q[G]$ includes as special cases, the bracket polynomial for link diagrams, and the dichromatic and Tutte polynomials for arbitrary graphs of constant sign.

Proposition 4.1. *Let G be a planar signed graph. Let $K(G)$ be the link diagram associated with G via the medial construction, as described in Section* 3. *Then $Q[G] = [K(G)]$ where the right-hand side denotes the bracket polynomial of Section* 3.

Proof. The proof follows immediately from the medial construction. The recursion formula for the bracket directly translates to formula (1) for $Q[G]$. A graph G consists only of isthmus and loop if and only if the corresponding link diagram $K(G)$ is irreducible (in the terminology of the previous section). A single curl will correspond to either an isthmus or a loop in the corresponding graph. In an irreducible, connected diagram the formula prior to Theorem 3.3 gives the same result as our formula (2) for $Q[G]$. This completes the proof. □

Proposition 4.2. *Let G be a signed graph all of whose edges receive positive signs. Let $Z[G](q, v)$ denote the dichromatic polynomial for the underlying unsigned graph as defined in Section 2. Let N denote the number of vertices of G. Then*

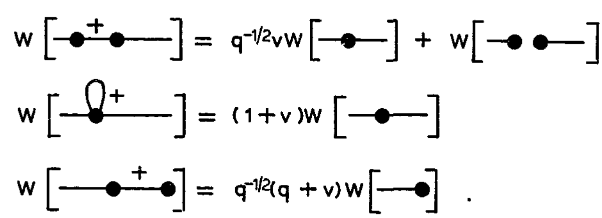
$$Z[G](q, v) = q^{(N+c)/2} Q[G](q^{-1/2}v, 1, q^{1/2}),$$

where c denotes the number of components in the graph G. Since the Tutte and dichromatic polynomials are reformulations of each other, this shows that the Tutte polynomial $T[G](x, y)$ is a special case of the polynomial $Q[G]$.

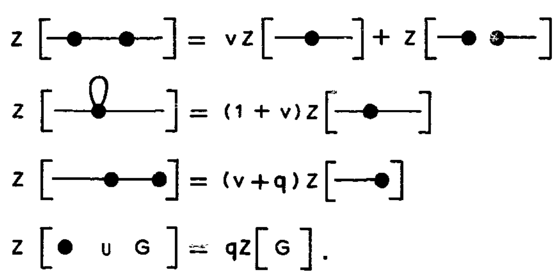
Proof. Note that with $B=1$, $A=q^{-1/2}v$ and $d=q^{1/2}$, we have

$$X = A + Bd = (q+v)q^{-1/2}, \qquad Y = Ad + B = (1+v).$$

Let $W[G]$ denote $Q[G](q^{-1/2}v, 1, q^{1/2})$. Then W satisfies the formulas



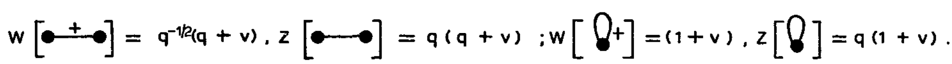
while $Z[G]$ is defined via



It is then easy to check inductively that these two definitions imply that

$$Z[G] = g^{(N+c)/2} W[G].$$

The most important verifications are for a single isthmus and a single loop. These are as follows:



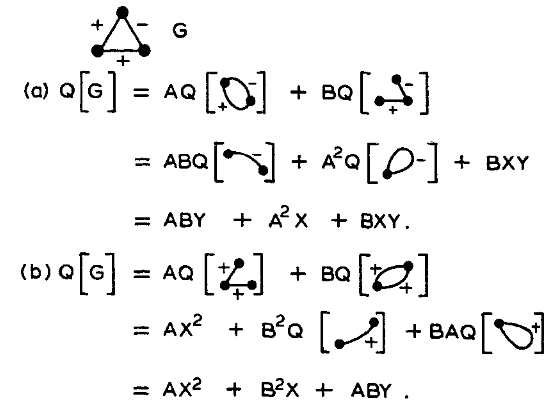
Remark. The existence of d such that

$$X = A + Bd, \qquad Y = Ad + B$$

is equivalent to the condition

$$AX - A^2 = BY - B^2.$$

It is easy to see that this restriction on the variables A, B, X, Y is necessary in order that the recursion for $Q[G]$ be well defined. For example, consider the following two computations of $Q[G]$ for a triangle graph with two positive and one negative signed edges.



These agree since

$$BY + A^2 = AX + B^2 \Rightarrow A^2X + BXY = AX^2 + B^2X.$$

In the next section we establish the existence of $Q[G]$ via a spanning tree expansion.

5. A spanning tree expansion for $Q[G]$

This spanning tree expansion is based on Tutte's notion of activities and weightings for labelled maximal trees as described in Section 2. The words spanning tree and maximal tree are used synonymously. A maximal tree is a tree in G, using every vertex of G and contained in no larger tree in G.

The Tutte weighting procedure depends upon an assignment of labels to the edges of G from an ordered set that we take to be the set $\{1, 2, \ldots, n\}$ when the graph G has n edges. *Relative to a given tree* an edge e of G is said to be *internally or externally active* (*inactive*) as stipulated in Definition 2.1.

I shall recall that definition here. Let H denote the given maximal tree in the connected graph G. Note that H falls into two components upon the deletion of any of its edges. Call these two components the *parts of H relative to the edge e in H.* An edge e is *internally active* if it belongs to the tree, and its label is smaller than the label on any edge outside the tree that connects the parts of H relative to e. An edge e is externally active if it does not belong to the tree, and it has a label smaller than the label on any other edge of the unique cycle formed by e and edges of H.

We designate an algebraic contribution to the polynomial $Q[G]$ from each of the maximal trees, and define $Q[G]$ to be the sum of these contributions from all the trees. It is necessary to show that the resulting sum is independent of the choice of labelling of the edges of G, and that it satisfies formulas (1)–(3) for the Q-polynomial as given in Section 4.

5.1. Contributions from the trees

Recall that the polynomial we wish to define has variables A, B, d and that we use auxiliary variables

$$X = A + Bd, \qquad Y = Ad + B.$$

Given a maximal tree H in G we shall define contributions from the edges of G (relative to H) via the following chart (in each row of the chart the contribution of the edge occurs after the colon):

$$\begin{cases} \text{internally active, sign} = +1\text{: } X, \\ \text{externally active, sign} = -1\text{: } X, \\ \text{internally active, sign} = -1\text{: } Y, \\ \text{externally active, sign} = +1\text{: } Y, \\ \text{internally inactive, sign} = +1\text{: } A, \\ \text{externally inactive, sign} = -1\text{: } A, \\ \text{internally inactive, sign} = -1\text{: } B, \\ \text{externally inactive, sign} = +1\text{: } B, \end{cases}$$

It is useful to abbreviate this chart as follows:

$$\begin{array}{ll} \text{ea}-: X & \text{ia}+: X \\ \text{ia}-: Y & \text{ea}+: Y \\ \text{e}-: A & \text{i}+: A \\ \text{i}-: B & \text{e}+: B \end{array} \qquad (\#)$$

The meaning of the abbreviations in the #-chart (as we shall refer to it) should be clear. For example, "i – : B" means that an internally inactive negatively signed edge contributes B. In general, activity is indicated by an a and inactivity is indicated by the absence of an a.

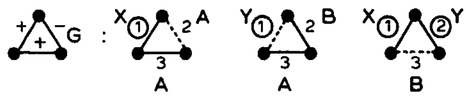
Now define the contribution of a maximal tree H, denoted $G(H)$, by

$G(H)$ = the product of the contributions of all the edges of G (relative to H).

And define the polynomial $Q[G]$ via

$$Q[G] = \sum_{\text{maximal trees } H} G(H).$$

Example. Here is a triangle graph with all positive signed edges. We show below the contributions from each maximal tree. The label on each active edge is encircled. Note that the sum is the same as the sum obtained via the recursive calculation of Section 4.



$$Q[G] = A^2X + ABY + BXY.$$

5.2. Independence of order

The key to this approach to order independence is the fact that any permutation of the set $\{1,2,\ldots,n\}$ can be accomplished by a sequence of interchanges of elements that differ by one unit. Call two such elements *adjacent* in the ordering. By letting the edge-labels for the Tutte weighting be chosen from a set $\{1,2,\ldots,n\}$ of consecutive integers, we can use this fact about permutations. Thus it suffices to show that an interchange of consecutive edge-labels leaves $Q[G]$ invariant. (Two edge-labels are said to be consecutive if they differ by one.)

For a given spanning tree H in G, let f and g be two edges labelled i and $i+1$ respectively. Let $\hat{f}$ and $\hat{g}$ denote the *same* edges but for the edge-labelling that switches i and $i+1$ so that $\hat{f}$ has label $i+1$, $\hat{g}$ has label i, and all other edges retain their labels. Tutte [14] observes the following facts:

(1) The activity of any edge h, not equal to f or to g, is unchanged by this interchange of labels.

(2) A change in the activity of f or g is possible only if the following three conditions hold:

- (a) One of these edges (say f) is in the tree H, while the other is not in the tree.
- (b) f is an edge on the cycle determined by g and the tree H. (This is equivalent to stating that g connects the parts of H relative to f.)
- (c) Each edge h (not equal to f or g) has the same activity with respect to H as it does with respect to the maximal tree, $s(H)$, obtained from H by deleting f and adding g.

Assuming conditions (a)–(c), the possible changes in activity are indicated in Fig. 8. In this figure we have indicated the activities of the edges e and f before and after the interchange with respect to the trees H and $s(H)$. The conventions are the same as in the #-chart.

It is now straightforward to check that in every case of signs for e and f the

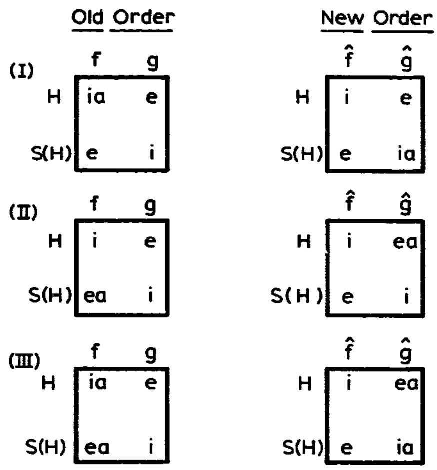


Fig. 8.

polynomial $Q[G]$ is unaffected by the interchange. For example in case I of Fig. 8 with sign$(f)=+1$, sign$(g)=-1$, we find (via the #-chart) that prior to the interchange H and $s(H)$ together contribute $(AX+B^2)K$ while after the interchange they contribute $(BY+A^2)K$ (same K). Since we know that $AX+B^2=BY+A^2$, this shows the invariance for this piece. The rest of the cases go through in similar fashion, and will be omitted. This completes the verification that $Q[G]$ is well defined as a spanning tree expansion.

5.3. *Formulas for* $Q[G]$

It is now routine to verify (using independence of order) that $Q[G]$ satisfies formulas (1) and (2) of Section 4 for connected graphs. Formula (3) defines Q for arbitrary graphs. I omit these verifications. Note in checking them that an edge which is neither a loop nor an isthmus can be made inactive by choosing for it a maximum label; an isthmus is always internally active and a loop is always externally active.

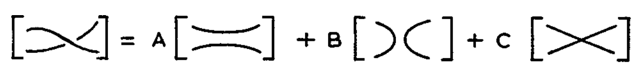
It is also easy to translate the spanning tree expansion for $Q[G]$ to a corresponding Jordan-Euler trail expansion (in the case of G signed, planar) thereby obtaining a proof of Theorem 3.3.

6. Conclusion

This paper began with a survey of the chromatic, dichromatic and Tutte polynomials. Then we showed how these ideas can be transferred into the category of link diagrams via the medial construction. The bracket polynomial $[K](A,B,d)$ arises naturally for link diagrams and has as a normalized special case the Jones polynomial, a topological invariant of knots and links. We then explained how the bracket has an expansion via Euler trails that formally resembles the spanning tree expansion of the Tutte polynomial. We then generalized the bracket to define a polynomial $Q[G]$ for signed graphs, showing that $Q[G]$ contains the Tutte polynomial *and* the bracket as special cases. Furthermore, the polynomial $Q[G]$ has a spanning tree expansion.

I have restricted this account to the bracket polynomial and its context. Even here it is remarkable that a construction that seems at the outset to depend upon planarity (planar signed graphs, link diagrams) is an essential part of a larger scheme defined for all (signed) graphs (the polynomial $Q[G]$).

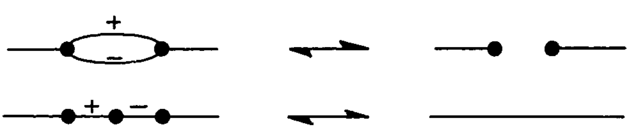
One of the simplest further generalizations that can be explored is an extended bracket satisfying a recursion of the form



Here the last replacement is a planar vertex, so that this extension must be told how to evaluate a 4-valent planar graph. Very simple evaluations in terms of selected cycles in the graph give as special cases the usual bracket and also chromatic enumerations of Penrose (see [10, 12], compare with [2]). More complex evaluations (see [11]) give the known topological generalizations of the Jones polynomial (the Homfly and Kauffman polynomials, compare [7, 8]).

Much more work remains to be done in this field.

One last remark: The type II Reidemeister move (Fig. 5) corresponds to the following two moves on signed graphs:



If we define

$$P[G](A) = Q[G](A, A^{-1}, -A^2 - A^{-2}),$$

then $P[G]$ is invariant under these graphical moves for any signed graph G (no planarity restriction). (It is also invariant under the "star/triangle" replacement corresponding to the Reidemeister type III move.)

One interpretation of this invariance is to let the graph represent a communications network for pure frequency signals, with the signs $(+1, -1)$ denoting positive and negative quarter-period phase shifts. Then the graphical moves shown above will not affect the transmission properties of the net (half-period phase shifted signals interfere destructively for pure frequencies) and the polynomial $P[G]$ will be an invariant of these network changes. Hence $P[G]$ can be used to discriminate inequivalent networks. This application marks the possibility of a deep interdisciplinary connection between topology and network theory.

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