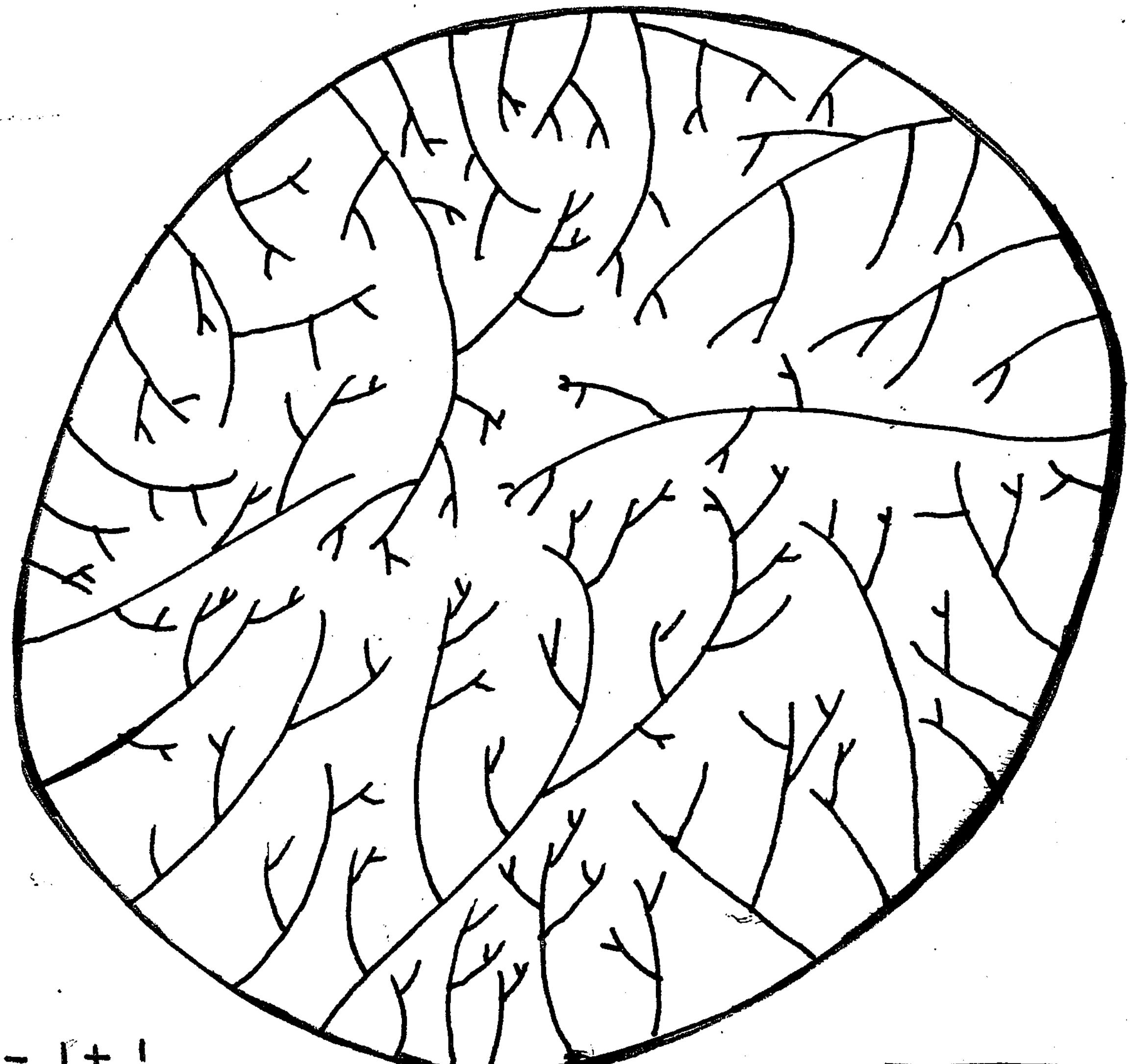
RECURSIVE FORMS

Louis H. Kauffman

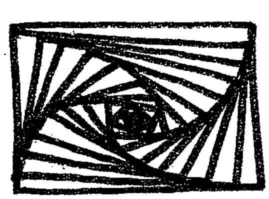


$$\frac{1+\sqrt{5}}{2} = 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}$$

$$\pi = \lim_{n\to\infty} 2^n \sqrt{2-\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n}}$$

$$\frac{1+\sqrt{5}}{2} = \left[1+\frac{1}{\quad}\right]$$

$$\pi = \left[2\cdot\quad\right]\sqrt{2-\sqrt{2+\quad}}$$

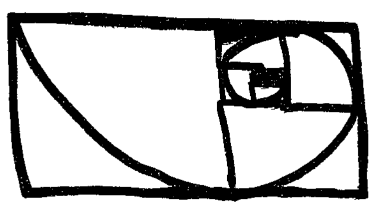


Recursive Forms

We call the geometric (and notational) forms in these notes <u>recursive forms</u>. They all have in common some sort of recursive or re-entering, looping process. Almost any such process can become a recursive form if translated into geometry. For example, here is the skeleton of the set theoretic counting sequence:

emptyset   1   2   3   . . .

Or a nest of golden rectangles:



Or an element of twist-form (see works of Joe Staley):

Many of the fractals of Benoit Mandelbröt or the graftals of the graphics community are also embraced by this concept of geometric recursion.

This is our topic for discussion and experiment.

LK

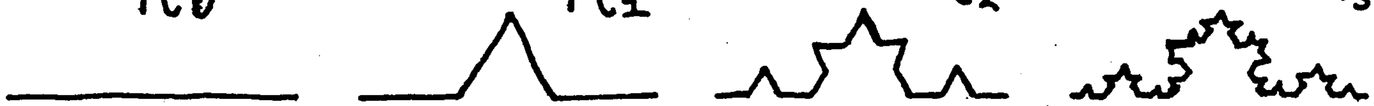
Fractals and Julia Sets

Louis H. Kauffman

I. Benoit Mandelbrot defines a fractal $S \subset \mathbb{R}^n$ as a set whose Hausdorff dimension is greater than its topological dimension.

Since Hausdorff dimension may be a bit unfamiliar, lets look at an example (the Koch snowflake):

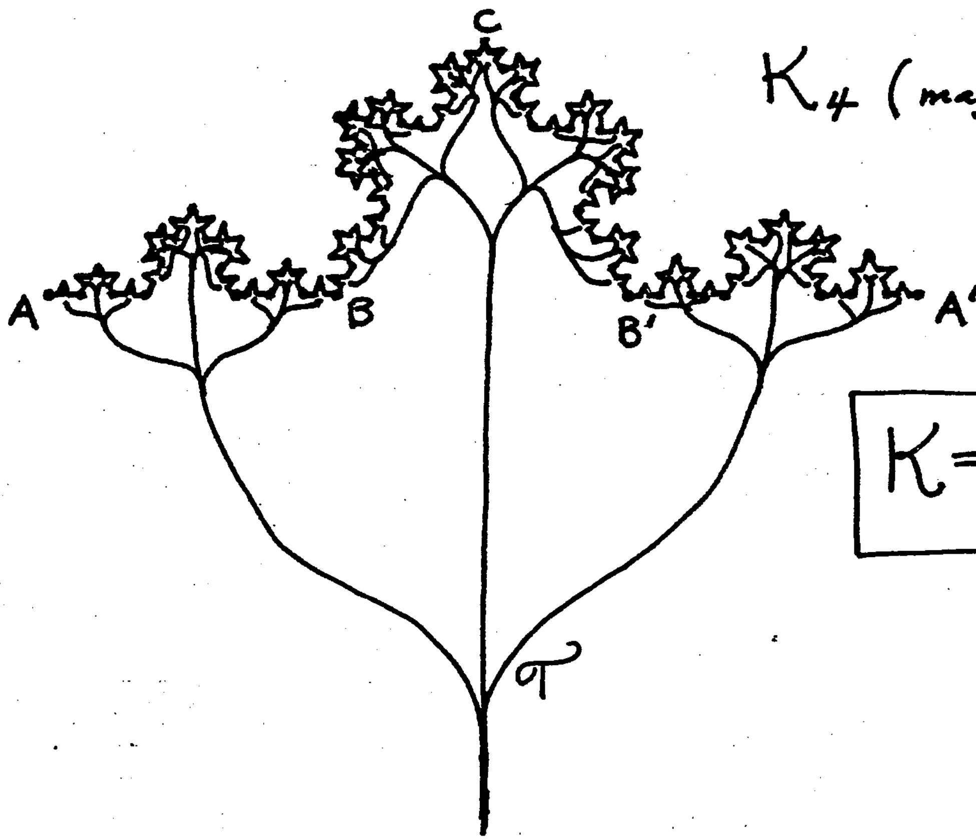
K_0 $\quad$ K_1 $\quad$ K_2 $\quad$ K_3



$\ell(K_0) = 1 \qquad \ell(K_1) = \frac{4}{3} \qquad \ell(K_2) = \left(\frac{4}{3}\right)^2 \qquad \ell(K_3) = \left(\frac{4}{3}\right)^3$

$$\boxed{\ell(K_n) = 4^n\left(\frac{1}{3^n}\right)}$$ Length of K_n

K_4 (magnified by 4)

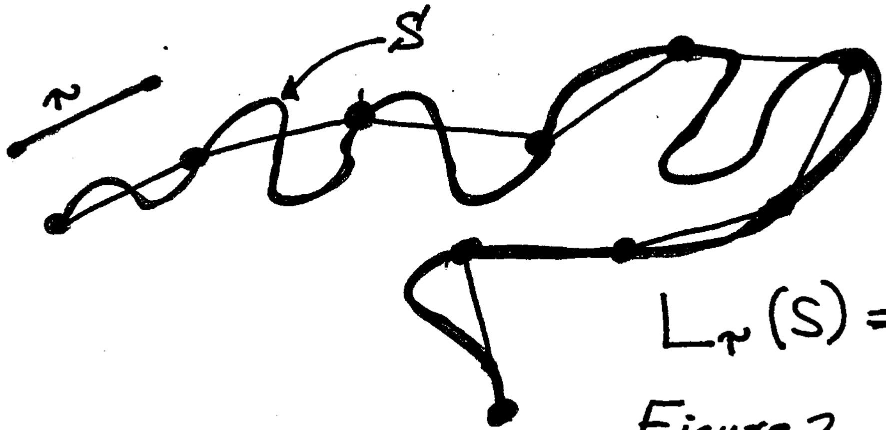


$$\boxed{K = \lim_{n\to\infty} K_n}$$

Figure 1

The Koch snowflake $K \subset \mathbb{R}^2$ is the limit of the sequence of approximating curves $\{K_n\}$. Each K_{n+1} is obtained from K_n by removing the middle third of each line segment and then replacing two copies of this middle third by forming two sides of a triangle as shown in Figure 1. As a consequence, the nth approximation (K_n) to K decomposes into 4^n segments, each of length $1/3^n$. Thus K_n has length given by the formula $\ell(K_n) = (4/3)^n$. Since this goes to ∞ as n approaches infinity, we see that the snowflake K is non-rectifiable.

The snowflake K is a simple mathematical version of natural patterns, such as coastlines, whose length is difficult to measure because <u>the measured length depends upon the basic scale of the measuring rod.</u> Thus, in measuring K, a ruler of length 1 ignores all variation within a unit length, and returns a measurement of 1. A ruler of length (1/3) can be laid out (4) times along K to return a measurement of 4/3; this is the length of K_1. In general, let $L_r(K)$ denote the length of K measured at scale r.



$L_r(S) = 9$

Figure 2

We have $L_{\left(\frac{1}{3}\right)^n}(K) = \left(\frac{4}{3}\right)^n$.

Thus for $r = \frac{1}{3^n}$ we have $L_r = 4^n r$. To see a functional dependence on r note that

$$\log L_r = n \log 4 + \log r$$
$$= n \log 4 - n \log 3$$
$$= \left(1 - \frac{\log 4}{\log 3}\right)(-n \log 3)$$
$$\log L_r = \left(1 - \frac{\log 4}{\log 3}\right)(\log r).$$

Hence $L_r = r^E$ where

$$E = 1 - \left[\frac{\log 4}{\log 3}\right].$$

It is clear from the geometry of the construction of K that $D = [\log 4/\log 3]$ is the significant term in this formula. <u>D is the Hausdorff dimension of K.</u>

This is, of course, <u>not</u> the definition of the Hausdorff dimension. It shows how the concept of generalized dimension is related to scaling measurement.

What is this concept? Imagine a fuzzy bit of curve appearing between two points A and B (Figure 3). Perhaps the light is dim and this fuzziness represents two-dimensionality. If so then the <u>area</u> so represented is approximated by $|A-B|^2$ where $|A-B|$ is the distance from A to B. If it is a one-dimensional curve, then its <u>length</u> is approximated by $|A-B|^1$. <u>The exponent is the dimension</u>.

Accordingly, we suggest that its D-dimensional area is approximated by $|A-B|^D$. What dimension is appropriate to a given curve? Using points A, B, C, B', A' for the Koch curve we see that we would want the formula

$$|A-A'|^D = 4|A-B|^D$$

since the Koch curve consists of 4 similar copies of itself, each of side-length $|A-B|$. [Any reasonable notion of D-dimensional area should add under disjoint unions and multiply under a similarity factor.]

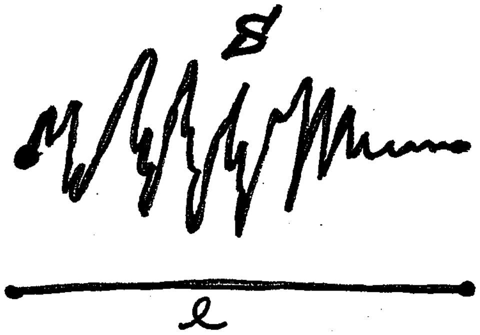
Let $\ell = |A-A'|$ so that $|A-B| = \ell/3$. Then the formula becomes

$$\ell^D = 4\left(\frac{\ell}{3}\right)^D$$

or

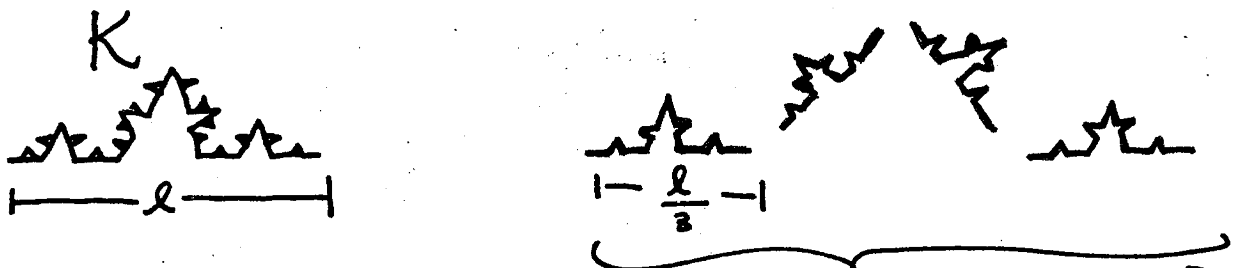
$$1 = 4/3^D .$$

Hence $3^D = 4$ and $D = \log 4 / \log 3$, as predicted. This explains the calculation of fractional dimension on the basis of the self-similarity property of K.



$$\mu_D(S) \approx l^D$$

Approximating D-Dimensional Area



$$\underbrace{\mu_D(K) \approx l^D} \qquad \underbrace{\mu_D(K) \approx 4\left(\frac{l}{3}\right)^D}$$

$$l^D = 4\left(\frac{l}{3}\right)^D$$
$$\Rightarrow D = (\log 4)/(\log 3)$$

Figure 3

Finally, the actual definition of fractional dimension runs as follows: Let $\mathcal{S} \subset \mathbb{R}^2$ be the set to be measured. Cover $\mathcal{S}$ with a finite collection of squares $\mathcal{U}$ (We assume $\mathcal{S}$ is bounded and closed.) Let $\mu_D(\mathcal{U}) = \sum_{u \in \mathcal{U}} \ell(u)^D$ where $\ell(u)$ is the length of the side of the square $u \in \mathcal{U}$. If there is a unique D such that the limit over covers $\mathcal{U}$ of of $\mathcal{S}$ goes neither to zero nor to infinity, then we say that $\mathcal{S}$ has Hausdorff dimension D.

It is a good exercise to see that this works out to give $D = \log 4/\log 3$ for the Koch Snowflake.

II. Not all fractals are self-similar but self-similar fractals form a rich sub-class, and the theme of self-similarity is quite pervasive in this subject. For example, the matter of different scales of measurement has its roots in antiquity. Consider the following ancient geometric proof of the irrationality of $\sqrt{2}$:

I shall prove that the diagonal of a square is <u>incommensurable</u> with its side. Recall that incommensurable means that the diagonal and the side have no common measure, that they cannot both be written as integer multiples of one length.

Refer to Figure 4.

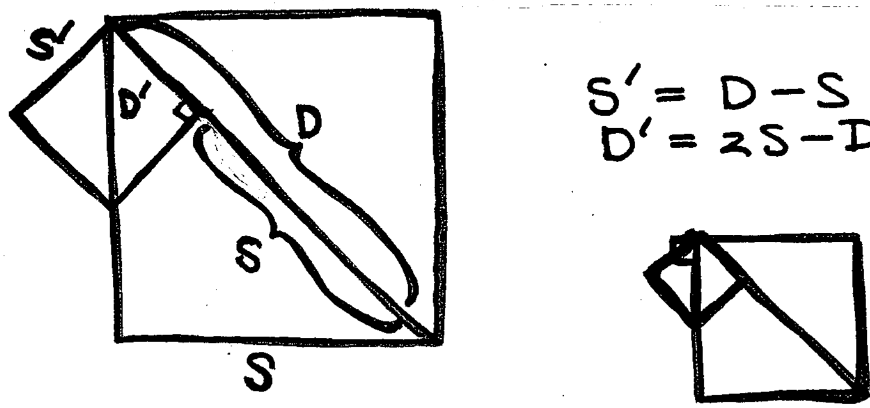


Figure 4

The larger square has side S and diagonal D, so that $D = \sqrt{2}\,S$ by the Pythagorean Theorem. A smaller square is built from the larger square as shown in the figure. The new side is obtained by excising a length S from the diagonal D. Hence $S' = D - S$ and $D' = \sqrt{2}\,S' = \sqrt{2}\,D - \sqrt{2}\,S = \sqrt{2}\sqrt{2}\,S - D = 2S - D$. We have $S' = D - S$, $D' = 2S - D$ and $S' < S$, $D' < D$. If the process is repeated then the lengths of the diagonals and sides go to zero. But these equations show that if D and S are commensurable by a length L then so are D' and S'. This leads at once to a contradition, since repetition of the process will eventually lead to squares of size (side) smaller than the L which is supposed to measure them! Self-similar process engenders an escape from rationality.

As if this were not all, consider the algebraic description:

$$\left\{\begin{array}{l} S' = D - S \\ D' = 2S - D \end{array}\right\} \Rightarrow \left\{\begin{array}{l} S = S' + D' \\ D = 2S' + D' \end{array}\right\}$$

$$\Rightarrow \quad D/S = \frac{(S'+D')+S'}{(S'+D')} = 1 + \frac{S'}{S'+D'}$$

$$\Rightarrow \quad D/S = 1 + \frac{1}{1+(D'/S')}$$

$$\Rightarrow \quad \sqrt{2} = 1 + \frac{1}{1+\sqrt{2}}$$

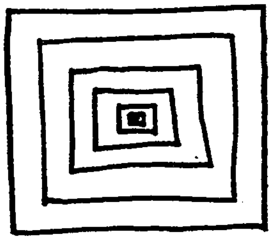
Iterating the process then yields the continued fraction expansion for $\sqrt{2}$:

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

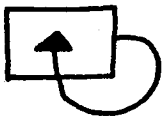
The interminable self-similarity of this expansion reflects the irrationality of root-two.

We shall return to the theme of quadratic iteration in section IV.

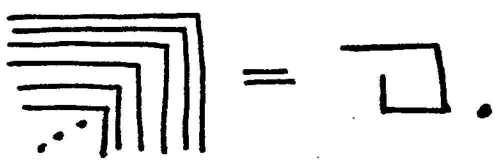
III. Self-Similar Forms

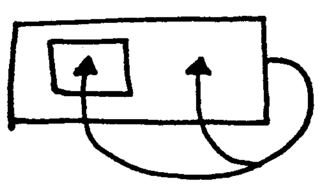
 $= X$ whence $X = \boxed{X}$

(Infinite concatenation of squares)

$\therefore$ $X =$  .

Here we are playing with notations that exhibit self-similarity by showing how a form re-enters its own indicational space. Sometimes this requires ingenuity and sometimes it leads to a very concise definition of a form.

[I get this idea from G. Spencer-Brown's book Laws of Form where he denotes  This is the simplest snake that eats its own tail.]

A second example is the Fibonacci Form $F = \boxed{\boxed{F}\ F} =$ 

So called, because in its depth levels are hidden the Fibonacci Series. See Figure 5. In this figure we

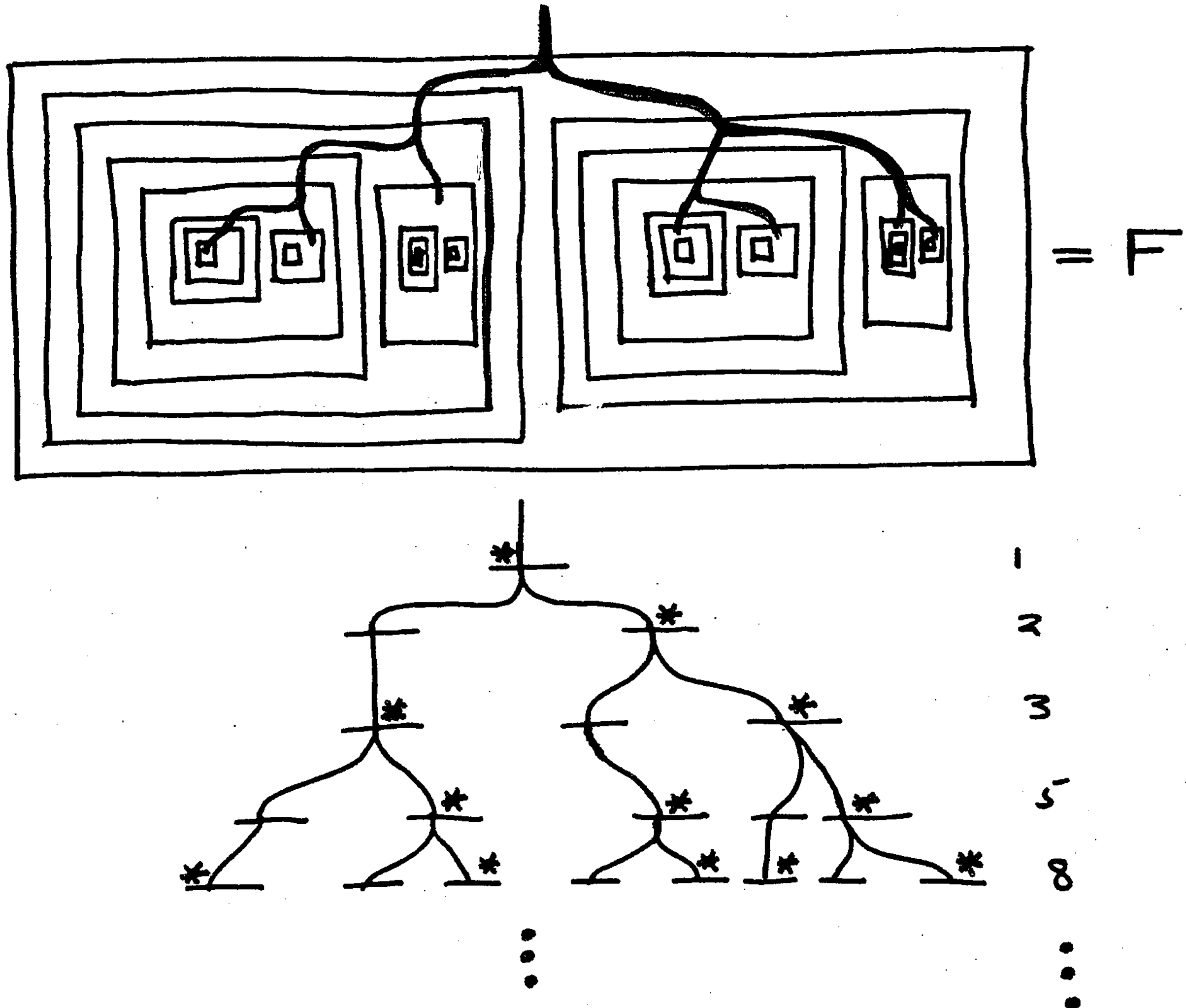
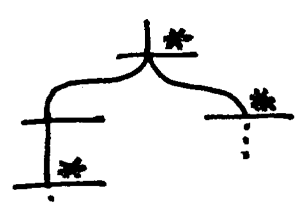


Figure 5

have elaborated the form F and drawn its characteristic structural tree. The tree branches according to the Fibonacci series 1, 1, 2, 3, 5, 8, ... with a Fibonacci number of nodes at each depth. Note how the tree is constructed from the nest of boxes by successive inward crossing. In the diagram a complete copy of the tree lies below each •. Thus



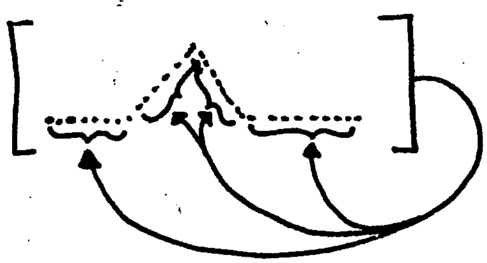
completely describes the tree just as

$F = \boxed{\boxed{F}\ F}$ defines the form.

With this introduction to self-similar forms, return to the Koch snowflake. We see that it is described by:

$K =$ 

or

$K =$  .

These attempts to indicate the re-entry are both leading and mis-leading! It is composed with an embedded tree that is an obvious relative of the form of re-entry in the drawing above. K is the set of ends of an embedded self-similar tree $\mathcal{T}$.

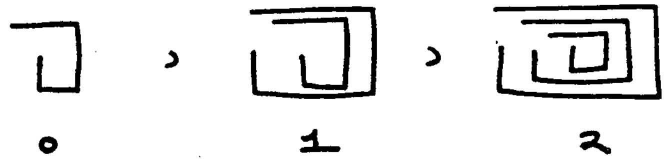
Remarkably, $\mathcal{T}$ is the tree associated with the form $T = \boxed{T\ \boxed{T\ T}\ T}$.
Just as the middle parts of K are singled out geometrically, in T they are singled out formally. <u>T embodies the abstract structure of K</u>.

<u>Other</u> <u>Examples</u>

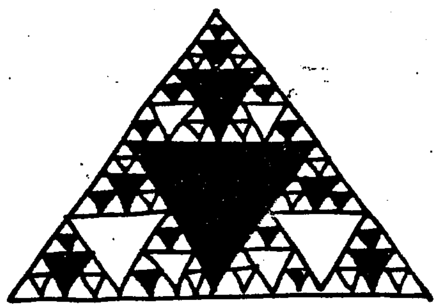
1. $z = a + \cfrac{b}{a + \cfrac{b}{a + \cdots}} = \left[a + \frac{b}{\ } \right]$

$z = a + \frac{b}{z}$ So "$\sqrt{-1} = \left[{}^{-1}\!/\ \right]$".

Sometimes This has numerical value.

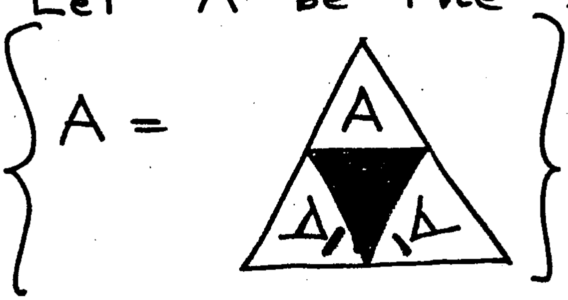
2. 

The tree for the n^{th} stage catalogs through its branching, the number of divisions of $\mathbb{R}^n$ by a set of hyperplanes.

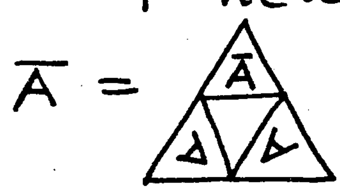
3. 

Let $\overline{A}$ = Photo-negative of A.

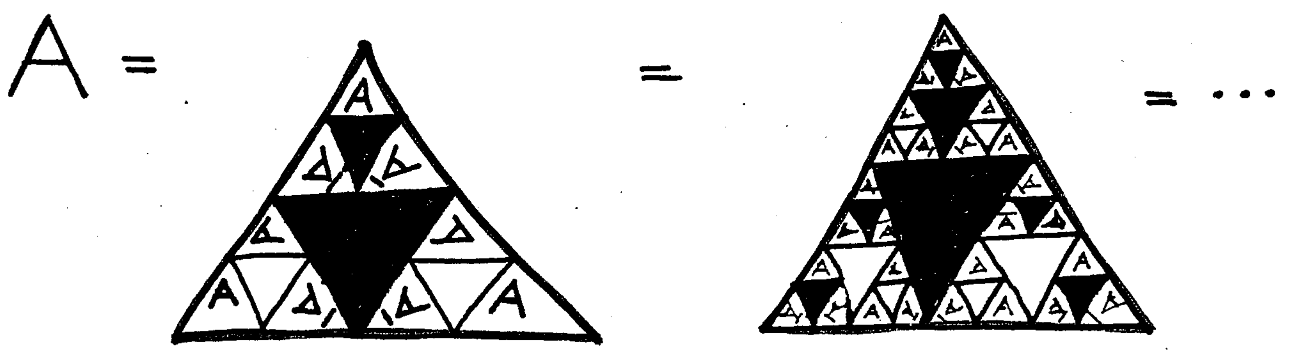
Let A be the form defined by



Then:



and



(See Mandelbrot —)

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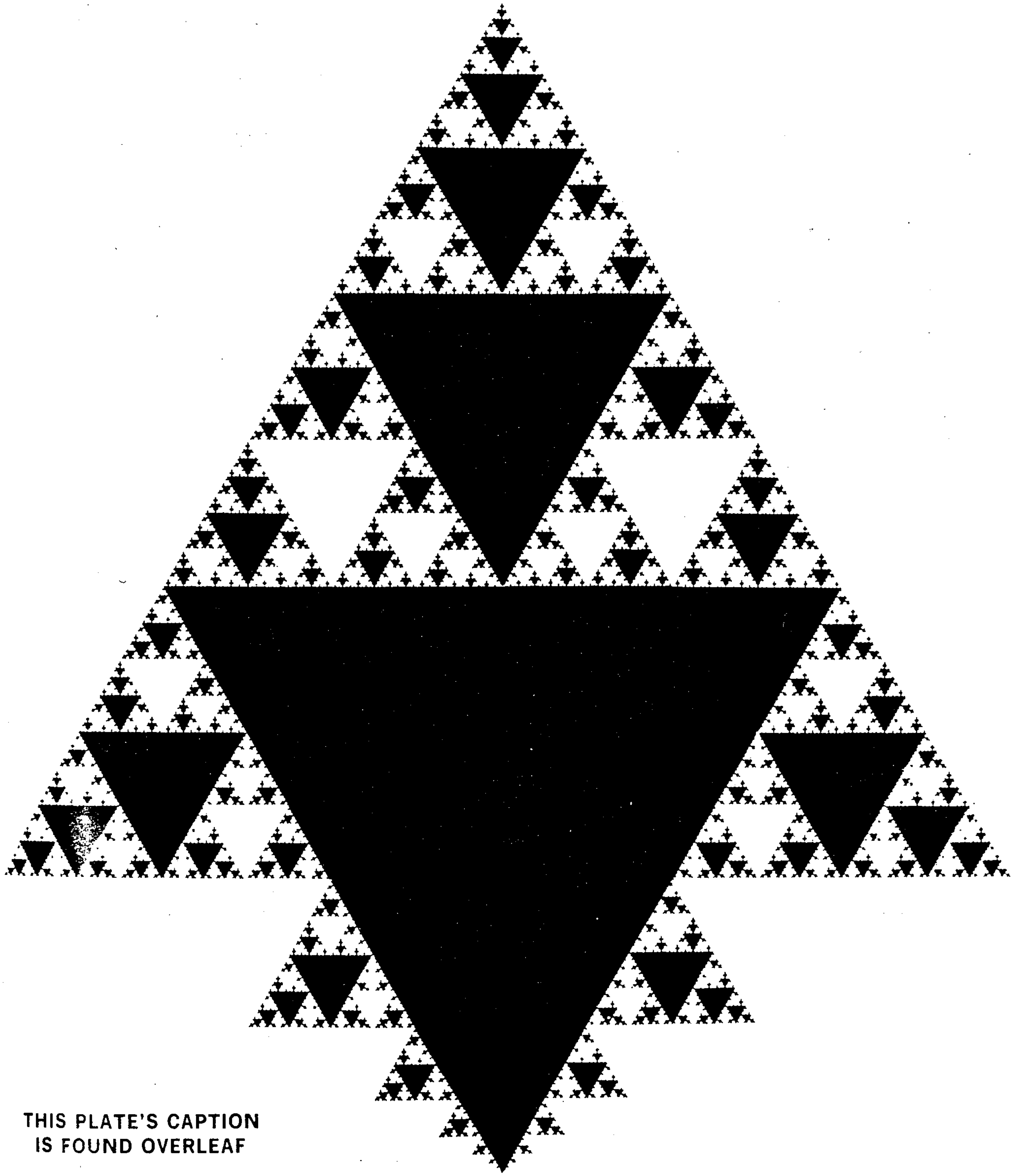


THIS PLATE'S CAPTION
IS FOUND OVERLEAF

4. Growth Rates

Given a re-entering form G.

G_n = # of divisions at depth n.

$$F = \boxed{\boxed{\boxed{\boxed{F}\ F}}\ \boxed{\boxed{F}\ F}} \Rightarrow \begin{aligned} F_0 &= 1 \\ F_1 &= 1 \\ F_2 &= 2 \\ &\text{etc...}\end{aligned}$$

Note: $\boxed{G}_n = G_{n-1}$.

Hence $F = \boxed{\boxed{F}\ F} \Rightarrow F_n = F_{n-2} + F_{n-1}$ ✓.

Define $\rho(G) = \underbrace{\lim_{n\to\infty} G_n / G_{n-1}}_{\text{The Growth Rate}}$

For F above: $\rho(F) = \lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$.

Now return to K (Koch curve) and its associated form $T = \boxed{T\ \boxed{T\ T}\ T}$

There are two associated forms:

$A = \boxed{A\ A\ A}$ indicating the three-fold cut.

$B = \boxed{B\ B\ B\ B}$ indicating the four-fold duplication.

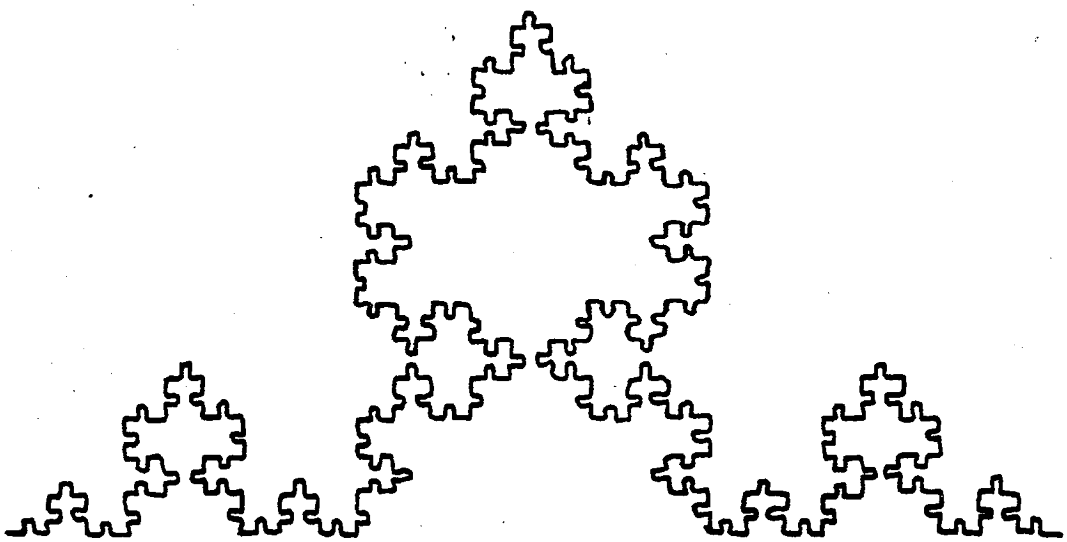
$\rho(A) = 3$, $\rho(B) = 4$.

Dimension $D(K) = \frac{\log \rho(B)}{\log \rho(A)} = \frac{\log(\text{duplication rate})}{\log(\text{cut rate})}$.

This holds for many simple recursions.
[And allows a defn of DIM for abstract forms.]

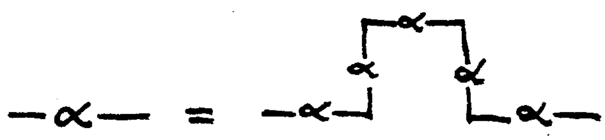
5.

$$\hat{\alpha} = \alpha \overset{\hat{\alpha}}{\alpha\rangle\ \langle\alpha} \alpha$$

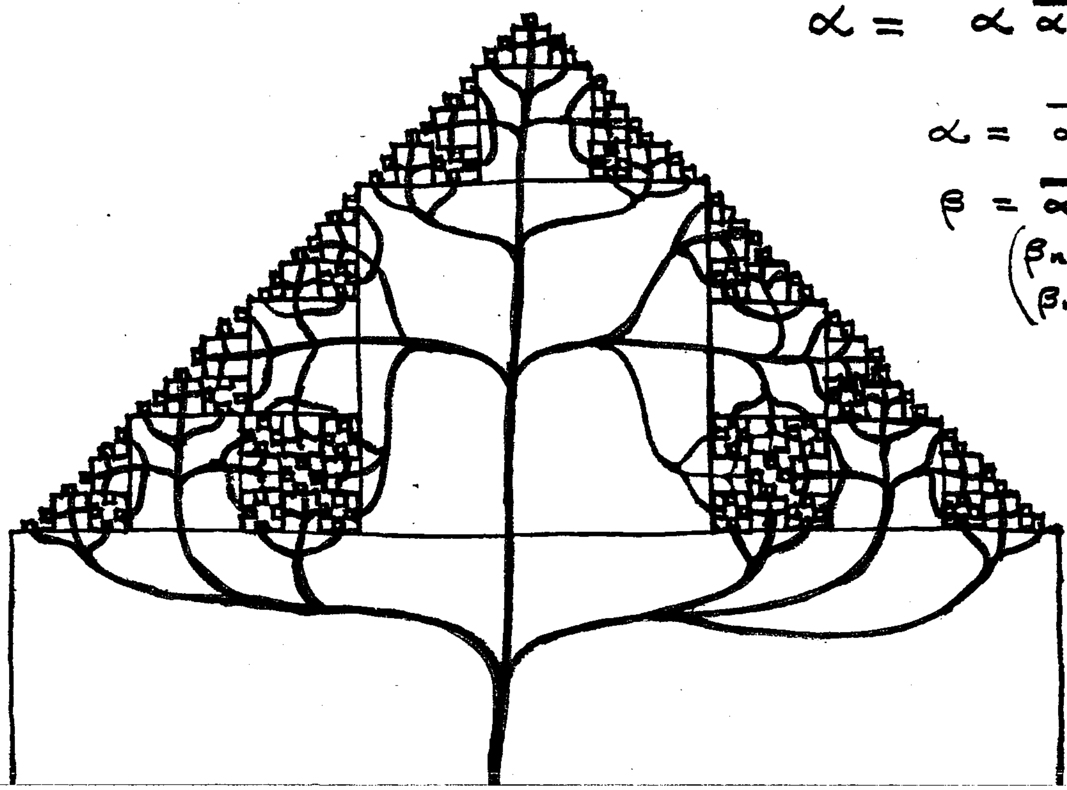


$D = \log 5 / \log 3$

$D = 1.464973521$



$$\alpha = \overline{\alpha\ \overline{\alpha\alpha\alpha}\,|\,\alpha}|$$



$$\alpha = \overline{\alpha\beta\alpha}|$$

$$\beta = \overline{\alpha\alpha\alpha}|$$

$$\begin{pmatrix} \beta_{n+1} = 3\alpha_n \\ \beta_{n+1}/\alpha_{n+1} = 3 \end{pmatrix}$$

Here we begin to see how the very simplest recursion can give rise to surprisingly complex patterns.

IV. Julia Sets

$$f(z) = z^2 + (A+Bi) = z^2 + c$$

$$J = \{ z \in \mathbb{C} \mid f^n(z) \text{ does not approach } \infty \text{ as } n \to \infty \}$$

Frontier

Algorithm: 1) Based on result that $\mathcal{D}_p = \{ f^{-n}(p) \mid n = 1, 2, \ldots \}$ is dense in J for any $p \in J$.

2) Let z_0 = root of $z = z^2 + c$ of largest absolute value.

3) Using

$$\pm\sqrt{a+bi} = \pm\sqrt{\frac{1+a}{2}} + \epsilon i \sqrt{\frac{1-a}{2}}$$

when $a^2+b^2=1$, $\epsilon = \text{sign}(b)$, form $f^{-1}(z_0)$, $f^{-1}f^{-1}(z_0)$, ...

4) One way to accomplish 3) is to use a random number generator to create a sequence of square roots. Plot these, then start back at z_0 and generate another sequence.

Better results are obtained by searching out all square roots to a certain depth and resolution.

The extraordinary forms of these sets underline how much more there is than meets the eye in the simple algorithm

$$\boxed{J = \pm\sqrt{-c \pm \;}}\,.$$

You may also enjoy thinking of Julia set for $f(z) = z^2 - \lambda$ as a set of points fixed under f. Hence it generalizes $z_0 = z_0^2 - \lambda$ or $z_0^2 = z_0 + \lambda$

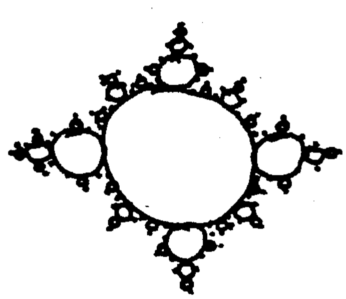
$$\boxed{z_0 = 1 + \lambda/z_0}$$

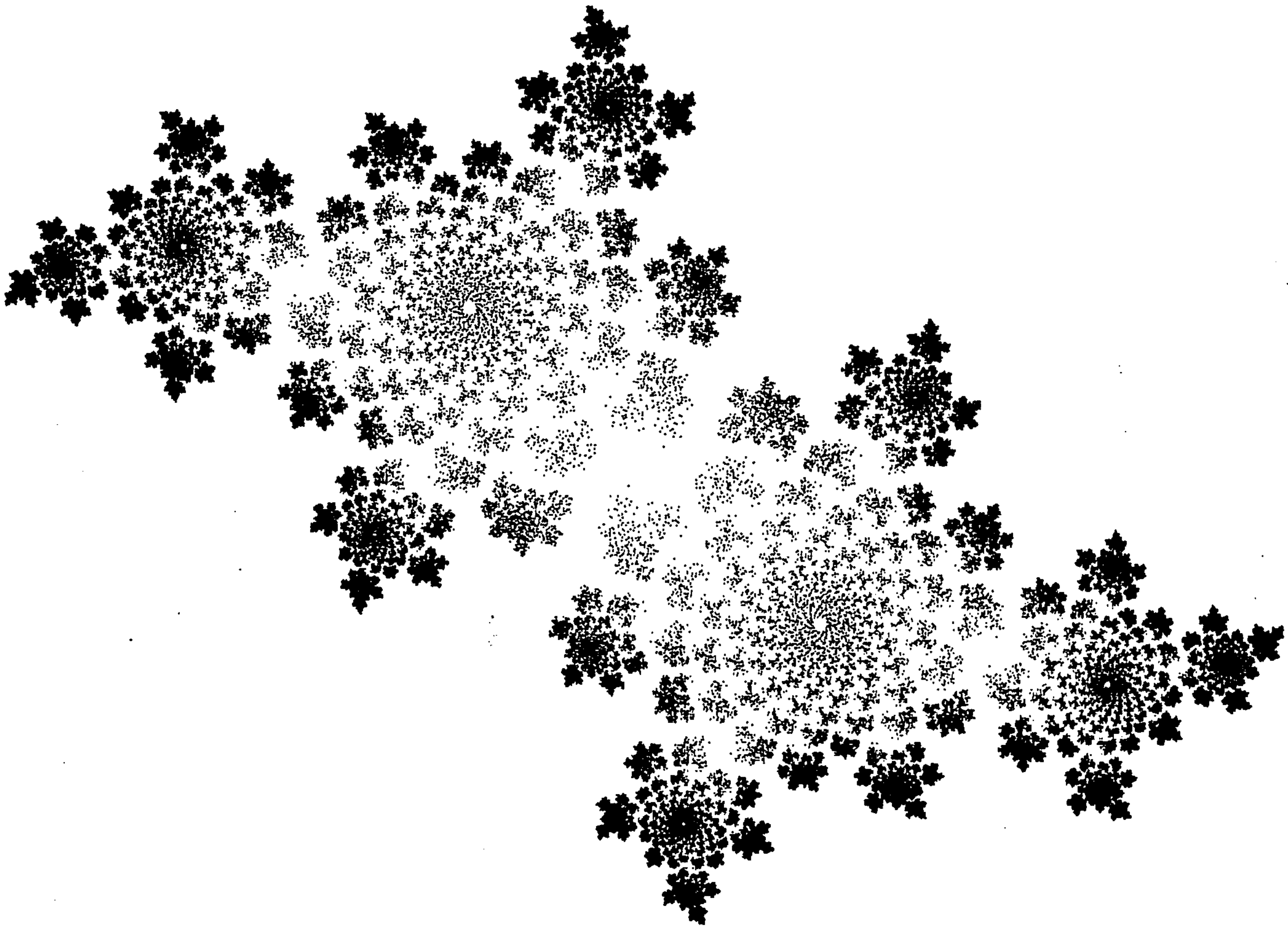
J is a "generalized continued fraction".

Refer to enclosed pictures.

Observe variety of forms.

<u>Question</u>: To what extent are the Julia sets for $z^2 - \lambda$ susceptible to simple combinatorial description?





$a = -0.400 + 0.600\, i$

number of points = 90888 stack depth = 450