Sample Exam 1 – Solutions

1. a) & b)

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 5 \\ 1 & 2 & 3 & 5 \end{bmatrix}}_{A}\left|\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}\right. \xrightarrow{r_2 \to r_2 - r_1} \left[\begin{matrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 3 & 5 \end{matrix}\left|\begin{matrix} 1 \\ 1 \\ 3 \end{matrix}\right.\right] \quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_3 \to r_3 - r_1} \left[\begin{matrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}\left|\begin{matrix} 1 \\ 1 \\ 2 \end{matrix}\right.\right] \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_1 \to r_1 - 3r_2} \left[\begin{matrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}\left|\begin{matrix} -2 \\ 1 \\ 2 \end{matrix}\right.\right] \quad E_3 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_1 \to r_1 - r_3} \left[\begin{matrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}\left|\begin{matrix} -4 \\ 1 \\ 2 \end{matrix}\right.\right] \quad E_4 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{r_2 \to r_2 - r_3} \left[\overbrace{\begin{matrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}^{R}\left|\begin{matrix} -4 \\ -1 \\ 2 \end{matrix}\right.\right] \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{\begin{matrix} x_4 = 2,\ x_3 = -1,\ x_2 = \alpha \\ x_1 = -2\alpha - 4 \end{matrix}} \qquad \boxed{\vec{x} = \begin{pmatrix} -2\alpha - 4 \\ \alpha \\ -1 \\ +2 \end{pmatrix}}$$

α any real number.

(c) $E = E_5 E_4 E_3 E_2 E_1$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underbrace{\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

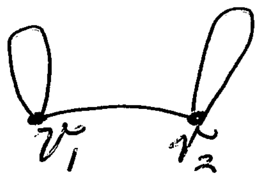
$$= \underbrace{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & -1 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 5 & -3 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}}}$$

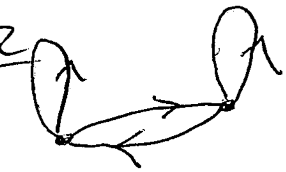
$$EA = \begin{bmatrix} 5 & -3 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 5 \\ 1 & 2 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = R \checkmark$$

2. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$



or



D

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2^1 & 2^1 \\ 2^1 & 2^1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 2^2 & 2^2 \\ 2^2 & 2^2 \end{bmatrix}$$

So $A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$, $n = 1, 2, 3, \ldots$

$\therefore$ The number of ~~(paths)~~ walks of length 137 from v_1 to v_1 is 2^{136}.

<u>Another question</u>: How many walks are there from v_1 to v_1 of length ≤ 137?

<u>Answer</u>: $1 + 2 + 2^2 + \cdots + 2^{136}$.

~~Let~~ $S = 1 + 2 + 2^2 + \cdots + 2^{136}$

$\Rightarrow 2S = \quad 2 + 2^2 + \cdots + 2^{136} + 2^{137}$

$\Rightarrow S = 2S - S = 2^{137} - 1$.

<u>Remark</u>. Note that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}}_{A^n} = \underbrace{\begin{bmatrix} 2^{n-1}+2^{n-1} & 2^{n-1}+2^{n-1} \\ 2^{n-1}+2^{n-1} & 2^{n-1}+2^{n-1} \end{bmatrix}}_{A^{n+1}} = \begin{bmatrix} 2^n & 2^n \\ 2^n & 2^n \end{bmatrix}$$

✓

Digression on Mathematical Induction

Suppose you know (I) Fact$_1$ is true
and you know that
(II) If Fact$_K$ is true then Fact$_{(K+1)}$ is true.

Then you can conclude that
Fact$_n$ is true for all $n = 1, 2, 3, \ldots$.

This is called the Principle of Mathematical Induction.

I. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow A^1 = A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2^0 & 2^0 \\ 2^0 & 2^0 \end{bmatrix}$.

Fact$_1$: $A^1 = \begin{bmatrix} 2^{1-1} & 2^{1-1} \\ 2^{1-1} & 2^{1-1} \end{bmatrix}$.

II. Fact$_K$: $A^K = \begin{bmatrix} 2^{K-1} & 2^{K-1} \\ 2^{K-1} & 2^{K-1} \end{bmatrix}$.

Suppose Fact$_K$ is true for some K.

Then $A^{K+1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} A^K = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{K-1} & 2^{K-1} \\ 2^{K-1} & 2^{K-1} \end{bmatrix}$

$$= \begin{bmatrix} 2 \cdot 2^{K-1} & 2 \cdot 2^{K-1} \\ 2 \cdot 2^{K-1} & 2 \cdot 2^{K-1} \end{bmatrix}$$

$$\Rightarrow A^{K+1} = \begin{bmatrix} 2^K & 2^K \\ 2^K & 2^K \end{bmatrix}.$$

Thus Fact$_K \Longrightarrow$ Fact$_{K+1}$.

$\therefore$ Fact$_n$ is true for all $n = 1, 2, \ldots$ //

3.

(a) $M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\underbrace{\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]}_{M \quad I} \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

with I on the left and M^{-1} on the right.

Check $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

(b) $B = ad(M) = \begin{bmatrix} +\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$

$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad B^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

✓ $B^T = M^{-1}$ ✓ (here $\det M = 1$)

4.) (a) $\det(AB) = \det(A)\det(B)$

See Week 4 notes.

(b) $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\det(C) = -1$

$\Rightarrow \det(C^n) = (-1)^n$.

5.(a) A $n \times n$, $tr(A) = \sum_{i=1}^{n} A_{ii}$

$tr\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 1+5+9=15$

$tr(A) = A_{11} + A_{22} + \cdots + A_{nn}$.

$tr(A) =$  (summing on indices for closed lines)

$(AB)_{ii} = \sum_K A_{iK} B_{Ki}$

$(AB)_{ij} = \sum_K A_{iK} B_{Kj}$

$$tr(AB) = \sum_i (AB)_{ii} = \sum_i \sum_K A_{iK} B_{Ki}$$

$$= \sum_K \sum_i B_{Ki} A_{iK}$$

$$= \sum_K \left(\sum_i B_{Ki} A_{iK} \right) = \sum_K (BA)_{KK}$$

$tr(AB) = tr(BA)$ ✓

(b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$AB \neq BA$ ✓

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
Note that $Be_1 = e_2$, $Be_2 = e_3$ and $Be_3 = \vec{0}$.

(c) $B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Then $B^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \neq 0$

$$B^3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$ ✓

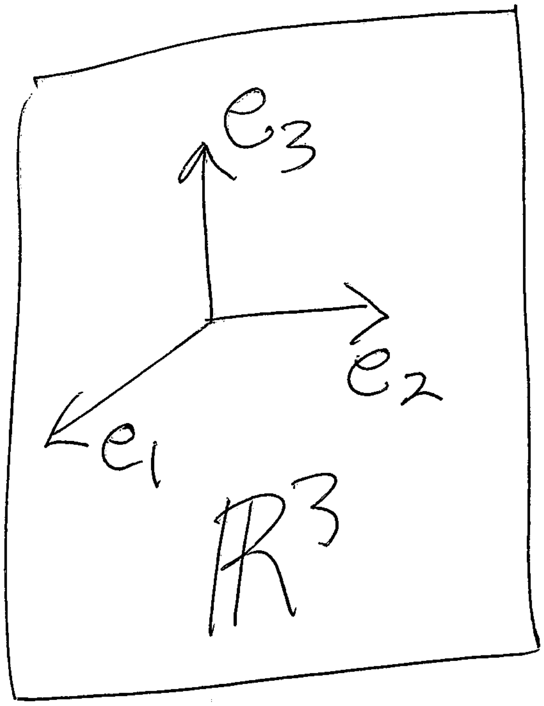
Problem. Let $B = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Show: $\det(B) = \lambda^2 - tr(A)\lambda + \det(A)$

Give an example of a matrix A s.t. $\det(B) = \lambda^2 - 5\lambda + 6$.

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$



$$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Be_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e_2$$

$$Be_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e_3$$

$$Be_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$