Special Relativity and a Calculus of Distinctions

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I. Introduction.

The purpose of this essay is to place the *mathematical pattern* of the theory of specal relativity in a new context. The pattern of the Lorentz Transformation, the Poincaré Group and related invariants arises historically from the ground of electromagnetic theory and Einstein's extraordinary formulation of these ideas in his special theory [6]. The next reformulation, due to Minkowski [14], led to the concept of *spacetime* as we know it today. Bondi [2] has pointed out the extraordinary simplicity of the theory, once it is formulated in light-cone or radar coordinates.

I will take this simplicity a step further and show that the mathematical pattern, and the pattern of ideas related to it can be regarded as arising from *consideration of the properties of a distinction.* These ideas, necessarily informal in the beginning, become more definite as we fit them out with mathematical clothing. In the course of this development, the mathematics of special relativity appears quite naturally, but with a non-physical interpretation. This provides a new ground and a new language for cradling the old physical ideas.

It is commonplace in mathematics to find one formalism holding a multitude of interpretations. Each such interpretation is potentially useful as a lens through which all the others may be seen.

For the reader unfamiliar with special relativity, I have included in the appendix to this paper a quick and self-contained introduction along the lines of [2] and [9]. In the body of the paper I have restricted my constructions to the context of distinction, and to the progression toward special relativity.

The basic idea of this paper is very simple. Let a distinction be given. Assume that the sides of the distinction are evaluated by A and by B. (A and B can be real numbers.) Denote this by $[A, B]$. Another observer may change the emphasis given by A and B. Thus we must consider transformations of the form $\mathcal{O}[A, B] = [RA, SB]$ where R and S denote the changes of emphasis created by the reference frame of the second observer. Then we see that choosing $RS = \rho^2$ and $R/S = \lambda^2$ we can re-write the transformation as $\mathcal{O}[A, B] = \rho[\lambda A, \lambda^{-1}B]$. If we are concerned only (projectively) with relative values, then it is sufficient to take $\rho = 1$. This leaves us to consider transformations of the form $\mathcal{O}[A,B] = [\lambda A, \lambda^{-1}B]$, *giving the structure of the Lorentz group to a calculus of distinctions!*

The paper is organized as follows. Section 2 discusses distinctions, indistinguishables and concepts of evaluating and weighting a distinction. This is considered with respect to the ideas of text and context, figure and ground. Section 3 re-examines these ideas in terms of description/perception of a ground-form that is susceptible to multiple interpretation. Along with this we construct an algebra of the iterants $[A,B]$.

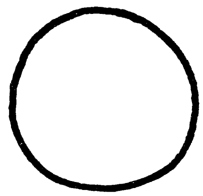
An *iterant* $[A,B]$ is a distinction viewed as an alternation or superposition of its two sides. (The iterant itself is neither one side nor the other.) Thus [Figure, Ground] as iterant becomes the process of viewing the figure and the ground and the figure and ... Similarly [Self, World] is the superposition of self-reference and world-seeing. Section 4 introduces the idea of considering an iterant as a superposition of *temperance* exemplified by $[1,1]$, and *polarity*, exemplified by $[1,-1]$. In *temperance* the sides of the distinction are equally emphasized. In polarity, they are oppositely emphasized. Any iterant, any ordered distinction, can be regarded as a superposition of temperance and polarity. On changing to the reference frame of another observer, these quantities transform in the *pattern* of time and space in special relativity. Time corresponds to the degree of temperance, and space to the degree of polarity. Thus the *the formal structure of special relativity is in fact present continually in the affairs of decision and evaluation that occur at all levels of human discourse and action.* That this is so, has surely been felt intuitively again and again. This paper provides a mathematical language that explicates this domain, giving conscious access to its implications.

The first appendix outlines the structure of physical relativity. The second appendix shows how spinors and Clifford algebras arise in this calculus of distinctions. Since it is quite significant that spinors (in the form or Pauli spin matrices and their generalizations) should appear so closely with the concept of distinction, I will give a quick sketch of how this comes about: After considering pairs $[A,B]$, it is natural to consider *pairs of pairs.* This arises at once if the pair or iterant is coupled with its conjugate $[B,A]$. This is the level of four-dimensional space. A single polarity such as $\sigma = [1,-1]$ becomes generalized into three basic double pairs

$$\sigma_1 = \begin{array}{cccccccc} \ddots & & & & & & & \\ & 1 & & & & & & \\ & & -1 & & 0 & & & \\ & & & 1 & & & & \\ & & 0 & & -1 & & & \\ & & & & & 1 & & \\ & & & & & & -1 & \\ & & & & & & & \ddots \end{array} \qquad \longleftrightarrow \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

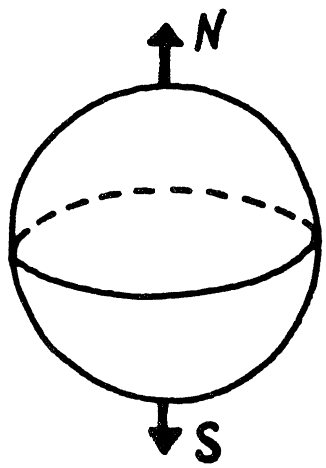
II. Considering a Distinction.

Consider a distinction. For example, consider the distinction between inside and outside that is made (indicated) by a circle drawn in the plane.



Here inside and outside are distinguished by difference in geometric form (bounded inside, unbounded (potentially unbounded) outside).

If the circle is drawn as an equatorial circle upon the surface of a sphere, then the two sides (topological disks) into which the sphere is divided by the equator are in all respects identical, except that one has the pole labelled N (north) while the other has the pole labelled S (south).



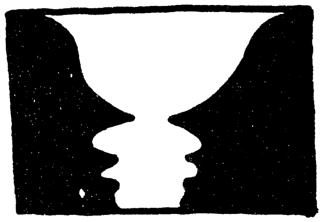
In this case, appropriate indication is required to let the division become a distinction. Without labelling, the upper and lower hemispheres are ***indistinguishable***, and this is underlined by the existence of a rotational symmetry (180° turn about an axis through antipodal points on the equator) that interchanges them.

In practice it is the observer who makes the difference, calling out a distinction between the symmetrical and indistinguishable hemi-spheres. This distinction may arise from context, such as the orientation and spin of that sphere.

In other circumstances a choice involving foreground and background is made. One side of the distinction is given prominence, and this may be agreed upon by a group of observers. Thus it is common to foreground the human figure against its three-dimensional background.

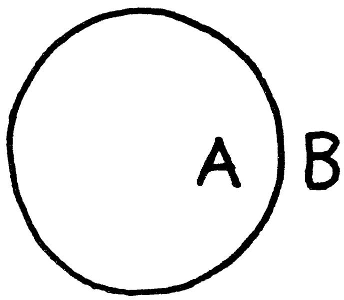
Only rarely, in ordinary speech, does one regard a person as identical to his/her surrounding space rather than the space of the body. Even in sketching, the formal outline of a profile is enough to bring forth the distinction ***head in space***.

It is by playing on this tendency that one creates situations where there is an alternation of figure and ground, as in the faces/vase illusion:

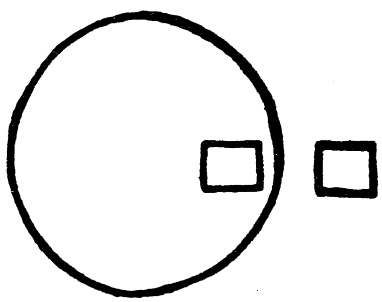


Rather than delineating the fine structure of distinctions (as is done in set theory) I wish to concentrate on a model for the pattern of valuing the sides of a distinction, and the possible alternation of sides.

Let a distinction be given, with sides labelled A and B.



These labels may be distinct, or they may themselves be indistinguishable. Thus the illustration below shows a distinction (the large circle) whose sides are labelled by indistinguishable squares.



Let the distinction with its labels A and B be indicated by the ordered pair $[A, B]$.

Let it be noted that an unlabelled, typographical ordered pair

$$[\quad , \quad]$$

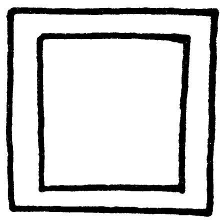
consists in a distinction between interior right-half ,] and interior left-half [, plus a distinction between inside and outside that is carried by common conventions about the use of brackets, parentheses and linear notation. Thus the given distinction can be taken to be the unlabelled ordered pair itself.

In fact (compare [3]) this entire discussion could be carried out using only the empty ordered pair. For example,

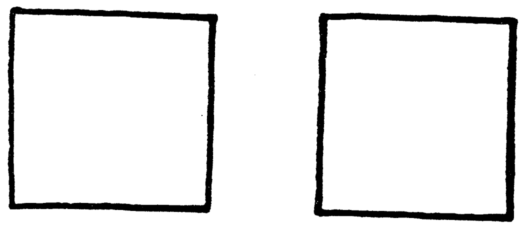
$$\Big[\ ,\ [\ ,\ ]\ \Big]$$

is a labelling that distinguishes the right-half from the left-half via the presence of an empty ordered pair in the right-hand compartment.

Note that while any two empty ordered pairs are indistinguishable, they may nevertheless become distinguished from one another through their mutual relationship in the indicational space. Thus any two boxes □ are typographically indistinguishable. Nevertheless, nested boxes



are clearly distinct from adjacent boxes



with regard to the context of the plane. In this way distinction and indication are mutually intertwined. (Compare [18].)

To return to the ordered pair $[A, B]$, we imagine that A and B are *evaluations* of the distinction. An evaluation may be denoted by word(s) in context (fine, precise, strange), by conventional symbols (!, *, A^+, ?), by numbers ($-3, 100\%, 3.14159$) or by another distinction (He wears his own picture on the security badge.).

Once the sides and evaluations of a distinction are symbolically or perceptually fixed, there is still lee-way for an observer to emphasize one side and (correspondingly) de-emphasize the other. Thus I shall model ***the action of an observer*** by the transformation

$$\mathcal{O}[A, B] = [\lambda A, \lambda^{-1} B]$$

where λ is a real or complex number. Implicit in this model is the "stationary observer" who sees $[A, B]$ and the "moving observer" who sees $[\lambda A, \lambda^{-1} B]$. This transformation is natural. Let me tender persuasions.

Note that $\lambda A / \lambda^{-1} B = \lambda^2 (A/B)$. The ratio λ^2 gives the change in the ratio A/B. If we are concerned always and only with relative evaluation then it is λ^2 that is the important quantity. A transformation of the form $[RA, SB]$ can be re-written as a constant multiple of one of the form $[\lambda A, \lambda^{-1}B]$. Since we are only concerned with relative changes of evaluation, the form $\mathcal{O}[A,B] = [\lambda A, \lambda^{-1}B]$ is sufficient at a numerical or algebraic level.

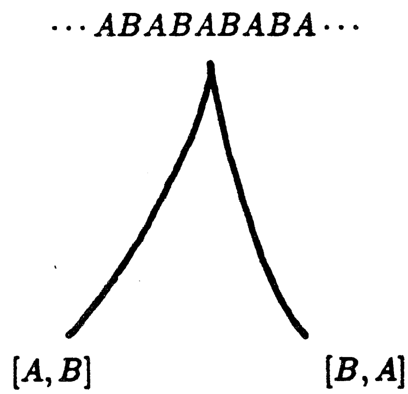
III. Iterants and Invariants.

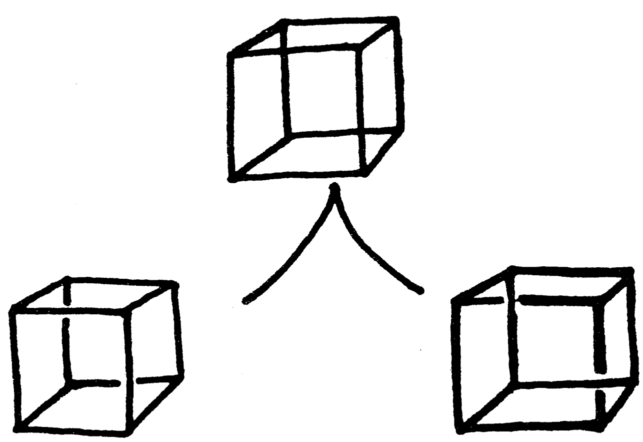
Let me summarize the discussion so far. We take as given a distinction and a context or "stationary observer" that assigns value to the sides of this distinction in the form $[A,B]$. We further assume that other observers will numerically or algebraically change the emphasis on these sides from $[A,B]$ to $[\lambda A, \lambda^{-1}B]$. Since we are only concerned with the ratio $\lambda A/\lambda^{-1}B = \lambda^2(A/B)$ it suffices to allow the transformation to take the form $\mathcal{O}[A,B] = [\lambda A, \lambda^{-1}B]$.

If A and B are themselves numerical evaluations, then *the product*

$$(\lambda A)(\lambda^{-1}B) = AB$$

is an invariant of the transformation. This remark re-opens the original source of the discussion in another way, for notice that $AB = BA$ implies that $[A,B]$ and $[B,A]$ receive the same invariant product. The forms $[A,B]$ and $[B,A]$ correspond to reversal of figure and ground as in





Proposition 4.1. Any pair $[A,B]$ can be written uniquely as a superposition (sum) of a *tempered iterant* $[t,t] = t1 = t$ and a *polar iterant* $[x,-x] = x[1,-1] = x\sigma$. That is, $[A,B] = t + x\sigma$ where

$$t = (A+B)/2 \text{ and } 1 = [1,1]$$
$$x = (A-B)/2 \qquad \sigma = [1,-1].$$

Hence

$$t + x = A$$
$$t - x = B.$$

PROOF: Certainly,

$$[t+x, t-x] = [t,t] + [x,-x] = t + x\sigma.$$

Conversely, if $t + x = A$ and $t - x = B$, then $t = \frac{1}{2}(A+B)$ and $x = \frac{1}{2}(A-B)$. This completes the proof.

The value t will be called the *temperance* of $[A,B]$ while $x = \frac{1}{2}(A-B)$ will be called the *polarity* of $[A,B]$. We wish to see how the temperance and polarity transform under an element $\mathcal{O}_\lambda$ of the *Iterant Group*

$$G = \{\mathcal{O}_\lambda | \mathcal{O}_\lambda[A,B] = [\lambda A, \lambda^{-1}B]\}.$$

To see this, note that $\mathcal{O}_\lambda$ may be regarded as multiplication by the element $[\lambda,\lambda^{-1}]$. That is $\mathcal{O}_\lambda[A,B] = [\lambda,\lambda^{-1}] * [A,B]$. ($[C,D] * [A,B] = [CA, DB]$).

Definition 4.2. The *velocity* v of a pair $[A,B]$ ***is the ratio*** $v = x/t$ of its ***polarity*** to its *temperance*: $v = \left(\frac{A-B}{A+B}\right)$. In this terminology,

$$v[1,1] = (1-1)/(1+1) = 0$$
$$v[1,-1] = 2/0 = \infty$$
$$v[-1,1] = -2/0 = -\infty$$
$$v[1,0] = 1$$
$$v[0,1] = -1.$$

Proposition 4.3. The quantity $t^2 - x^2$ is invariant under transformations of the iterant group $G = \{[\lambda,\lambda^{-1}]\}$.

PROOF: $[A,B] = [t+x, t-x]$.

$$AB = (t+x)(t-x) = t^2 - x^2$$

is invariant. Q.E.D.

Proposition 4.4. Let $v = v(\lambda)$ denote the velocity of the pair $[\lambda, \lambda^{-1}]$. Then

$$[\lambda, \lambda^{-1}] = \frac{1 + \sigma v}{\sqrt{1 - v^2}}$$

where $1 = [1, 1]$, $\sigma = [1, -1]$ are the basic tempered and polar iterants.

PROOF: Let $[\lambda, \lambda^{-1}] = [t + x, t - x]$. Then $1 = \lambda\lambda^{-1} = (t + x)(t - x) = t^2 - x^2$. Hence $\frac{1}{t^2} = 1 - (x^2/t^2) = 1 - v^2$. Hence $t = 1/\sqrt{1 - v^2}$. Thus

$$\begin{aligned}[\lambda, \lambda^{-1}] &= t + \sigma x \\ &= t(1 + \sigma v) \quad (v = x/t) \\ &= \frac{1 + \sigma v}{\sqrt{1 - v^2}}.\end{aligned}$$

Proposition 4.5. Let $[A, B] = t + \sigma x$ be any iterant with temperance t and polarity x. Let $\mathcal{O}_\lambda = [\lambda, \lambda^{-1}] \in G$ be an element of the iterant group with velocity v. Then

$$\mathcal{O}_\lambda[A, B] = t' + \sigma x'$$

where

$$\begin{aligned}t' &= (t + vx)/\sqrt{1 - v^2} \\ x' &= (x + vt)/\sqrt{1 - v^2}.\end{aligned}$$

PROOF: From 4.4 we have

$$\mathcal{O}_\lambda = [\lambda, \lambda^{-1}] = \frac{1 + v\sigma}{\sqrt{1 - v^2}}.$$

Thus

$$\begin{aligned}\mathcal{O}_\lambda[A, B] &= \left(\frac{1 + v\sigma}{\sqrt{1 - v^2}}\right) * (t + x\sigma) \\ &= \frac{t + vx\sigma * \sigma + (x + vt)\sigma}{\sqrt{1 - v^2}} \\ &= \left(\frac{t + vx}{\sqrt{1 - v^2}}\right) + \left(\frac{x + vt}{\sqrt{1 - v^2}}\right)\sigma \quad (\sigma * \sigma = 1).\end{aligned}$$

Hence

$$t' = \frac{t + vx}{\sqrt{1 - v^2}}, \quad x' = \frac{x + vt}{\sqrt{1 - v^2}}.$$

As the reader will undoubtedly recognize, the transformations in 4.5 are exactly the classical Lorentz transformation when the speed of light is normalized to unity. In conjunction with the remarks in Appendix 1 to this paper, this constitutes a derivation of these transformations based on the itera $\imath$t calculus.

The mathematical ground of our discussion has been the concept of distinction, and how the evaluation of distinctions leads naturally to the concepts of iterant, iterant group, temperance and polarity.

Remark. It is worth noting the following formula for an element $[\lambda, \lambda^{-1}] \in G$. If v is the velocity of $[\lambda, \lambda^{-1}]$, then $v = \frac{\lambda - \lambda^{-1}}{\lambda + \lambda^{-1}} = \frac{\lambda^2 - 1}{\lambda^2 + 1}$. Hence $\lambda^2 = \frac{1+v}{1-v}$. Since λ is real we must have $|v| \leq 1$. Only by extending G to complex values can we obtain velocities greater than 1.

It may be quite interesting epistemologically to compare the concept of tachyon ($v >$ light speed) in physics with the concept of imaginary values ($\sqrt{-1}$) in the evaluation of distinctions. Note also that an iterant with $v > 1$ has more polarity than temperance. Real transformations live in the realm where temperance dominates polarity.

Remark. It is easy to see that if $v_1 = \text{velocity}[\lambda_1, \lambda_1^{-1}]$ and $v_2 = \text{velocity}[\lambda_2, \lambda_2^{-1}]$ then $v = \text{velocity}[\lambda_1, \lambda_1^{-1}] * [\lambda_2, \lambda_2^{-1}]$ is given by the formula $v = \frac{v_1 + v_2}{1 + v_1 v_2}$. It is worth thinking about this formula for the relativsitic addition of velocities in the context of velocity as ratio of polarity to temperance. In fact, the velocity of an arbittary pair $[A, B]$ is given by the formula

$$v[A, B] = \frac{\text{polarity}[A, B]}{\text{temperance}[A, B]} = \frac{(A - B)/2}{(A + B)/2} = \frac{A - B}{A + B}$$

and we have the

Proposition 4.6. Let $v_1 = v[A, B]$, $v_2 = v[C, D]$ and $v_3 = v([A, B] * [C, D])$ Then

$$v_3 = \frac{v_1 + v_2}{1 + v_1 v_2}.$$

PROOF:

$$\frac{v_1 + v_2}{1 + v_1 v_2} = \frac{\left(\frac{A-B}{A+B}\right) + \left(\frac{C-D}{C+D}\right)}{1 + \left(\frac{A-B}{A+B}\right)\left(\frac{C-D}{C+D}\right)}$$

$$= \frac{(A - B)(C + D) + (A + B)(C - D)}{(A + B)(C + D) + (A - B)(C - D)}$$

$$= \frac{AC + AD - BC - BD + AC - AD + BC - BD}{AC + AD + BC + BD + AC - AD - BC + BD}$$

$$= \frac{2AC - 2BD}{2AC + 2BD} = \frac{AC - BD}{AC + BD}$$

$$= v[AC, BD]$$

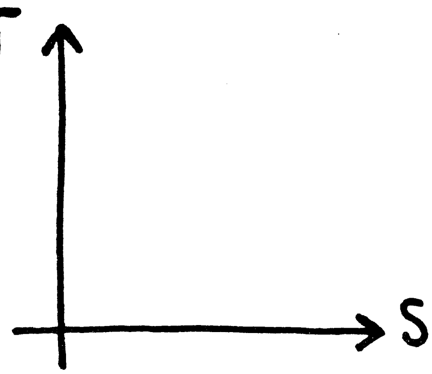
$$= v([A, B] * [C, D])$$

$$= v_3.$$

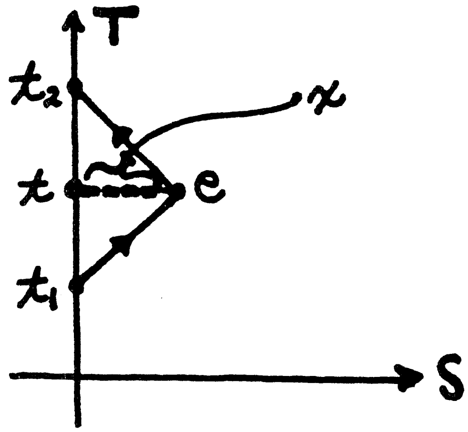
Q.E.D.

Appendix 1. Physical Relativity.

Let



denote a two-dimensional time-space frame. Let the speed of light be taken conventionally to be unity. An event e is determined, for a given observer, by two times $[t_2, t_1]$. The time t_1 is the time at which a signal is sent. The signal moves outward at light-speed, encounters the event and is reflected back to be received by the observer at time t_2.

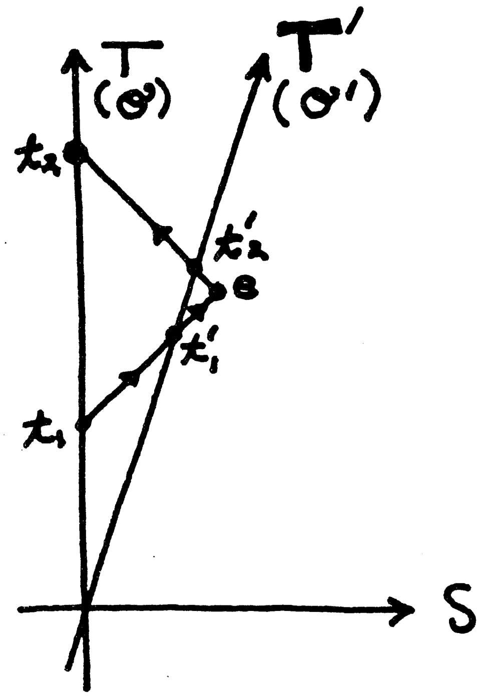


$$e = [t_2, t_1] = [t + x, t - x]$$
$$e = t + x\sigma$$

It is manifest from the diagram above that $t_2 = t + x$ and $t_1 = t - x$. Thus the event e may be identified as the iterant $[t_2, t_1]$ and the *time* t *of* e is the *temperance of* e while the *distance* x *of* e is the *polarity of* e. This gives a physcial interpretation of iterants in terms of space time coordinates.

That the Lorentz transformation has the form $T[t_2, t_1] = [\lambda t_2, \lambda^{-1} t_1]$ follows from the principle of relativity: (See [2], [6].) If observers O and O' have a constant velocity v of separation then light signals separated by time Δt in O's frame will be received in separation $\Delta\imath'$ in O''s frame. The ratio $K = \Delta t'/\Delta t$ is dependent only on the relative velocity. $K = \frac{\Delta(\text{signals received})}{\Delta(\text{signals sent})}$. It then follows as shown in the diagram below that

$$[t_2', t_1'] = [K^{-1} t_2, K t_1].$$



$$t_1' = Kt_1$$
$$t_2 = Kt_2'$$
$$(\Rightarrow t_2' = K^{-1}t_2)$$

In this diagram an event e receives a signal sent by the observer $\mathcal{O}$ at time t_1. This same signal can be regarded as sent by $\mathcal{O}'$ at time t_1'. Similarly, the reflected signal is received by $\mathcal{O}'$ at t_2' and by $\mathcal{O}$ at t_2. Here t_1, t_2 refer to times in $\mathcal{O}$'s frame and t_1', t_2' refer to times in $\mathcal{O}'$'s frame. Then we see that $t_1' = Kt_1$ and $t_2' = K^{-1}t_2$. Thus the group of Lorentz transformations is identical to the group of iterant transformations.

Appendix 2. Spinors and Clifford Algebra.

In this appendix I wish to discuss how the mathematical structures of spinors and Clifford algebras arise naturally from our discussion of distinctions and iterants. To see this, recall that an iterant $[A, B]$ is resolved into its temporal (t) and polar (x) components via

$$\begin{aligned}[A, B] &= [t + x, t - x] \\ &= t[1, 1] + x[1, -1] = t + x\sigma.\end{aligned}$$

We let 1 denote the iterant $[1, 1]$ since this iterant acts as an identity element in the algebra of iterants. We let $\sigma = [1, -1]$ and note that $\sigma * \sigma = [1, -1] * [1, -1] = [1, 1] = 1$.

In special relativity σ *corresponds to a direction in space.* This provides the clue for a generalization. Let V^n be an n-dimensional vector space over the real numbers. Let $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be a basis for V^n and let us suppose that elements $\sigma = x_1\sigma_1 + \ldots + x_n\sigma_n \in V^n$ are part of an algebra structure $\mathcal{A}^n \supset V^n$ satisfying the following properties:

1. $\mathcal{A}^n$ is closed under multiplication and addition.

2. If $\sigma = x_1\sigma_1 + \ldots + x_n\sigma_n$ and $||\sigma||^2 = x_1^2 + x_2^2 + \ldots + x_n^2 = 1$ then $\sigma * \sigma = 1$.

3. $\mathcal{A}^n$ is associative under multiplication and addition. Multiplication distributes over addition. For real numbers $x, y : (xv)(yw) = (xy)(vw)$ where $v, w \in V^n$. Scalars such as 1 form an extra dimension so that $\{1, \sigma_1, \ldots, \sigma_n\}$ is linearly independent.

Call $\mathcal{A}^n$ a *direction algebra for* V^n. Any direction σ in V^n will have square equal to 1. Thus 1 and σ can be used as an iterant basis for any spatial direction σ.

Proposition. $\mathcal{A}^n$ will be a direction algebra exactly when

(i) $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_n^2 = 1$

and

(ii) $\sigma_i\sigma_j = -\sigma_j\sigma_i$ for $i \neq j$.

Hence $\mathcal{A}^n$ is a Clifford algebra.

PROOF: (i) follows immediately from condition number 2. To see (ii) note that if $\sigma = x_1\sigma_1 + \ldots + x_n\sigma_n$ with $x_1^2 + \ldots + x_n^2 = 1$ then

$$1 = \sigma * \sigma = x_1^2\sigma_1^2 + \ldots + x_n^2\sigma_n^2 + \sum_{i \neq j} x_i x_j \sigma_i \sigma_j$$

$$1 = 1 + \sum_{i \neq j} x_i x_j \sigma_i \sigma_j.$$

Hence $0 = \sum_{i<j} x_i x_j(\sigma_i\sigma_j + \sigma_j\sigma_i)$. Since this equality is true for all choices of x_i and x_j, it follows that $\sigma_i\sigma_j + \sigma_j\sigma_i = 0$ whenever $i < j$. This completes the proof.

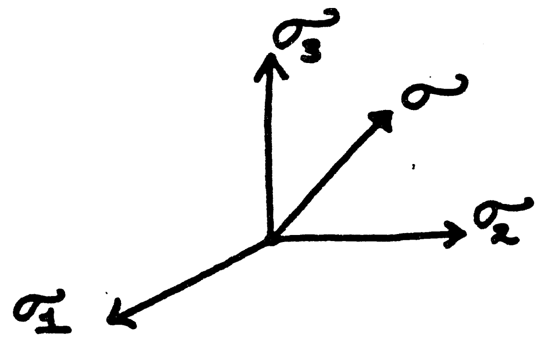
This argument is formally identical to Dirac's production of spinor formalisms in his theory of the electron [5]. Here however, we have made the argument on the ground of distinctions, iterant algebra and motivations from special relativity. In this generalization we have seen how to associate vectorial elements $t + x\sigma$ where σ ranges over an n-dimensional vector space.

Curiously, the basic iterant structure has now become parametrized by the space V^n. We have weighted distinctions or iterants of the form

$$[A, B]_\sigma = \left(\frac{A+B}{2}\right) + \left(\frac{A-B}{2}\right)\sigma.$$

Each distinction has become equipped with a vector direction in a higher dimensional space.

All of this is perfectly understandable in the context of special relativity. Here $n = 3$ and we can even obtain closure in $V = \mathbf{R}^3$ via $\sigma_1\sigma_2 = -\sqrt{-1}\sigma_3$, forming the Pauli algebra.



$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1$$
$$\sigma_1\sigma_2 = -\sqrt{-1}\sigma_3$$
$$\sigma_2\sigma_3 = -\sqrt{-1}\sigma_1$$
$$\sigma_3\sigma_1 = -\sqrt{-1}\sigma_2$$

In fact the Pauli algebra is represented by

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}$$

so that

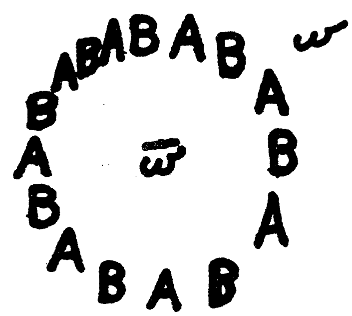
$$e = t + x\sigma_1 + y\sigma_2 + z\sigma_3 = \begin{pmatrix} t+x & y+\sqrt{-1}z \\ y-\sqrt{-1}z & t-x \end{pmatrix}.$$

In this way an event is 4-dimensional spacetime and is expressed as a Hermitian matrix

$$H = \begin{pmatrix} t+x & y+\sqrt{-1}z \\ y-\sqrt{-1}z & t-x \end{pmatrix}.$$

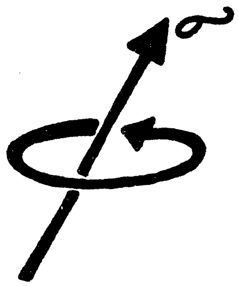
This is an extraordinary form. We see, within it, the original iterant structure $[t+x, t-x]$ displayed as the main diagonal. This diagonal is flanked by a complex number $w = y + \sqrt{-1}z$ and its conjugate $\overline{w} = y - \sqrt{-1}z$. Thus the original iterant becomes the boundary dividing the sides of a distinction labelled by $\overline{w}$ and w:

$$\begin{array}{ccc} & \vdots & \\ & A & \\ & B & \\ & A & \\ \overline{w} & B & w \\ & A & \\ & B & \\ & A & \\ & \vdots & \end{array} \quad \text{or}$$



Here we move to the realm on pattern. The boundary has resolved into vibration, and the sides are conjugate imaginaries.

This is but one take, one possible viewpoint, on the event as complex pattern. By associating a direction or *axis* σ to a distinction D,



$$D = [A, B]_\sigma$$

that distinction partakes of a larger domain of interaction, and of the possible combination and recombination into new patterns in a hierarchy of form. (It would be well to compare this abstraction with the image of a gyroscope creating its own distinct axis in space.)

It is here that this discussion touches the structure of the combinatorial hierarchy [1], but I have only indicated the general principles of this correspondence. More work is needed in this domain.

Appendix 3. Language.

This appendix begins a discussion of certain issues in linguistics that are related to our relativistic calculus of distinctions, and to the algebra of iterants. Iterant pairs occur continually in language. For example: "scotch and soda." One does not say "soda and scotch." One can make a list:

scotch/soda
observer/observed
here/now
heaven/earth
name/that which is named
...

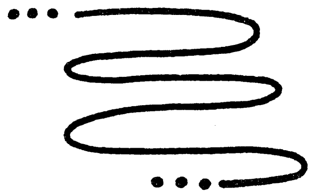
It is fascinating to speculate on the nature of these orderings. They do not constitute a preference for one member of the pair - only the fact of ordering. This provides the first non-numerical level of our structure. It remains to be seen whether the patterns of algebraic/numerical relativity apply in the domain of speech.

In terms of iterants, one may speculate that a sentence is a freeze on the process exemplified by its own unfolding - through repetition, sounding, and the associations of meaning. The simplest unfolding is pure repetition. Thus "I am that." unfolds to become the process

...I am that I am that I am that I am...

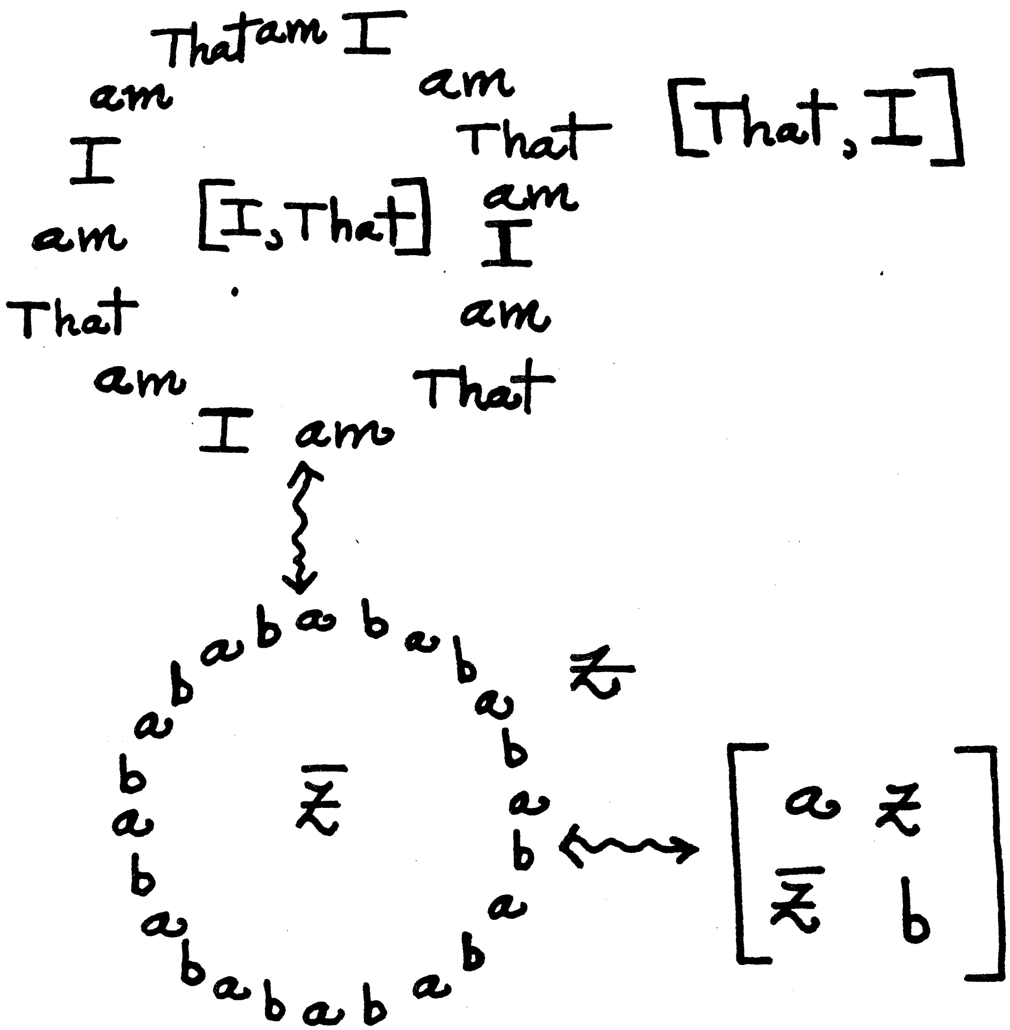
In this sense our speech is a necessary freeze or projection of the vibratory biological process of being in the world.

What process does the mind echo in response to each fragment of speech? The simple sentence "I am that." tempts this author to read it forward, then backward, then forward, in a weaving process



that forms

...I am that am I that am I that...

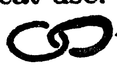


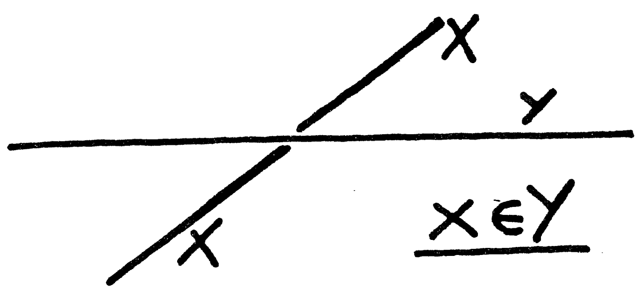
In this explication of the sentence, we already see the advent of the higher dimensional event-structure exemplified by the Hermitian matrix of the last section. The specifics of *that* and *I* can be chosen, ordered and frozen out. But being (am) is verbal, vibratory. To tell the story well it had best remain symbolized as boundary, glue, process and divider.

In this sense even a single sentence becomes the possibility for an event that is a conversation, a conversation among those embodiments of mind that would pick up its multifold components and discuss them in the chorus of its parts.

It is in the arena of *conversation* that the ideas and formalisms discussed in this paper come forward most strongly. A conversation is a taking of turns, a passing back and forth between the viewpoints of two observers. Yet, in the form of the conversation, these two viewpoints become unified into a context that is the conversation. The multitude of transformations of emphasis become one pattern of agreement that unfolds into the particpants. The participants themselves become sides of the distinction that the conversation is. And "they" move back and forth in the change of emphasis that is the rhythm of exchange, the taking of turns.

In the conversation there is ample room for a multidimensional patterned space of relativity, less restricted than the more numerically-based space for physics. These are only hints in the direction of linguistic context, yet even in the domain of hints the ideas beckon to be pursued and articulated. I believe that the key to this articulation will arise through the dvelopment of appropriate non-numerical (diagrammatic, geometric, topological) mathematics, and through the interest generated by these patterns of interconnection. The world of our speaking is the world of our being.

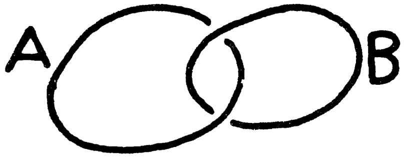
A few words about the non-numerical approach: Diagrammatic formalisms can be of great use. Thus the *linking* of participants A and B in conversation can be diagrammed as A  ^{B}. This is actually the diagram of a link in 3-space, but it can also be interpreted as the mutual relationship that each participant has to the conversation as a whole. In fact, we may formally let $X \in Y$ denote the relation that X ***undercrosses*** Y:



This gives rise to a non-standard set theory where a set may belong to itself:

$$X \in X$$
$$X = \{X\}$$

and two sets may each be members of the other:



$$A = \{B\}$$
$$B = \{A\}$$

Here is a non-numerical context for the structures of conversation. And within this context the transformations and patterns of relativity will appear again. (Note that the two viewpoints $A = \{B\}$, $B = \{A\}$ correspond to A ***listening to*** B and to B ***listening to*** A - two possible interpretations of $[A, B]$ and $[B, A]$. Compare [9], [11], [13].)

Appendix 4. Binocular Vision.

I believe that it is worthwhile to compare the calculus of distinctions given here with a model for binocular visual perception outlined in ***Foundations of Cyclopean Perception*** [7] by Bela Julez. Julez proposes a model for individual eye's perceptions as a grid or lattice of polarity choices. Thus at each lattice point is chosen $[1, -1]$ or $[-1, 1]$. The two grids (left eye and right eye) are compared by a superposition that (in analogy to magnetic fields of the magnets $[1, -1]$, $[-1, 1]$) tends to *rotate* each of the individual polarities in the lattice - trying to bring the two views in alignment. In our terms, the result is to replace each individual polarity $[1, -1]_\sigma$ by a *transform* $[\lambda_\sigma, -\lambda_\sigma^{-1}]$ with the value of λ_σ corresponding to the apparent *depth* of this point in the perceptual field. Here is seen a deeply contextual version of our model, and much possibility for further questions and creations.

Appendix 5. Observations and Quantum Mechanics.

We have used an abstraction of an observer as an operator that changes the emphasis on two sides of a distinction as in $\mathcal{O}[A, B] = [\lambda A, \lambda^{-1} B]$. In this sense the (abstract) observer is also a distinction, and we have made the identification $\mathcal{O} = [\lambda, \lambda^{-1}]$ in the iterant algebra so that $\mathcal{O}[A, B] = \mathcal{O} * [A, B] = [\lambda, \lambda^{-1}] * [A, B]$. In this epistemology the observer is an operator ***at the same level*** as that which is observed.

It is also possible to consider observation of the sides of a distinction. Thus, in the case of ordered pairs, we can define $\pi_1[A, B] = A$ and $\pi_2[A, B] = B$. When the elements of the ordered pairs are themselves distinctions, this is an appropriate move. Note that this sort of operation can also be identified with the extreme case of an observer who gives ***zero-emphasis*** to one side or the other as in

$$P_1[A, B] = [A, 0]$$
$$P_2[A, B] = [0, B].$$

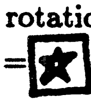
We have tacitly excluded operations of this type from the discussion because they are not invertible - information about the contents of the side that is assigned zero is lost. Let us call operations of type P_1 and P_2 *projections*. Note that

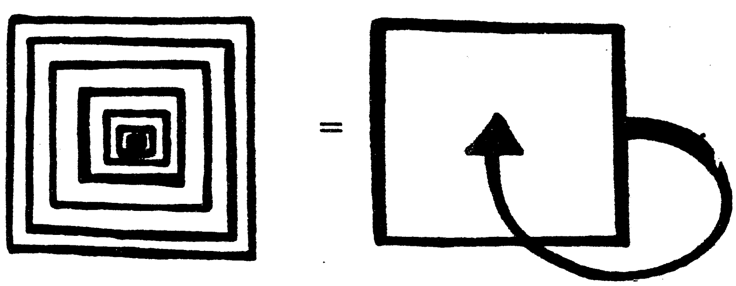
$$P_1^2 = P_1$$
$$P_2^2 = P_2 \qquad \text{and}$$
$$P_1 + P_2 = 1$$

in the sense that $P_1[A,B] = [1,0] * [A,B]$ and $P_2[A,B] = [0,1] * [A,B]$ so that $P_1 \equiv [1,0]$, $P_2 \equiv [0,1]$ and $[1,0] * [1,0] = [1,0]$ etc. Projections of this type form the basis of the model for observation in quantum theory [5]. (That is, in quantum theory an observable is an operator H on a Hilbert space and the process of observation corresponds to the projections to the eigenspaces of this operator.)

In the realm of value and distinction this spearation into "relativistic" and "quantum mechanical" operators is clearly a matter of choice. Both are special cases of multiplication in the iterant algebra. Nevertheless, even here the meaning takes a subtle shift. Since projection loses information it must be treated differently than a balanced change of emphasis. It is this basic mathematical difference that informs the greater differences in the corresponding physical theories.

At the level of patterns of distinction the consideration of an operator P such that $P(P(X)) = P(X)$ for all X ($P^2 = P$) leads to the more general idea of solutions to $P(V) = V$. If V is non-numerical, then P may be a rotation or other self-similarity transformation. Thus Rot$[\rightarrow] = \leftarrow$ (rotate by 180°) implies that Rot$[-] = -$ since an undirected segment is *indistinguishable from itself* after a rotation by 180°.

Similarly, if $P =$ "put a box around it" as in P  $=$ , then $PJ = J$ when J is an infinite nest of boxes:

$J =$ 

J is an *eigenform* for the operator P. The notion of eigenform goes back to Heinz von Foerster in the cybernetic domain [16]. See also [7] and [9], [11], [12], [13].

The point I want to make is that as soon as we begin to consider projection operators, other domains begin to open up before us. In the realm of distinction and pattern the development is natural and rapid - producing many-valued logics, fractal geometry and recursions of all kinds. In physics the pattern of the projection operation is at the heart of quantum theory, and these formal developments have been restricted by the requirement of numerical eigenvalues (versus eigenforms) to ***hints in the formalism*** such as Dirac's formal projection operators $|\phi\rangle\langle\psi|$. (Here $\langle\psi||\phi\rangle = \langle\psi|\phi\rangle = 1$ so that if $P = |\phi\rangle\langle\psi|$ then $P^2 = |\phi\rangle\langle\psi||\phi\rangle\langle\psi| = |\phi\rangle 1\langle\psi| = |\phi\rangle\langle\psi| = P$ whence $P^2 = P$.) In the patterned realm we can go ahead and investigate the entire panoply of distinctions: creating, evaluating and projecting process and possibility. The resonance with physics will resound throughout in a great fugue.

One last comment about eigenvalues: In Appendix 2 I pointed out how the event in 4-dimensional spacetime naturally takes the form

$$H = \begin{bmatrix} a & z \\ \bar{z} & b \end{bmatrix} = \begin{bmatrix} T+X & Y+\sqrt{-1}\,Z \\ Y-\sqrt{-1}\,Z & T-X \end{bmatrix},$$

and that H could be viewed as a second-level iterant (pairing conjugates across the vibratory pattern boundary $\ldots ababa \ldots$).

However, H is a Hermitian matrix and, as such, one can enquire of it its numerical eigenvalues. They are: $T \pm \sqrt{X^2+Y^2+Z^2}$. In other words, the event ***when operating on*** (observing) *itself* can determine its time T and geometrical radius $R = \sqrt{X^2+Y^2+Z^2}$. In so-doing the event becomes a local observer. In this sense, the local observer is linked with quantum mechanics. (Compare [9]).

On extending consideration to include the eigenforms, the relations of our discussion with patterns of language and communication become more significant. A word, a sentence, a paragraph or a book, each is a structure whose implicit reference is unfolded (projected) by context and observer. Thus distinction and pattern work throughout. Each form may be seen as a pattern of patterns just as the sentence unfolds into words and these words into ideas that inform the sentence that inform the idea of the sentence, that inform the words. This implicate structure unfolds in deep resonance with creation of eigenform from its own operator. For more about this context see [9], [11], [12], [13], [17].

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