ON THE FOUR-COLOUR CONJECTURE

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1. *Introduction*

The maps discussed in this paper are dissections of surfaces into simple polygons, called *regions*. In each map it is supposed that the regions are finite in number and that each vertex of a region is common to just three regions. Sides and vertices of the regions will be called *edges* and *vertices* of the map respectively.

A *colouring* of a map M is defined as a set of four mutually exclusive subclasses, called *colour-classes*, of the regions of M such that each region belongs to some colour-class and no two regions of the same colour-class have any edge in common. If Z is a colouring whose colour-classes are C_1, C_2, C_3 and C_4, we write

$$Z = (C_1, C_2, C_3, C_4).$$

The union $A \cup B$ of two colour-classes A and B of a colouring Z we call a *colour-dyad*. The regions of a colour-dyad may or may not form a connected set. In any case we call the disjoint internally connected components *Kempe chains* and denote their number by $c_0(A \cup B)$.

If U is a Kempe chain of $A \cup B$ and V is the remainder of $A \cup B$, then it is clear that the four sets

$$((A \cap U) \cup (B \cap V)), \quad ((B \cap U) \cup (A \cap V)), \quad C, \quad D,$$

where $Z = (A, B, C, D)$ and $A \cap U$, for example, denotes the intersection of A and U, constitute a colouring Z_1 of the map concerned and that Z_1 differs from Z if and only if

$$c_0(A \cup B) > 1.$$

We say that Z_1 is derived from Z by an *exchange operation* on the Kempe chain U. The set of all colourings of the map that can be derived from Z by a finite sequence of exchange operations we call the *colour-system* containing Z and denote it by $\Pi(Z)$. Clearly, if Y is any colouring in $\Pi(Z)$, then

$$\Pi(Y) = \Pi(Z).$$

The problem with which this paper is concerned is as follows.

PROBLEM. Let M be a map on the sphere, and let M contain a pentagon P. Let any colouring Z of that part M_P of M which is exterior to P be said to be of type I if it has a colour-class A which contains no region of M_P adjacent to P, and of type II otherwise.

The problem is to find M_P and Z such that all the colourings of $\Pi(Z)$ are of type II.

A demonstration that this problem is insoluble would complete the verification of the four-colour conjecture by enabling us to deduce from the existence of a colouring of M_P the existence of a colouring of type I of M_P and thence (by assigning P to A) the existence of a colouring of M.†

The contribution of this paper to the problem is the deduction of some new limitations on the structure of $\Pi(Z)$ which must be satisfied in any solution that may exist.

Use is made of an elementary general theorem on the colourings of spherical maps, to which I have not seen any reference in the literature. This is the "Parity Theorem" proved below.

Reference may be made to a paper by Kittel‡ on the above problem. In comparing this paper with his it should be borne in mind that he counts as distinct colourings which differ only by a redistribution of "colours" among the colour-classes—a distinction which has no meaning with the definitions used here.

2. *The parity theorem*

Let $Z = (A, B, C, D)$ be a colouring of the spherical map M of α_2 regions. Let $\alpha_2(X)$ and $\alpha_2(X \cup Y)$ denote the number of regions in the colour-class X and the colour-dyad $(X \cup Y)$ respectively, and let $\beta_1(X \cup Y)$ be the number of edges in which regions of X meet regions of Y. Then, if $c_1(X \cup Y)$ is the connectivity of the set of regions $X \cup Y$, we have by elementary topology§

$$c_0(A \cup B) - c_1(A \cup B) = \alpha_2(A \cup B) - \beta_1(A \cup B) \tag{1}$$

and

$$c_1(A \cup B) = c_0(C \cup D) - 1. \tag{2}$$

From these equations and the corresponding ones for the colour-dyads $A \cup C$ and $A \cup D$ it follows that

$$\begin{aligned} &c_0(A \cup B) + c_0(A \cup C) + c_0(A \cup D) - c_0(C \cup D) - c_0(B \cup D) - c_0(B \cup C) \\ &\quad = 3\alpha_2(A) + \alpha_2(B) + \alpha_2(C) + \alpha_2(D) - \beta_1(A \cup B) - \beta_1(A \cup C) - \beta_1(A \cup D) - 3 \\ &\quad = 2\alpha_2(A) - \sum_{R \in A} f(R) + \alpha_2 - 3, \end{aligned} \tag{3}$$

where $f(R)$ denotes the number of sides of the polygon R.

† Heawood, *Quart. J. of Math.* 24 (1890), 332–338.

‡ Kittel, *Bull. American Math. Soc.* 41 (1935), 407–413.

§ See, for example, Newman, *Topology of Plane Sets* (C.U.P. 1939), 194–199. The regions must be dissected into triangles before his results are applied, but this introduces no real difficulty.

Now the right-hand side of (3) depends only on the colour-class A. Hence we have

THEOREM I. *The quantity on the left of equation* (3) *has the same value for all colourings of M for which A is a colour-class.*

We say that two colourings Z_1 and Z_2 of a map are *connected* if there exists a finite sequence of colourings of the map beginning with Z_1 and ending with Z_2 and such that each pair of consecutive members have a colour-class in common. Clearly any two colourings which belong to the same colour-system are connected. (But the converse does not follow.)

For any colouring Z, let $J(Z)$ denote the sum of the quantities $c_0(X \cup Y)$ over the six colour-dyads. We call the parity of $J(Z)$ the *parity of Z*.

THEOREM II (the parity theorem). *If Z_1 and Z_2 are connected colourings of a spherical map M, then*

$$J(Z_1) \equiv J(Z_2), \quad (\text{mod } 2). \tag{4}$$

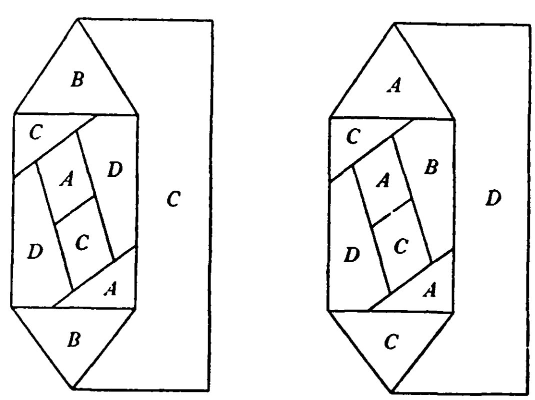
For by (3), for any colouring of M

$$J(Z) \equiv \sum_{R \in A} f(R) + \alpha_2 + 1, \quad (\text{mod } 2),$$

where A is any colour-class of Z.

Hence (4) is true whenever Z_1 and Z_2 have a colour-class in common, and therefore it is true whenever Z_1 and Z_2 are connected.

It may be worth mentioning that the colourings of a particular spherical map need not be restricted to one parity. For example the two colourings shown in Fig. 1 have opposite parities.



A B

Fig. 1

3. *The pentagon problem*

Let us now return to the problem of the partially coloured map stated in the introduction.

Let F denote the set of regions of M which meet P. We denote these regions by F_1, F_2, F_3, F_4, F_5 in the cyclic order of incidence with the edges of P. We may suppose that they are taken in clockwise order as seen from the centre of the sphere. We have not yet assumed that they are all distinct.

In a colouring of type II each colour-class contains a region of F. Three colour-classes therefore each meet P in one edge only, and the fourth meets it in just two edges, E_1 and E_2 say. E_1 and E_2 do not meet, for if they did the third edge incident with their common vertex would be common to two regions of the fourth colour-class. There is therefore an edge E_3 of the pentagon adjacent to both E_1 and E_2. We call the set of regions of F which have E_1 or E_2 as an edge the *norm* of the colouring, and the region of F having E_3 as an edge the *apex* of the colouring.

The following well-known result will be needed.†

THEOREM III. *If Z is a colouring of M_P such that all the members of* $\Pi(Z)$ *are of type II, then if any Kempe chain contains the norm of Z it contains also the apex of Z.*

For let $Z = (A, B, C, D)$ where A contains the norm and B the apex of Z. Then if the theorem is false for Z one of the remaining two regions of F, $\epsilon\, C$ say, must be separated in M_P from the apex by a Kempe chain of $A \cup D$ containing the norm. Hence, by an exchange operation on that Kempe chain of $B \cup C$ which contains the apex, a type I colouring of $\Pi(Z)$ can be obtained, contrary to hypothesis.

It follows from theorem III that if all the colourings of $\Pi(Z)$ are of type II, then the five regions F_i are all distinct. We therefore assume their distinctness in what follows.

If F_i is the apex of Z, then the norm of Z is the pair of regions F_{i+1}, F_{i+4}. (The addition in the suffices is modulo 5.) We then denote the Kempe chain containing F_{i+1} and F_{i+2} by $g(Z)$ and the colour-dyad to which it belongs by $G(Z)$. We denote the Kempe chain of $G(Z)$ which contains the other member F_{i+4} of the norm by $h(Z)$. It follows from theorem III that $g(Z)$ and $h(Z)$ are distinct.

We define a *λ-operation* on Z as the application of the exchange operation to each member of a subset Λ of the Kempe chains of $G(Z)$, where Λ contains $g(Z)$ but not $h(Z)$.

† Heawood, loc. cit.

(*Note.* A similar operation on $G(Z)$ affecting $h(Z)$ but not $g(Z)$ would be equivalent to the λ-operation affecting just those components of $G(Z)$ which the former operation left unchanged.)

We define a λ-*circuit* in $\Pi(Z)$ as a cyclic sequence of colourings belonging to $\Pi(Z)$ such that each member of the sequence can be derived from its immediate predecessor by a single λ-operation. λ-operations and λ-circuits can of course be defined in the same way when $\Pi(Z)$ contains type I colourings but they are of particular interest in the other case, for then it has been shown that every colouring in $\Pi(Z)$ is a member of a λ-circuit.†

If a λ-operation is applied to a colouring Z of apex F_i the norm of the new colouring will be the pair F_{i+2}, F_{i+4}, and its apex will therefore be F_{i+3}. The colour-classes of the old and new apices must be common to both colourings, since neither belongs to $G(Z)$. Each λ-operation thus advances the apex three places in the cyclic sequence of the F_i. Hence if the number of members of any λ-circuit is n, then

$$n \equiv 0, \quad (\text{mod}\, 5). \tag{5}$$

The object of this paper is to improve upon this result by establishing the following two theorems:

THEOREM IV. $n \equiv 0, \quad (\text{mod}\, 10).$

THEOREM V. $n \neq 10.$

4. *Proof of theorem IV*

Let λZ be the colouring obtained from Z by application of the λ-operation λ. The intersections of the colour-classes with F will be as shown in the diagrams (i) and (ii) of Fig. 2 for Z and λZ respectively. The colour-classes B and D are common to both colourings.

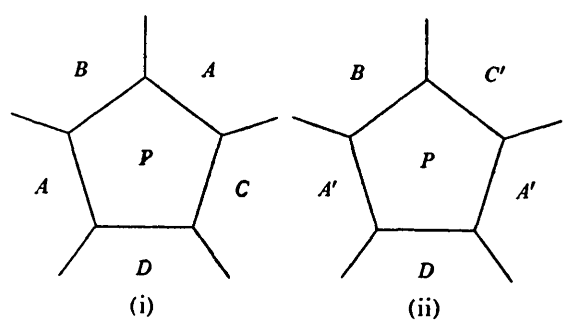


Fig. 2

† Errera, "Du Coloriage des Cartes et de Quelques Questions d'Analysis Situs", Thesis (Gauthier-Villars, 1921). See also Kittel, loc. cit.

Since the regions of F are distinct, a new spherical map N can be formed from M by incorporating P in that region of F which belongs to B in the colourings Z and λZ. If the new region is assigned to B, the colourings Z and λZ will be transformed into colourings Z_N and $(\lambda Z)_N$ respectively of N. These two colourings of N are connected, for they have the colour-class D in common. Hence by the parity theorem

$$J(Z_N) \equiv J((\lambda Z)_N), \quad (\text{mod } 2). \tag{6}$$

Now in the change from Z to Z_N the only two of the six quantities $c_0(X \cup Y)$ not obviously unaltered are $c_0(B \cup C)$ and $c_0(B \cup D)$. But for one of these, let us say $c_0(B \cup C)$, to be altered it is necessary for the two regions in which the corresponding colour-dyad of Z meets F to be in different Kempe chains of that colour-dyad. This colour-dyad $(B \cup C)$ of Z would then contain the apex and would not separate the members of the norm in M_P. The members of the norm would therefore belong to the same component of $(A \cup D)$ in Z, which contradicts theorem III, since the apex of Z is in B and therefore not in $(A \cup D)$. Hence

$$J(Z_N) = J(Z). \tag{7}$$

Again in the change from λZ to $(\lambda Z)_N$ the only quantities $c_0(X \cup Y)$ not obviously unaffected are $c_0(B \cup D)$ and $c_0(B \cup A')$. But the first of these is unchanged, by the last paragraph, since $B \cup D$ is the same in Z as in λZ and in Z_N as in $(\lambda Z)_N$. However, by theorem III the two members of the norm in λZ are in different Kempe chains of $(B \cup A')$ (which does not contain the apex). $c_0(B \cup A')$ therefore decreases by 1 and so

$$J((\lambda Z)_N) = J(\lambda Z) - 1. \tag{8}$$

From (6), (7) and (8) we have

$$J(\lambda Z) \equiv J(Z) + 1, \quad (\text{mod } 2). \tag{9}$$

It follows that the number of members of any λ-circuit must be even. Hence by (5),

$$n \equiv 0, \quad (\text{mod } 10). \tag{10}$$

5. *Proof of theorem V*

Assume that a λ-circuit of 10 members exists. Let its members be in order $Z_0, Z_1, Z_2, \ldots, Z_9$.

Let A_i denote that colour-class of Z_i which contains the apex. By the paragraph immediately preceding equation (5), A_{i-1}, A_i and A_{i+1} are all colour-classes of Z_i. Moreover they are distinct, for the apices of Z_{i-1}, Z_i and Z_{i+1} are distinct regions F_j, F_{j+3} and F_{j+1}. (Addition in the suffices of the A_i and the Z_i is mod 10.)

The ten colourings Z_i are therefore completely determined when the ten colour-classes A_i are given, for when three colour-classes of a colouring are given the fourth is uniquely determined.

If R is any region of M_P we write

$$\begin{aligned} v_i = v_i(R) &= 1 \quad \text{if} \quad R \text{ is in } A_i, \\ &= 0 \quad \text{if} \quad R \text{ is not in } A_i. \end{aligned} \tag{11}$$

Then the set of 10-vectors

$$V(R) = (v_0(R), v_1(R), \ldots, v_9(R))$$

given for all R completely determines the A_i and therefore the Z_i.

As an example we note that F_j is in A_i if and only if it is the apex of Z_i and that this happens just once in any five consecutive members of the λ-circuit. So we may write

$$V(F_1) = (1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0), \tag{12}$$

a vector which we denote hereafter by e.

We then have, by the properties of λ-operations,

$$V(F_1) = Qe, \quad V(F_2) = Q^2e,$$

and in general,
$$V(F_{1+r}) = Q^{2r}e, \tag{13}$$

where Q is a cyclic permutation defined by

$$Q(v_0, v_1, \ldots, v_9) = (v_9, v_0, v_1, \ldots, v_8). \tag{14}$$

We denote the group of cyclic permutations Q^i by $\mathfrak{Q}$, and say that two 10-vectors are *equivalent* when they can be transformed into one another by operations of $\mathfrak{Q}$.

A 10-vector whose components are restricted to the values 0 and 1 we call a *V-vector*.

We say that a V-vector V is *admissible* if it satisfies the following conditions: (i) no three consecutive signs of V (regarded as a cyclic sequence) include more than one 1; and (ii) there exist three other V-vectors satisfying (i) such that their sum with V is $I = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$.

The other three V-vectors of (ii) are clearly admissible. We call a set of four admissible V-vectors whose sum is I a *tetrad*.

We have at once

LEMMA I. *If V_1 and V_2 are equivalent V-vectors and V_1 is admissible, then V_2 is admissible.*

For the properties (i) and (ii) are invariant under the transformation Q.

LEMMA II. *If R is any region of M_P, then $V(R)$ is an admissible V-vector.*

For first, by (11), $V(R)$ is a V-vector.

Secondly, suppose two of $v_{i-1}(R)$, $v_i(R)$, and $v_{i+1}(R)$ have the value 1. Then, by (11), R belongs to two of the colour-classes A_{i-1}, A_i, A_{i+1} which is impossible since these are distinct colour-classes of the same colouring Z_i. Hence $V(R)$ satisfies (i).

Thirdly, any region R_0 of M_P has some vertex not a vertex of P. For if this were false for F_j then F_{j-1} and F_{j+1} would not be distinct. At this vertex R_0 meets two other regions, R_1 and R_2 say, of M_P. Consider the matrix whose three rows are the vectors $V(R_0)$, $V(R_1)$, $V(R_2)$.

No column of this matrix contains more than one 1, since no two of R_0, R_1 and R_2 belong to the same colour-class in any colouring of M_P. Hence $V' = I - V(R_0) - V(R_1) - V(R_2)$ is a V-vector.

Consider the $(i-1)$th, ith and $(i+1)$th columns (addition mod 10). If two of them consist entirely of 0's, then Z_i has a colour-dyad containing none of the regions R_0, R_1 and R_2 by (11). But this is impossible for it requires that two of these mutually contiguous regions shall belong to the same colour-class of Z_i. It follows that the V-vector V' satisfies (i) and so by the previous result that $V(R)$ is a V-vector satisfying (i), and by the definition of V', it follows that $V(R_0)$ satisfies (ii). This proves the lemma.

COROLLARY. *If three regions R_0, R_1 and R_2 of M_P meet at a vertex, then the four vectors $V(R_0)$, $V(R_1)$, $V(R_2)$, $I - V(R_0) - V(R_1) - V(R_2)$ are admissible V-vectors and constitute a tetrad.*

If V is a V-vector, we denote by $\sigma(V)$ the number of its 1's. By considering in turn the cases $\sigma(V) = 0$, $\sigma(V) = 1$, etc., we find that every V-vector satisfying (i) is equivalent to a member of the following set.

$$\left.\begin{aligned} a &= (0\,0\,0\,0\,0\,0\,0\,0\,0\,0), \\ b &= (1\,0\,0\,0\,0\,0\,0\,0\,0\,0), \\ c &= (1\,0\,0\,1\,0\,0\,0\,0\,0\,0), \\ d &= (1\,0\,0\,0\,1\,0\,0\,0\,0\,0), \\ e &= (1\,0\,0\,0\,0\,1\,0\,0\,0\,0), \\ f &= (1\,0\,0\,1\,0\,0\,1\,0\,0\,0). \end{aligned}\right\} \tag{15}$$

Clearly no two members of this set are equivalent.

If x is one of the vectors satisfying (i) we define $m(x)$ as the smallest integer m not 0 such that $Q^m x = x$. We then have

$$\begin{aligned} m(x) &= 1 \text{ if } x \text{ is equivalent to } a, \\ &= 5 \text{ if } x \text{ is equivalent to } e, \\ &= 10 \text{ otherwise.} \end{aligned} \tag{16}$$

The next step in the proof is the determination of all the tetrads. We note that if the vectors V_i ($i = 1, 2, 3, 4$) constitute a tetrad, then

$$\sum_i \sigma(V_i) = 10 \tag{17}$$

by the definition of a tetrad; and for each $\sigma(V_i)$ (by (15))

$$0 \leqslant \sigma(V_i) \leqslant 3. \tag{18}$$

The only sets of four integers satisfying (17) and (18) are

$$(3, 3, 3, 1) \quad \text{and} \quad (3, 3, 2, 2). \tag{19}$$

The V-vector a therefore is not admissible.

Now the only vector x of (15) for which $\sigma(x) = 1$ is b and the only one for which $\sigma(x) = 3$ is f. Hence by (19) any tetrad involving b is of the form

$$T = (b, Q^i f, Q^j f, Q^k f).$$

Now f contains just one block of three consecutive 0's. Consider the matrix whose rows are the vectors of T. The three columns which contain the corresponding block in $Q^i f$ must contain three 1's altogether (definition of a tetrad) and this is possible only if they contain the non-zero component of b (by (i)) which implies that $i = 1$, 2, or 3. The same argument applies with i replaced by j or k. Hence if b is contained in a tetrad, that tetrad is

$$(b, Qf, Q^2 f, Q^3 f),$$

and it is easily verified that these four vectors do indeed constitute a tetrad.

If we apply an operation Q^i to each of the four V-vectors of any tetrad T we shall clearly obtain a new tetrad. We denote this by $Q^i T$ and say that it is *equivalent* to T. In listing the tetrads it will suffice to give one member of each set of equivalent tetrads. We may therefore proceed to those tetrads which contain no vector equivalent to b. By (15) and (19) such a tetrad involves just two vectors equivalent to f, and is therefore equivalent to a tetrad involving a pair f, $Q^j f$. We can further suppose j less than 6, for the operation Q^{10-j} transforms the above pair into the pair f, $Q^{10-j} f$ (by (16)).

By comparing the first of the six V-vectors f, Qf, $Q^2 f$, $Q^3 f$, $Q^4 f$, $Q^5 f$ with the other five, we find that the cases $j = 3$ and $j = 4$ are impossible since for each of these there is an s such that

$$v_s(f) = v_s(Q^j f) = 1,$$

contrary to the definition of a tetrad, but that the other cases cannot be ruled out in this way. We may suppose therefore that $j = 1$, 2 or 5.

Now
$$\begin{aligned} I - f - Qf &= (0\,0\,1\,0\,0\,1\,0\,0\,1\,1) \\ &= Q^5 d + Q^8 d \\ &= Q^5 c + Q^9 c, \end{aligned}$$

and these are the only two ways in which the vector (0 0 1 0 0 1 0 0 1 1) can be expressed as the sum of two vectors equivalent to members of the set (15) and satisfying $\sigma(x) = 2$.

Again $$I - f - Q^2 f = (0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1),$$

which can be expressed subject to the above conditions only as

$$Q^7 d + Q^4 e.$$

Finally $$I - f - Q^5 f = (0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1),$$

which must be expressed either as

$$Q^2 e + Q^4 e \quad \text{or as} \quad Q^4 c + Q^9 c.$$

Every tetrad therefore is equivalent to a member of the following set:

$$\left.\begin{aligned} T_1 &= (b,\ Qf,\ Q^2 f,\ Q^3 f), \\ T_2 &= (f,\ Qf,\ Q^5 d,\ Q^8 d), \\ T_3 &= (f,\ Qf,\ Q^5 c,\ Q^9 c), \\ T_4 &= (f,\ Q^2 f,\ Q^7 d,\ Q^4 e), \\ T_5 &= (f,\ Q^5 f,\ Q^2 e,\ Q^4 e), \\ T_6 &= (f,\ Q^5 f,\ Q^4 c,\ Q^9 c). \end{aligned}\right\} \tag{20}$$

The significance of the set W of all admissible V-vectors can best be understood in terms of the dual map M^* of M. The regions of M^* are all triangles and so M^* is a simplicial dissection of the sphere. If R is any region of M, we denote the corresponding vertex of M^* by R^*. The dual map M_P^* of M_P may be defined as the set of all simplexes of M^* which do not have P^* as a vertex. It is therefore a simplicial dissection of a part of the sphere bounded by a simple closed curve F^* which consists of the vertices $F_1^*, F_2^*, \ldots, F_5^*$ and the edges $F_1^* F_2^*, F_2^* F_3^*, \ldots, F_5^* F_1^*$ dual to the five distinct edges of M_P which meet the pentagon P. (If there were not five such distinct edges of M_P, some two vertices of the pentagon would be joined by an edge E in M_P and then at least one of the two regions of M_P incident with E would be incident with two of the edges of P. This would contradict the requirement that the F_i must be distinct.)

It follows that the formal sum

$$\sum_{i=1}^{5} (F_i^*, F_{i+1}^*) = K \text{ say}, \tag{21}$$

where (F_i^*, F_{i+1}^*) is a 1-simplex of F^* with an orientation given by the order of the terms F_i^* and F_{i+1}^*, is a bounding 1-cycle on M_P^*.

We can treat W as a simplicial 3-complex of which the V-vectors are the 0-simplexes and the tetrads the 3-simplexes (and in which each i-simplex is incident with an $(i+1)$-simplex for $i<3$). The correspondence $R^* \to V(R)$ defines a mapping of M_P^* into W which maps vertices on to vertices, and, by the corollary to lemma II, 2-simplexes on to 2-simplexes.

It follows at once that if the correspondence maps the 2-chain U of M_P^* on to the 2-chain U_W of W, where U and U_W have ordinary integers as coefficients, then it maps the boundary of U on to the boundary of U_W. Consequently bounding cycles on M_P^* are mapped on to bounding cycles on W.

Now the bounding 1-cycle K on M_P^* maps on to a 1-cycle

$$K_W = \sum_{i=1}^{5} (V(F_i),\ V(F_{i+1})) \quad \text{by (21)}$$

$$= ((e, Q^2e) + (Q^2e, Q^4e) + (Q^4e, Qe) + (Qe, Q^3e) + (Q^3e, e)) \quad \text{by (13) and (16).}$$

But it can be shown that this is a non-bounding cycle of W. By proving this we shall show that the hypothesis of the existence of a λ-circuit of 10 members leads to a contradiction and so establish theorem V. To do this we first define a function $\Delta(V_1, V_2)$ for each 1-simplex (V_1, V_2) of W by means of the equations

$$\Delta(V_1, V_2) = -\Delta(V_2, V_1), \tag{22}$$

$$\Delta(Q^iV_1, Q^iV_2) = \Delta(V_1, V_2) \tag{23}$$

and the following table.

TABLE I

Reference number	V_1	V_2	$\Delta(V_1, V_2)$	Reference number	V_1	V_2	$\Delta(V_1, V_2)$
1	b	Qf	-1	12	d	Q^2f	-2
2	b	Q^2f	0	13	d	Q^3f	-1
3	b	Q^3f	1	14	d	Q^5f	1
4	c	Q^4c	-1	15	d	Q^6f	2
5	c	Q^5c	0	16	e	Q^2e	2
6	c	Qf	0	17	e	Qf	-1
7	c	Q^2f	1	18	e	Q^3f	1
8	c	Q^5f	-1	19	f	Qf	1
9	c	Q^6f	0	20	f	Q^2f	2
10	d	Q^3d	3	21	f	Q^5f	0
11	d	Q^2e	0				

It may readily be verified (with the help of (16)) that no two of the pairs of this table, even when regarded as unordered, are equivalent under the operations of $\mathfrak{Q}$. The definition of $\Delta(V_1, V_2)$ is therefore consistent.

Other assertions verifiable from the table with the help of equations (16), (22) and (23) are: (i) $\Delta(V_1, V_2)$ is defined for every 1-simplex (V_1, V_2), that is for every ordered pair of V-vectors both contained in the same tetrad; and (ii) if V_1, V_2 and V_3 are distinct members of the same tetrad, then

$$\Delta(V_1, V_2) + \Delta(V_2, V_3) + \Delta(V_3, V_1) = 0. \tag{24}$$

It is sufficient to verify these two assertions for each of the tetrads of (20) by (23). This verification is set out in tabular form in Table II. The numbers in the last column of this table are the references, in order, to the rows of Table I.

Equation (24) states that the sum of the function $\Delta(V_1, V_2)$ over the boundary of any 2-simplex of W is 0. It follows from this, and (22) that the sum of the function over any bounding 1-cycle of W is 0. But its sum over the 1-cycle K_W is, by (23) and the preceding expression for K_W,

$$5\Delta(e, Q^2e) = 10 \quad \text{(by Table I)}.$$

Thus K_W is shown to be non-bounding and the proof of theorem V is complete.

TABLE II

Tetrad	V_1	V_2	V_3	$\Delta(V_1, V_2)$	$\Delta(V_2, V_3)$	$\Delta(V_3, V_1)$	Sum	References
T_1	b	Qf	Q^2f	-1	1	0	0	1, 19, 2
	b	Q^2f	Q^3f	0	1	-1	0	2, 19, 3
	b	Qf	Q^3f	-1	2	-1	0	1, 20, 3
	Qf	Q^2f	Q^3f	1	1	-2	0	19, 19, 20
T_2	f	Qf	Q^5d	1	-2	1	0	19, 15, 14
	f	Qf	Q^8d	1	1	-2	0	19, 13, 12
	f	Q^5d	Q^8d	-1	3	-2	0	14, 10, 12
	Qf	Q^5d	Q^8d	-2	3	-1	0	15, 10, 13
T_3	f	Qf	Q^5c	1	0	-1	0	19, 9, 8
	f	Qf	Q^9c	1	-1	0	0	19, 7, 6
	f	Q^5c	Q^9c	1	-1	0	0	8, 4, 6
	Qf	Q^5c	Q^9c	0	-1	1	0	9, 4, 7
T_4	f	Q^2f	Q^7d	2	-1	-1	0	20, 14, 13
	f	Q^2f	Q^4e	2	-1	-1	0	20, 18, 17
	f	Q^7d	Q^4e	1	0	-1	0	13, 11, 17
	Q^2f	Q^7d	Q^4e	-1	0	1	0	14, 11, 18
T_5	f	Q^5f	Q^2e	0	-1	1	0	21, 18, 18
	f	Q^5f	Q^4e	0	1	-1	0	21, 17, 17
	f	Q^2e	Q^4e	-1	2	-1	0	18, 16, 17
	Q^5f	Q^2e	Q^4e	-1	2	-1	0	18, 16, 17
T_6	f	Q^5f	Q^4c	0	0	0	0	21, 6, 9
	f	Q^5f	Q^9c	0	0	0	0	21, 9, 6
	f	Q^4c	Q^9c	0	0	0	0	9, 5, 6
	Q^5f	Q^4c	Q^9c	0	0	0	0	6, 5, 9

The simple method used in the proof of theorem V will not suffice to prove the analogous theorem for $n = 20$. For Errera† has given a map M_P and a colouring Z such that a certain sequence of 20 λ-operations transforms Z into itself. Perhaps it is significant that this map contains a region (the central one in Kittel's diagram) whose 20-vector V satisfies $Q^4(V) = V$, where Q and V are defined by equations analogous to (14) and (11) respectively. Even in this case $\Pi(Z)$ contains colourings of type I.

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† Errera, loc. cit. and Kittel, loc. cit.