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PRODUCTS OF KNOTS, BRANCHED FIBRATIONS AND SUMS OF SINGULARITIES

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INTRODUCTION

A *KNOT* $\mathscr{K} = (S^k, K)$ is an oriented k-sphere with an oriented 2-codimensional submanifold $K \subset S^k$. We say that a knot $\mathscr{L} = (S^l, L)$ is *fibered* if a fibration $S^l - L \xrightarrow{b} S^1$ is given such that the closure $F_t = b^{-1}(t)$ of any fiber of b is a closed submanifold of S^l with boundary $\partial F_t = L$. We shall discuss a product operation that associates to two knots $\mathscr{K}$ and $\mathscr{L}$, one of which is fibered, a new knot $\mathscr{K} \otimes \mathscr{L} = (S^{k+l+1}, K \otimes L)$. This construction has many useful properties. It is commutative (up to orientation), associative, and distributes over connected sums. The product of fibered knots is again fibered. Algebraic information about a product may be deduced easily from the factors. For example, the monodromy of a product of fibered knots is the tensor product of the monodromy of the factors. The Seifert pairing for a product is the tensor product of the Seifert pairings for the factors.

A particularly interesting class of fibered knots is the class of links of isolated complex polynomial singularities. That is, let $f: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a polynomial mapping such that $0 \in \mathbf{C}^n$ is an isolated critical point of f. Then the *link of* f is the knot $\mathscr{L}(f) = (S_\epsilon^{2n+1}, L(f))$, where $\epsilon > 0$ is sufficiently small and $L(f) = S_\epsilon^{2n+1} \cap f^{-1}(0)$. Milnor[17] showed that $\mathscr{L}(f)$ has a natural fibered structure. If $g: \mathbf{C}^{m+1} \to \mathbf{C}$ is another such mapping, then so is $f + g: \mathbf{C}^{n+1} \times \mathbf{C}^{m+1} \to \mathbf{C}$ given by $(f+g)(x, y) = f(x) + g(y)$. We show that $\mathscr{L}(f+g)$ and $\mathscr{L}(f) \otimes \mathscr{L}(g)$ are isomorphic as fibered knots. Thus the product operation gives a geometric construction for sum of singularities.

We show the corresponding result also for isolated singularities of real polynomial maps $f: \mathbf{R}^{k+1} \to \mathbf{R}^2$, $g: \mathbf{R}^{l+1} \to \mathbf{R}^2$, except that in certain low dimensions it is still an open question whether the isomorphism of $\mathscr{L}(f+g)$ and $\mathscr{L}(f) \otimes \mathscr{L}(g)$ preserves fibered structure. This isomorphism is also proved in a yet more general situation—"tame" singularities (Definition 1.3)— except that in low dimensions we now only get an h-cobordism of fibered knots.

The properties of knot product generalize and put in a clearer perspective many known results about complex polynomial singularities. In particular, Thom and Sebastiani[20] showed that the monodromy for $\mathscr{L}(f+g)$ is the tensor product of the monodromies of f and g. The same result was conjectured for Seifert pairings by A. Durfee and first proved by Sakamoto[19]. Our results generalize the Thom–Sebastiani theorem and the Sakamoto theorem to (and in fact beyond) the real polynomial case. Note that the low dimensional problems mentioned above are irrelevant here, since these homological invariants only depend on the h-cobordism class of a fibered knot. It is well known that the class of links of real polynomial singularities is a much more extensive class than that of links of complex polynomial singularities (see [15] and [17]). Every fibered knot is the link of a tame singularity. Thus these generalizations are very non-empty.

G. Bredon in [2] gave a suspension construction for knots, using O (n)-manifolds, which he used to give a geometric version of knot cobordism periodicity. His results also generalized results of Hirzebruch[9] and Erle[7] about K. Jänich's knot manifolds[10]. We observe that Bredon's construction corresponds in our context to forming $\mathscr{K} \otimes \mathscr{L}(z_1^2 + \cdots + z_n^2)$ and our results generalize Bredon's results. Our construction generalizes also results [12] and [18]. Some of the results were announced in [13].

In a final section we indicate how the product construction, its properties, and its relation to isolated singularities, generalizes to arbitrary codimension.

The paper is organized as follows. §1 discusses fibered knots, open books, and branched

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fibrations over D^2. A branched fibration over D^2 is a mapping $\tau\colon D^{l+1}\to D^2$ so that $\tau^{-1}(0)$ is homeomorphic to the cone over a knot $L\subset S^l$ and $\tau|D^{l+1}-\tau^{-1}(0)$ is a smooth fibration so that its further restriction to the boundary S^l-L gives a fibered structure for the knot $\mathscr{L}=(S^l, L)$. Thus a branched fibration is an analog of a degenerating family of algebraic varieties with degenerate fiber over the origin. Given a fibered knot $\mathscr{L}=(S^l, L)$ there is a naturally associated branched fibration over D^2.

In §2 we discuss branched fibrations along codimension two submanifolds of a given manifold and consider specific branched fibrations obtained by pull-back constructions. These generalize cyclic branched coverings.

§3 uses branched fibrations to define products of knots. This done as follows: Let $\mathscr{K}=(S^k, K)$ be a knot and (D^{k+1}, F) a codimension two embedding of an oriented spanning surface for K (that is, $\partial F=K$). Choose $\alpha\colon D^{k+1}\to D^2$ transverse to $0\in D^2$ so that $\alpha^{-1}(0)=F$. Form the pull-back

$$\begin{tikzcd} M \arrow[r] \arrow[d] & D^{l+1} \arrow[d, "\tau"] \\ D^{k+1} \arrow[r, "\alpha"] & D^2 \end{tikzcd}$$

where τ is a branched fibration for $\mathscr{L}$. Then $(S^{k+l+1}, K\otimes L)\cong(\partial(D^{k+1}\times D^{l+1}), \partial M)$. We show that the product construction is commutative and give a cut-and-paste description of $\mathscr{K}\otimes\mathscr{L}$.

If $\mathscr{K}$ and $\mathscr{L}$ are both fibered, with associated branched fibrations $\tau'\colon D^{k+1}\to D^2$ and $\tau\colon D^{l+1}\to D^2$, then $\mathscr{K}\otimes\mathscr{L}$ is isomorphic to the pair $(\partial(D^{k+1}\times D^{l+1}), T^{-1}(0)\cap\partial(D^{k+1}\times D^{l+1}))$ where $T\colon D^{k+1}\times D^{l+1}\to R^2$ is the mapping defined by the equation $T(x, y)=\tau'(x)-\tau(y)$. Thus, for fibered knots the product is directly analogous to the link of a sum of singularities.

§4 relates the product construction to links of algebraic singularities. We first show that $\mathscr{L}(f+g)$ and $\mathscr{L}(f)\otimes\mathscr{L}(g)$ are isomorphic for complex polynomial singularities. Then by more careful vector field arguments we prove this for isolated real polynomial singularities.

§5 shows that the product of fibered knots is fibered and gives an explicit description of the fibration.

In §6 we show that $\mathscr{K}\otimes\mathscr{L}$ has a spanning manifold with the homotopy type of the join of appropriate spanning manifolds for $\mathscr{K}$ and $\mathscr{L}$. This leads to the result about the monodromy of a product of fibered knots and to the calculation of a Seifert pairing for $\mathscr{K}\otimes\mathscr{L}$.

In §7 we verify that $\otimes$ is associative and that it distributes over connected sums.

§8 discusses group actions on knot products and Bredon's work mentioned above.

§9 is the promised discussion of other codimensions.

In this paper we have intentionally avoided too much detail on questions of smoothing corners and the like, since we feel this would only have added length and opacity to the exposition. We could of course appeal directly to general smoothing theory for this, since the smoothing needed is all of sufficiently low codimension. However the smoothing questions are in fact much more elementary than this: whatever object $\mathscr{X}$ we construct, its smooth structure can always be obtained by standard angle straightening techniques (see also remark at end of §3) and the independence of smooth structure (up to isomorphism) from the choices in the construction of $\mathscr{X}$ can always be seen by applying the same angle straightening to a suitable version of $\mathscr{X}\times I$. We hope we have included sufficient detail that any unconvinced reader can fill in the remaining details. A convenient reference for angle straightening is [4].

Throughout the paper, the symbol $\cong$ denotes diffeomorphism (possibly of pairs) after any necessary smoothing; the symbol $\approx$ denotes homeomorphism.

§1. FOUNDATIONS

In this section we discuss the general setting for branched fibrations and their relationship to open book structures and to singularities of mappings.

An *open book structure* on a closed compact manifold M is a decomposition $M=E\cup N$ where E is a fiber bundle over S^1 with a trivialization of its boundary so that $\partial E\cong K\times S^1\cong\partial N$, and $N\cong K\times D^2$. Here K is a codimension two submanifold of M. We refer to K as the *binding* of the book. The fibers of the bundle $E \to S^1$ are the *leaves*. For our purposes the following definition will be most convenient.

Definition 1.1. An open book structure is a map $b: M \to D^2$ such that zero is a regular value and $\bar{b} = b/\|b\|: M - b^{-1}(0) \to S^1$ is a smooth fibration. Two books (M, b) and (M', b') are *equivalent* if there exists a diffeomorphism $h: M \to M'$ such that $b' \circ h$ agrees with b on a neighborhood of $b^{-1}(0)$ and $b' \circ h/\|b' \circ h\| = b/\|b\|$ on $M - b^{-1}(0) = M - (b' \circ h)^{-1}(0)$.

This definition is equivalent to our first description. Given $b: M \to D^2$ as above, then $K = b^{-1}(0)$ and we may take $N = b^{-1}(D_\epsilon^2)$ where D_ϵ^2 denotes a sufficiently small subdisc about zero.

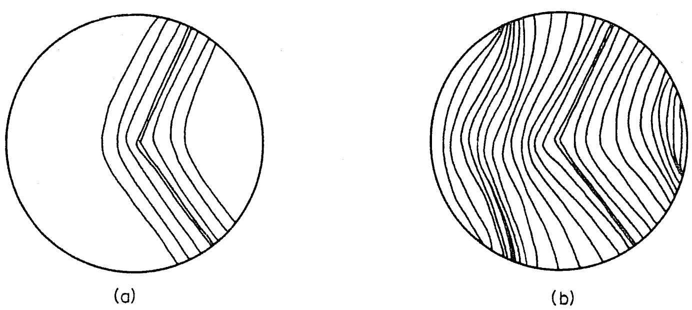
A *knot* $\mathcal{K} = (S^k, K)$ is a pair $K \subset S^k$ where K is an oriented, compact closed codimension two submanifold of the (oriented) sphere S^k. We say that $\mathcal{K}$ is a *fibered knot* if S^k has an open book structure with binding K. The book $b: S^k \to D^2$ will be referred to as the *fibered structure* of $\mathcal{K}$.

Definition 1.2. A smooth map $\tau: D^{n+1} \to D^2$ of the $(n+1)$-ball to D^2 is a *branched fibration along* $0 \in D^2$ if

(i) $\tau | \tau^{-1}(D^2 - \{0\})$ and $\tau | \partial D^{n+1} \cap \tau^{-1}(\mathring{D}^2)$ are smooth fibrations;
(ii) $\tau^{-1}(0)$ is homeomorphic to CK where $K = \tau^{-1}(0) \cap \partial D^{n+1}$;
(iii) the cone point of CK is the only singular point of τ.

Observe that in this situation $(D^{n+1}, \partial D^{n+1} \cap \tau^{-1}(0))$ is a fibered knot with fibered structure given by the restriction of τ to the boundary of the $(n+1)$-ball). Hence each branched fibration gives rise to a fibered knot. Conversely, we associate to each fibered knot a branched fibration as follows:

Let (S^n, K) be a fibered knot with fibered structure $b: S^n \to D^2$ and fiber (leaf) F. Let $cb: D^{n+1} \to D^2$ be the coned map $cb(rx) = rb(x)$ for $0 \le r \le 1$ and $x \in S^n$. One can smooth cb at the origin, for instance by composing with the map $\lambda: D^2 \to D^2$ given by $\lambda(rx) = e^{1-1/r^2} rx$ for $0 \le r \le 1$ and $x \in S^1$; call such a smoothed version τ. If we restrict τ to $E = \tau^{-1}(D_\epsilon^2)$ for ϵ sufficiently small, then τ is a fiber bundle over $D_\epsilon^2 - \{0\}$ with fiber F and $\tau^{-1}(0) = CK$, the cone over K. E of course has a corner in its boundary, but, as is well known, such a corner can be smoothed uniquely up to diffeomorphism. Such a smoothed version of E is diffeomorphic to D^{n+1}, as can be seen by pushing out along linear rays through the origin, and $(E, \tau | E)$ gives the desired branched fibration (see Fig. 1(a)). In fact, for most purposes the corner will not worry us, and it is maybe more intuitive to leave it in. Another way of avoiding worry about the corner is to first alter the map $b: S^n \to D^2$ to have only regular values in the interior of D^2, in which case we can take (D^{n+1}, τ) as the branched fibration. The fibration of D^{n+1} then looks something like Fig. 1(b).



(a) (b)

Fig. 1.

Thus we associate to any fibered knot (S^n, K, b), a smooth branched fibration $\tau: D^{n+1} \to D^2$. We call this mapping τ, a *branched fibration corresponding to* (S^n, K, b).

We wish to include certain low dimensional cases under these definitions. Thus if $[a]: S^1 \to S^1$ is given by $[a](x) = x^a$, we call $[a]$ the *empty* (*fibered*) *knot* of degree a (since this book has no binding).

If $[a]: S^1 \to S^1$ denotes the empty knot of degree a, then the corresponding branched fibration is $\mu_a: D^2 \to D^2$ where $\mu_a(z) = z^a$. That is, it is the standard a-fold ramified cover of D^2 with the origin as branch point.

Another way to obtain examples is as follows. Let $f: \mathbf{R}^{n+1} \to \mathbf{R}^2$ be a smooth map with an isolated singularity at $\mathbf{0} \in \mathbf{R}^{n+1}$ (and $f(\mathbf{0}) = 0$). Assume also that $f^{-1}(0)$ is transverse to sufficiently small spheres about $\mathbf{0} \in \mathbf{R}^{n+1}$. Choose $0 < \epsilon \ll 1$ and let $0 < \delta \ll \epsilon$. Set $E = \{x \in D_\epsilon^{n+1} \mid \|f(x)\| \leq \delta\}$. Then we map E to the disk of radius δ by $\pi: E \to D_\delta^2$, where π is defined by $\pi(x) = f(x)$. If, after smoothing corners, E is diffeomorphic to D^{n+1}, then (E, π) is a branched fibration (in fact one can check that E, after smoothing corners, is diffeomorphic to the cone over its boundary, and deduce that E is diffeomorphic to D^{n+1}, at least for $n \neq 3, 4$.).

Definition 1.3. Let $f: \mathbf{R}^{n+1} \to \mathbf{R}^2$ be a smooth map (germ) with $f(\mathbf{0}) = 0$ and an isolated singularity at the origin. If $f^{-1}(0)$ is transverse to sufficiently small spheres about the origin and E, as defined above, is diffeomorphic to D^{n+1}, we say that f is a *tame isolated singularity*. For $0 < \epsilon \ll 1$, define the *link of f* by $L(f) = S_\epsilon^n \cap f^{-1}(0)$.

Thus we have shown that for a tame isolated singularity there is a branched fibration $\pi: E \to D^2$ and an associated open book $\pi|: \partial E \to D^2$ so that $L(f)$ is the binding of the book and $(E, \partial E) \cong (D^{n+1}, S^n)$. In general, this book structure will not occur naturally on a standardly embedded sphere of radius ϵ in R^{n+1}. However, if $f: \mathbf{C}^{n+1} \to \mathbf{C}$ is a (complex) polynomial mapping with an isolated singularity at the origin, then it follows from Milnor[17] that f is tame and that S_ϵ^{2n+1} inherits the appropriate fibered structure. In fact, Milnor proves that the fibration of the complement of $L(f)$ is given by $\phi: S_\epsilon^{2n+1} - L(f) \to S^1$ where $\phi(z) = f(z)/\|f(z)\|$.

It is of interest to know when two tame isolated singularities define the same book structure. To this end, we make the following definition.

Definition 1.4. Let $f, g: \mathbf{R}^{n+1} \to \mathbf{R}^2$ be two tame isolated singularities. We say that f and g are *tame topologically equivalent* if there exists a diffeomorphism (germ at 0) $h: \mathbf{R}^{n+1} - \{0\} \to \mathbf{R}^{n+1} - \{0\}$ such that $g \circ h = f$ and so that for ϵ arbitrarily small there exists $\epsilon' \ll \epsilon$ such that $f^{-1}(0) \cap D_\epsilon^{n+1} - h^{-1}(g^{-1}(o) \cap \mathring{D}_{\epsilon'}^{n+1})$ is a trivial (product) h-cobordism. Thus equivalence preserves the cone structure in a weak sense.

Remark. That $f^{-1}(0) \cap D_\epsilon^{n+1} - h^{-1}(g^{-1}(0) \cap \mathring{D}_{\epsilon'}^{n+1})$ is an h-cobordism follows from the next lemma.

LEMMA 1.5. *Let M and N be closed manifolds of equal dimension and $CN \subset CM$ an embedding of cones such that the cone points coincide. Then, if the embedding is smooth except at the cone point, the difference $\overline{CM - CN}$ is an h-cobordism.*

Proof. We can embed $CM \subset CN \subset CM \subset CN$ such that $CM \subset CM$ and $CN \subset CN$ are standard embeddings. Let B', A, B, be as in Fig. 2. Then $B' \cup A \cong M \times I$ and $A \cup B \cong N \times I$. Hence $B' \cong B' \cup A \cup B$ and $B' \cup A \cup B \cong B$, so $B' \cong B$. Now we have homotopy equivalences

$$A \simeq A \cup (M \times [0, \infty)) = A \cup (B \cup A \cup B \cup A \cup \cdots) =$$
$$(A \cup B) \cup (A \cup B) \cup \cdots = N \times [0, \infty) \qquad \text{and}$$
$$A \simeq (N \times (-\infty, 0]) \cup A = (\cdots \cup B \cup A \cup B) \cup A =$$
$$\cdots (B \cup A) \cup (B \cup A) \simeq \cdots (B' \cup A) \cup (B' \cup A) = (-\infty, 0] \times M.$$

Hence A is homotopy equivalent to each of its ends.

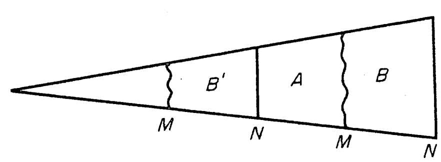


Fig. 2.

Call two fibered knots *isomorphic* if they have equivalent book structures in the sense of 1.1 Definition 1.4 is tailored to make the following theorem hold.

THEOREM 1.6 *The set of tame topological equivalence classes of tame isolated singularity* *germs* $f: \mathbf{R}^{n+1} \to \mathbf{R}^2$ *is in* 1–1 *correspondence with the set of isomorphism classes of fibered knots in* S^n.

We leave the proof to the reader.

In practice, we shall be concerned with a stronger equivalence relation for fibered knots. Let (S^n, K, b) denote the fibered knot with fibered structure $b: S^n \to D^2$.

Definition 1.7. Two fibered knots (S^n, K, b) and (S^n, K', b') are *isotopic* if b and b' are equivalent in the sense of Definition 1.1 by a diffeomorphism $h: S^n \to S^n$ which is isotopic to the identity.

The corresponding equivalence relation for tame isolated singularities is then as follows.

Definition 1.8. Two tame isolated singularities $f: \mathbf{R}^{n+1} \to \mathbf{R}^2$ and $g: \mathbf{R}^{n+1} \to \mathbf{R}^2$ are *tamely isotopic* if there is a tame topological equivalence $h: \mathbf{R}^{n+1} - \{0\} \to \mathbf{R}^{n+1} - \{0\}$ as in 1.4 between them such that the germ h is isotopic to the germ at 0 of $1_{\mathbf{R}^{n+1}-\{0\}}$.

It is also helpful to have the corresponding terminology for branched fibrations.

Definition 1.9. Two branched fibrations $\tau: D^{n+1} \to D^2$ and $\tau': D^{n+1} \to D^2$ are *topologically equivalent* if there is a diffeomorphism $h: D^{n+1} - \{0\} \to D^{n+1} - \{0\}$ with $\tau \circ h = \tau'$. They are *isotopic* if the diffeomorphism h may be chosen to be isotopic to the identity.

With these definitions at hand we are prepared to state a theorem comparing the categories of fibered knots, tame isolated singularities, and branched fibrations over D^2.

THEOREM 1.10. *The following sets are in* 1–1 *correspondence*:

(a) *The set of tame topological equivalence classes of tame isolated singularity germs* $f: \mathbf{R}^{n+1} \to \mathbf{R}^2$.

(b) *The set of isomorphism classes of fibered knots in* S^n.

(c) *The set of topological equivalence classes of branched fibrations* $\tau: D^{n+1} \to D^2$.

The same result holds if topological equivalence is replaced by isotopy in (a) *and* (c), *while isomorphism is replaced by isotopy in* (b).

The proof will be left to the reader.

§2. BRANCHED FIBRATIONS

We now continue the discussion of branched fibrations, extending the concept to branching over an arbitrary codimension two submanifold.

Definition 2.1. Let (S^n, K, b) be a fibered knot with fiber F. Let M^m be a smooth manifold with a codimension two submanifold $V^{m-2} \subset M^m$. If M has boundary, then we assume that V is properly embedded. We say that a smooth mapping $\pi: N \to M$ is a *b-branched* (or K-branched) *fibration along* V if the following conditions are satisfied:

(i) $N - \pi^{-1}(V) \to M - V$ is a smooth locally trivial fibration with fiber F.

(ii) for each $v \in V$, $\pi^{-1}(v)$ is homeomorphic to CK.

(iii) on normal disks to V, π is topologically equivalent to (a smoothing of) cb. (See 1.9).

In order to include the case of standard cyclic branched coverings (that is, (S^n, K, b) an empty knot of degree a), we use the convention that the cone on the empty set is a single point.

This definition generalizes Definition 1.2. For example, if $\tau: D^{n+1} \to D^2$ is any smoothing of cb, then τ is b-branched along $0 \in D^2$.

Many useful examples are obtained by a pull-back construction. Given M and V as in Definition 2.1, suppose that there is a smooth map $\alpha: M \to D^2$, transverse to $0 \in D^2$ with $\alpha^{-1}(0) = V$. Then we may form the pull-back

$$\begin{tikzcd} N \arrow[r, "\hat{\alpha}"] \arrow[d, "\pi"] & D^{n+1} \arrow[d, "\tau"] \\ M \arrow[r, "\alpha"] & D^2. \end{tikzcd}$$

That is, $N = \{(m, d) \in M \times D^{n+1} | \alpha(m) = \tau(d)\}$.

The manifold N with the differentiable structure described in the next theorem is b-branched along $V \subset M$. The mapping $\pi: N \to M$ is given by the equation $\pi(m, d) = m$. In this example, if $\tau = \mu_a: D^2 \to D^2$, then N is a cyclic branched cover of M along V.

THEOREM 2.2 *Let $\tau: D^{n+1} \to D^2$ be a smooth branched fibration along $0 \in D^2$. Let M^m be a smooth manifold with a properly embedded codimension two submanifold $V^{m-2} \subset M$. Suppose that there is a smooth map $\alpha: M \to D^2$, transverse to $0 \in D^2$, with $\alpha^{-1}(0) = V$. Let $\tau(M, \alpha) = N$, the pull-back construction described above. Then*

(i) *$\tau(M, \alpha)$ has a well-defined differentiable structure so that the mapping $\pi: N \to M$ is a smooth branched fibration.*

(ii) *If τ is a smoothing of cb for some fibered knot (S^n, K, b), then $\tau(M, \alpha)$ does not depend (up to diffeomorphism) on the smoothing we take. In this case we shall write $\tau(M, \alpha) = b(M, \alpha)$.*

(iii) *Suppose that $\tau, \tau': D^{n+1} \to D^2$ are branched fibrations along 0. Then $\tau(M, \alpha)$ and $\tau'(M, \alpha)$ are homeomorphic whenever τ and τ' are topologically equivalent; they are diffeomorphic if τ and τ' are isotopic (see Definitions* 1.4–1.8).

(iv) *Let (S^n, K, b) and (S^n, K', b') be fibered knots. Then $b(M, \alpha)$ and $b'(M, \alpha)$ (see* (ii) *above) are diffeomorphic whenever these fibered knots are isotopic.*

Proof. We shall describe a differentiable structure on N that is dependent only on α and τ. We need only specify this structure in a neighborhood of $\pi^{-1}(V)$, since $N - \pi^{-1}(V)$ has a natural differentiable structure as the pull-back of a smooth fibration. Note that V has a trivial tubular neighborhood since the map α is transverse to 0 in D^2, and 0 is framed in D^2. In a neighborhood of $\pi^{-1}(V)$ the situation is

$$\begin{tikzcd} & W \arrow[r, "\hat{\alpha}"] \arrow[d] & D^{n+1} \arrow[d, "\tau"] \\ V \times D^2 & U \arrow[l, "\cong"'] \arrow[r] & D^2 \end{tikzcd}$$

with $W = \pi^{-1}(U)$, U a tubular neighborhood of V in M. Specify a differentiable structure on W by requiring that the maps $W \xrightarrow{\hat{\alpha}} D^{n+1}$ and $\varphi: W \longrightarrow U \xrightarrow{\cong} V \times D^2 \xrightarrow{pr} V$ be submersions (and hence define a diffeomorphism $(\hat{\alpha}, \varphi): W \to D^{n+1} \times V$).

We now show that this structure is well-defined. Denote by N_f, N with the differentiable structure defined as above by the tubular neighborhood $f: U \to V \times D^2$. Similarly, let W_f denote W with the differentiable structure corresponding to this tubular neighborhood. Let $g: U' \to V \times D^2$ be another tubular neighborhood. Without loss of generality, we may assume that $U' \subset U$. Note that $W_f \xrightarrow{\pi} U$ is smooth, so if $W' = \pi^{-1}(U')$ then $\varphi': W'_f \xrightarrow{\pi} U' \xrightarrow{g} V \times D^2 \xrightarrow{pr} V$ is smooth, so $(\alpha, \varphi'): W'_f \to V \times D^{n+1}$ is smooth. But this is the map which defines the differentiable structure W'_g, so we have shown id: $N_f \to N_g$ is smooth. Similarly id: $N_g \to N_f$ is smooth, so the differentiable structures are the same, proving (i).

We can now describe this differentiable structure in a different way: after choosing a tubular neighborhood $U \xrightarrow{\cong} V \times D^2$ of V in M compatible with $\alpha: M \to D^2$ (i.e. $\alpha \circ f^{-1}$ is projection onto $D_\epsilon^2 \subset D^2$ for some ϵ) we can write $M = M_0 \cup (V \times D^2)$ pasted along the boundary, and this induces a splitting $N = N_o \cup (V \times D^{n+1})$ of N. This determines the differentiable structure on N up to diffeomorphism (by uniqueness up to diffeomorphism of pasting of manifolds along boundaries). Here we are no longer using anything about τ except $\tau|\partial D^{n+1}$, which determines the pasting map. Since we only need the pasting map up to isotopy, it follows that N is determined up to diffeomorphism already by the isotopy class (see definition 1.9) of τ. This proves (ii) and (iii) (The statement on homeomorphism in (iii) is trivial.). Property (iv) is a restatement of (ii). This completes the proof of the theorem.

The next few lemmas delineate further properties of the pull-back construction.

LEMMA 2.3. *Let $V^{m-2} \subset M^m$ be a proper embedding of smooth oriented manifolds. If M is 2-connected then*

(a) *There exists a smooth map $\alpha: M \to D^2$ which is a fibration over a neighborhood of zero, and $V = \alpha^{-1}(0)$ as an oriented submanifold of M.*

(b) *α is unique up to smooth homotopy preserving property* (a).

Remarks. We do not assume compactness. Proper embedding means that V is closed in M and V is transversal to ∂M with $\partial V \subset \partial M$. If M is compact it suffices for (a) that 0 be a regular value of α and of $\alpha|\partial M$ and that $V = \alpha^{-1}(0)$ with correct orientation. Note that the orientation for D^2 gives, via α, an orientation for the normal bundle of V and hence for V.

The proof actually shows more than the above: For any M (not necessarily 2-connected), the existence of α is equivalent to the dual cohomology class in $H^2(M)$ of V being zero, and uniqueness is equivalent to $H^1(M) = 0$.

Proof. Let N be a closed tubular neighborhood of V in M. We can identify N with the total space of the normal disc bundle of V in M. Let ∂N denote the sphere bundle of this disc bundle. (The notation is potentially confusing, since this is only a portion of the boundary of N if V has boundary.) The existence of the map α is clearly equivalent to the existence of a map $\beta\colon \overline{M-N} \to S^1$ which restricts to a bundle trivialization of ∂N, for a suitable tubular neighborhood N. Since the bundle N is oriented, there is a natural Thom class $U \in H^2(N, \partial N)$. Note that $H^2(N, \partial N) = H^2(M, \overline{M-N})$ by excision and $H^1(M, \overline{M-N}) = H^1(N, \partial N) = 0$ by excision and the Thom isomorphism. We shall consider the exact sequence

$$
\begin{array}{c}
0 \longrightarrow H^1(M) \longrightarrow H^1(\overline{M-N}) \xrightarrow{\ \delta\ } H^2(M, \overline{M-N}) \longrightarrow H^2(M) \longrightarrow \cdots \\
\| \\
H^2(N, \partial N)
\end{array}
$$

A map β as above induces also a map $M \to D^2$ and then naturality of the Thom class and the commutative square

$$
\begin{tikzcd}
H^1(S^1) \arrow[r] \arrow[d, "\beta^*"'] & H^2(D^2, S^1) \arrow[d] \\
H^1(\overline{M-N}) \arrow[r, "\delta"] & H^2(M, \overline{M-N})
\end{tikzcd}
$$

shows that β, considered as an element of $[\overline{M-N}, S^1] = H^1(\overline{M-N})$, goes to the Thom class $U \in H^2(M, \overline{M-N})$ under δ. Conversely, if we have a class $[\beta] \in [\overline{M-N}, S^1] = H^1(\overline{M-N})$ with $\delta[\beta] = U$, then commutativity of

$$
\begin{tikzcd}
H^1(\overline{M-N}) \arrow[r, "\delta"] \arrow[d] & H^2(M, \overline{M-N}) \arrow[d] \\
H^1(S^1) \arrow[r] & H^2(D^2, S^1)
\end{tikzcd}
$$

(vertical arrows induced by inclusion $D^2 \longrightarrow N \subset M$ of a fiber of N) shows that β restricts to a homotopy trivialization of the bundle ∂N. Since differentiable and homotopy classifications of S^1-bundles agree, we can homotop $\beta|\partial N$ to be a genuine trivialization of ∂N. This homotopy can be extended via a collar to all of $M-N$, so β then has the desired form.

The existence statement of the Lemma is thus equivalent to the existence of a $[\beta] \in [\overline{M-N}, S^1] = H^1(\overline{M-N})$ with $\delta[\beta] = U$. It thus follows from the exact sequence, since $H^2(M) = 0$. To see uniqueness, observe first that the exact sequence shows that β is unique up to homotopy, since $H^1(M) = 0$. Given two different maps α and α' as in the lemma, it follows from this that we can assume that α and α' agree on $M-N$. They hence determine the same homotopy trivialization of the normal bundle of V, hence the same genuine trivialization, so by uniqueness of tubular neighborhoods up to isotopy we can in addition assume that α and α' agree on some smaller tubular neighbourhood $N_0 \subset N$ of V. Now the "difference" $\gamma(x) = \alpha'(x)\alpha^{-1}(x)$, (group structure in S^1) is a map $(\overline{N-N_0}, \partial(\overline{N-N_0})) \to (S^1, 1)$ whose homology class $[\gamma] \in H^1(\overline{N-N_0}, \partial(\overline{N-N_0}))$ is the obstruction to homotoping α to α' without altering $\alpha|N_0$ and $\alpha|M-N$. But $H^0(\partial\bar{N}_0) = H^0(\partial(\overline{N-N_0}), \partial\bar{N}) \xrightarrow{\ \cong\ } H^1(\overline{N-N_0}, \partial(\overline{N-N_0}))$, by the exact sequence of the triple $(\overline{N-N_0}, \partial(\overline{N-N_0}), \partial\bar{N})$, and it follows that by applying a homotopy to α which rotates each component of the tubular neighbourhood N_0 a suitable number of turns, we can reduce the obstruction $[\gamma] \in H^1(\overline{N-N_0}, \partial(\overline{N-N_0}))$ to zero. This completes the proof; we have in fact shown that maps α classified up to homotopies as in the lemma are in 1–1 correspondence to the set $\delta^{-1}(U) \in H^1(M-N)$, so the stronger statements of the remark follow on observing in the exact sequence that the image of U in $H^2(M)$ is the dual cohomology class of V, so U is in the image of δ if and only if this dual class vanishes; also δ is injective if and only if $H^1(M) = 0$.

LEMMA 2.4. *Let $V_1^{m-4} \subset V_2^{m-2} \subset M^m$ be proper embeddings of smooth manifolds. Assume that M and V_2 are each 2-connected. Let $i\colon V_2 \to M$ denote the given inclusion. Then there exists an embedding of pairs $j\colon (V_2, V_1) \to (M, V_2)$ so that*

(a) *$j(V_2)$ is transverse to V_2 with $j(V_2) \cap i(V_2) = V_1$.*

(b) *j is isotopic to i through maps satisfying* (a).

(c) *j is unique up to isotopy through maps satisfying* (a) *and* (b).

Proof. For the proof, we shall assume that the manifolds are compact. With a little more work one can also prove the non-compact case. Note that V_2 has a tubular neighborhood $T\colon V_2 \times D^2 \subset M$. Let $\alpha\colon V_2 \to D^2$ be a map as in Lemma 2.3 so that $\alpha^{-1}(0) = V_1$. Define $\hat{\alpha}\colon V_2 \to M$ by $\hat{\alpha}(x) = T(x, \alpha(x))$. Then $\hat{\alpha}\colon (V_2, V_1) \to (M, V_2)$ satisfies properties (a) and (b). To see (b), define $i_t\colon V_2 \to M$ by $i_t(x) = T(x, t\alpha(x))$ for $0 \le t \le 1$. Then $i_0 = i$ and $i_1 = \hat{\alpha}$.

Given $j_1\colon V_2 \to M$ satisfying (a) and (b) and an isotopy $J\colon V_2 \times I \to M$ as in (b), then for $s > 0$ sufficiently small one has $J(x, s) = T(h_3(x), f_s(x))$ where $h_s\colon V_2 \to V_2$ is a diffeomorphism (this needs compactness). Here f_s satisfies Lemma 2.3 and h_s is isotopic to the identity by h_r, $0 \le r \le s$. Uniqueness then follows from the uniqueness in Lemma 2.3.

COROLLARY 2.5. *Let $V^{m-2} \subset M^m$ be a proper embedding of compact smooth oriented manifolds. Let M be 2-connected and let $\alpha\colon M \to D^2$ be the map constructed in Lemma 2.3. Let (S^n, K, b) be a fibered knot. Then the pull-back $b(M, \alpha)$ is actually independent of α, up to diffeomorphism.*

Remark. Under the hypotheses of this corollary, we may now denote the pull-back $b(M, \alpha)$ by $b(M, V)$. The branched fibration $\pi\colon b(M, V) \to M$ is uniquely determined by $V \subset M$ and the fibered knot (S^n, K, b).

Proof. Let α and α' be two maps from M to D^2, constructed as in Lemma 2.3, and let $A\colon M \times I \to D^2$ be a homotopy between them, also constructed as in Lemma 2.3. It then follows from Ehresmann's theorem (see [17], p. 98) that $b(M \times I, A)$ is a smooth fiber bundle over the interval I with fiber $b(M, \alpha)$. Hence $b(M \times I, A)$ is diffeomorphic to $b(M, \alpha) \times I$. The technique of the proof of Ehresmann's theorem actually shows that this diffeomorphism can be chosen compatible with the maps to M. This fact suffices to prove the corollary.

COROLLARY 2.6. *Given a fibered knot (S^n, K, b) and inclusions $V_1 \subset V_2 \subset M$ as in Lemma 2.4, there is an inclusion of branched fibrations.*

$$\begin{array}{ccc} b(V_2, V_1) & \xrightarrow{f} & b(M, V_2) \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ V_2 & \xrightarrow{j} & M \end{array}$$

where j is the map satisfying (b) *of Lemma* 2.4.

Proof. Restrict the branched fibration π to $j(V_2)$. Since $j(V_2) \cap V_2 = V_1$, we see that the restricted fibration branches along V_1. It is then easy to check that the restriction is $b(V_2, V_1)$.

Thus we obtain canonical inclusions of branched fibrations. The empty knots give an interesting special case. Let $[a]$ denote the fibered structure for the empty knot of degree a (that is $[a] = \mu_a$). Let $K \subset S^n \subset S^{n+2}$ where (S^n, K) is any knot and $S^n \subset S^{n+2}$ is the standard (unknotted) inclusion. Then, from 2.3, we have $j\colon S^n \to S^{n+2}$ so that $j(S^n) \cap S^n = K$. Hence we have the diagram

$$\begin{array}{ccc} [a](S^n, K) & \xrightarrow{f} & [a](S^{n+2}, S^n) \\ \downarrow & & \downarrow \\ S^n & \xrightarrow{j} & S^{n+2}. \end{array}$$

But $[a](S^n, K)$ is the a-fold cyclic branched cover of S^n along K, and $[a](S^{n+2}, S^n) \cong S^{n+2}$ since branching along an unknotted sphere has no effect. Therefore $\hat{f}$ embeds the branched covering into S^{n+2}. Letting $K \otimes [a]$ stand for $j([a](S^n, K))$, we obtain a new knot $(S^{n+2}, K \otimes [a])$. This will be referred to as the *a-fold cyclic suspension* of (S^n, K). (See [18]).

Another application of Lemma 2.3 is the well-known construction of a spanning manifold for a knot. Given a knot (S^n, K) there is a homotopy unique (as in Lemma 2.3) map $\alpha: S^n \to D^2$ with $\alpha^{-1}(0) = K$. Let F denote the closure of $\bar{\alpha}^{-1}(t)$ where $\bar{\alpha} = \alpha/\|\alpha\|: S^n - K \to S^1$, and t is a regular value of $\bar{\alpha}$. Then $F^{n-1} \subset S^n$ and $\partial F = K$ as oriented manifolds. Every spanning surface can be obtained in this way.

§3. PRODUCTS OF KNOTS

Given a knot $\mathcal{K} = (S^k, K)$ and a fibered knot $\mathcal{L} = (S^l, L, b)$ we will define a *product knot* $\mathcal{K} \otimes \mathcal{L} = (S^{k+l+1}, K \otimes L)$. This will generalize the cyclic suspension discussed at the end of §2. We have already explained some of the motivation behind our definition in the introduction. While it might seem most natural to give an initial definition of the product that is an obvious analog of the link of a sum of singularities, we have chosen instead to define the product by using branched fibrations; the embedding of $K \otimes L$ in S^{k+l+1} is obtained by using the technique of Lemma 2.4 and Corollary 2.6. This approach has the advantage that the product is at once well-defined in the differentiable category. Furthermore one sees clearly that only one knot need be fibered. In Lemmas 3.4 and 3.5 we derive the versions of the product described in the introduction. We may write $\mathcal{K} \otimes \mathcal{B}$ for $\mathcal{K} \otimes \mathcal{L}$ and $K \otimes b$ for $K \otimes L$ in order to emphasize the fibered structure of $\mathcal{L}$.

We shall have need of the boundary of a branched fibration. Suppose $V \subset M$ with M 2-connected so that we can form $\pi: b(M, V) \to M$. The reader can easily verify that the boundary of $b(M, V)$ decomposes as follows:

$$\partial(b(M, V)) \cong b(\partial M, \partial V) \cup (M \times L)$$

where L is the binding of the fibered knot (S^l, L, b). The union in this equality is taken along boundaries; it is also clear that $\partial b(\partial M, \partial V) \cong \partial M \times L$. This decomposition of the boundary reflects an essential difference between branched fibrations and branched coverings: The fiber of a branched covering map is a discrete collection of points. Hence the boundary of a branched covering is a branched covering of the boundary. For branched fibrations, the fibers of the branched fibration are manifolds with boundary. In the case under consideration each fiber has boundary diffeomorphic to L, and this contributes the extra term $M \times L$.

Definition 3.1. Let $\mathcal{K}$ and $\mathcal{L}$ be knots as above, and choose a properly embedded codimension two submanifold $F \subset D^{k+1}$ with $\partial F = K$. Then we define $K \otimes b$ to be the boundary of $b(D^{k+1}, F)$. That is,

$$K \otimes b = \partial(b(D^{k+1}, F)) = b(S^k, K) \cup (D^{k+1} \times L).$$

For example, if $\mathcal{L}$ is the empty knot $(S^1, [a])$, then

$$K \otimes [a] = [a](S^k, K) \cup (D^{k+1} \times \phi) = [a](S^k, K),$$

the a-fold cyclic branched cover of S^k along K.

Note that our definition is independent of the choice of submanifold $F \subset D^{k+1}$ (by the second equality in the definition), and that $K \otimes b$ has a well-defined differentiable structure via the results of §2.

To obtain the embedding $K \otimes b \subset S^{k+l+1}$, we proceed by analogy with the cyclic suspension. A cyclic branched cover of a sphere along the unknot is again a sphere; the next lemma gives the corresponding result for branched fibrations.

LEMMA 3.2. *Let $D^{k+1} \subset D^{k+3}$ be the standard (unknotted) embedding. Then for any fibered knot (S^l, L, b) we have*

$$b(D^{k+3}, D^{k+1}) \cong D^{k+l+2}.$$

Proof. Regard $D^{k+3} = D^{k+1} \times D^2 \supset (D^{k+1} \times 0) = D^{k+1}$, and choose the map $\alpha: D^{k+3} \to D^2$ to be projection on the second factor. Pulling back the branched fibration over D^2 yields $b(D^{k+3}, D^{k+1}) \cong D^{k+1} \times D^{l+1} \cong D^{k+l+2}$, proving the lemma.

Definition 3.3. Under the same hypotheses as 3.2 let $D^{k+1} \subset D^{k+3}$ be the standard inclusion. Apply Lemma 2.4 to the triple $F \subset D^{k+1} \subset D^{k+3}$, obtaining an embedding $j: b(D^{k+1}, F) \hookrightarrow b(D^{k+3}, D^{k+1})$. Taking boundaries, and using Lemma 3.2, we obtain an embedding $K \otimes b \hookrightarrow S^{k+l+1}$. This defines the pair $\mathcal{K} \otimes \mathcal{B} = (S^{k+l+1}, K \otimes b)$.

Once again, this construction is independent of the choice of F.

Our definition of the product $\mathcal{K} \otimes \mathcal{L}$ is precisely the a-fold cyclic suspension of $\mathcal{K}$ when $\mathcal{L}$ is the empty knot of degree a.

Here is a more symmetrical description of $\mathcal{K} \otimes \mathcal{B}$.

LEMMA 3.4. *Let (S^k, K) be a knot and (S^l, L, b) a fibered knot. Let $F \subset D^{k+1}$ be a spanning manifold for K as in Definition 3.1. Use Lemma 2.3 to obtain $\gamma: D^{k+1} \to D^2$ with $\gamma^{-1}(0) = F$, 0 a regular value of γ. Let $\tau: D^{l+1} \to D^2$ be a smoothing of cb. Use these maps to form the pullback*

$$\begin{tikzcd} b(D^{k+1}, F) \arrow[r] \arrow[d] & D^{l+1} \arrow[d, "\tau"] \\ D^{k+1} \arrow[r, "\gamma"] & D^2 \end{tikzcd}$$

Thus $b(D^{k+1}, F) \subset D^{k+1} \times D^{l+1}$. Then the product is obtained by taking boundaries from this embedding. That is, $(S^{k+l+1}, K \otimes b) \cong (\partial(D^{k+1} \times D^{l+1}), \partial(b(D^{k+1}, F)))$.

Proof. We have already identified $K \otimes b$ as the boundary of the pull-back. Therefore it suffices to check the embedding. Recall the method of Lemma 2.4. We have the triple $F \subset D^{k+1} \subset D^{k+3} = D^{k+1} \times D^2$. The map γ gives rise to a map $\hat{\gamma}: D^{k+1} \to D^{k+3}$ with defining equation $\hat{\gamma}(x) = (x, \gamma(x))$. Let $\alpha: D^{k+3} \to D^2$ be defined by the equation $\alpha(x, z) = z$. Thus $\alpha \circ \hat{\gamma} = \gamma$. Let $p: D^{k+3} \to D^{k+1}$ be the map defined by the equation $p(x, z) = x$, for $(x, z) \in D^{k+1} \times D^2$. We then obtain a diffeomorphism $\hat{p}: b(D^{k+3}, D^{k+1}) \to D^{k+1} \times D^{l+1}$ via the equation $\hat{p}(v, y) = (p(v), y)$, where $b(D^{k+3}, D^{k+1}) = \{(v, y) \in D^{k+3} \times D^{l+1} | \alpha(v) = \tau(y)\}$.

By definition, $K \otimes b \hookrightarrow S^{k+l+1}$ is obtained by taking boundaries from $j: b(D^{k+1}, F) \hookrightarrow b(D^{k+3}, D^{k+1})$ where this embedding is defined by the equation $j(x, y) = (\hat{\gamma}(x), y)$. Using our identification of $b(D^{k+3}, D^{k+1})$ with $D^{k+1} \times D^{l+1}$, we have an embedding $\hat{p} \circ j: b(D^{k+1}, F) \hookrightarrow D^{k+1} \times D^{l+1}$. However, $\hat{p} \circ j(x, y) = \hat{p}(\hat{\gamma}(x), y) = \hat{p}((x, \gamma(x)), y) = (p(x, \gamma(x)), y) = (x, y)$. Thus $\hat{p} \circ j$ is just the standard embedding $b(D^{k+1}, F) \hookrightarrow D^{k+1} \times D^{l+1}$ arising from the pull-back construction. This completes the proof of the lemma.

If both knots are fibered, then we can give an even more symmetrical description of the product.

LEMMA 3.5. *Let (S^n, K, b) and (S^m, K', b') be two fibered knots. Let $\tau: D^{n+1} \to D^2$ and $\tau': D^{m+1} \to D^2$ be branched fibrations corresponding to b and b' respectively. Then $\mathcal{B} \otimes \mathcal{B}' \cong (\partial(D^{n+1} \times D^{m+1}), \partial X)$ where X is the pull-back corresponding to the diagram below:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & D^{m+1} \arrow[d, "\tau'"] \\ D^{n+1} \arrow[r, "\tau"] & D^2. \end{tikzcd}$$

In this diagram X is not a manifold, but its singularity occurs away from the boundary. Note that if we let $T: D^{n+1} \times D^{m+1} \to R^2$ be the map defined by the equation $T(x, y) = \tau(x) - \tau'(y)$, then $\partial X = T^{-1}(0) \cap \partial(D^{n+1} \times D^{m+1})$.

Proof. Let $f_{\epsilon'}: D^2 \to D^2$ be a diffeomorphism of $D^2 = \{z \in \mathbf{C} | |z| = 1\}$ which fixes a collar about the boundary and so that $f_{\epsilon'}(\epsilon') = 0$ (for some ϵ' with $0 < \epsilon' \ll 1$). We may assume that $f_{\epsilon'}$ is isotopic to the identity through maps $f_\epsilon: D^2 \to D^2$ $(0 \leq \epsilon \leq \epsilon')$ so that $f_0 = 1_{D^2}$ and $f_\epsilon(\delta) = 0$. Let $\tau_\epsilon = f_\epsilon \circ \tau$. Then for $0 < \epsilon \leq \epsilon'$, $\tau_\epsilon: D^{n+1} \to D^2$ has the origin as a regular value, and $\tau_\epsilon^{-1}(0) \subset D^{n+1}$ is a manifold with boundary, isotopic to the fiber of b. Consequently, we may use τ_ϵ and τ' to form $\mathcal{B} \otimes \mathcal{B}'$ from the pull-back:

$$\begin{tikzcd} X_\epsilon \arrow[r] \arrow[d] & D^{m+1} \arrow[d, "\tau'"] \\ D^{n+1} \arrow[r, "\tau_\epsilon"] & D^2. \end{tikzcd}$$

By Lemma 3.4, $\mathcal{B} \otimes \mathcal{B}' \cong (\partial(D^{n+1} \times D^{m+1}), \partial X_\epsilon)$. Letting ϵ approach zero isotopes ∂X_ϵ in $\partial(D^{n+1} \times D^{m+1})$. Of course X_0 acquires a singularity, but this occurs away from its boundary. Therefore $\mathcal{B} \otimes \mathcal{B}'$ may be obtained from the pull-back diagram for τ and τ'. This proves the lemma.

COROLLARY 3.6. *Let (S^n, K, b) and (S^m, K', b') be two fibered knots. Then $\mathcal{B} \otimes \mathcal{B}' = (-1)^{(n-1)(m-1)}\mathcal{B}' \otimes \mathcal{B}$. (Here the minus sign refers to knot orientations; $-(S, K) = (-S, -K)$.)*

Proof. By the lemma we may obtain $\mathcal{B} \otimes \mathcal{B}'$ from the pull-back diagram for τ and τ'. Since this is symmetric in τ and τ', the only difference between $\mathcal{B} \otimes \mathcal{B}'$ and $\mathcal{B}' \otimes \mathcal{B}$ lies in the orientation change resulting from switching factors. This proves the corollary.

We shall see later that the two fibered structures combine to make $\mathcal{B} \otimes \mathcal{B}'$ a fibered knot also.

In accordance with 3.6, we make the following definition: Let (S^n, K) be any knot and (S^m, L, b) be a fibered knot. Then $\mathcal{B} \otimes (S^n, K) = (-1)^{(n-1)(m-1)}(S^n, K) \otimes \mathcal{B}$. Thus products of knots commute (in the graded sense).

The rest of this section is devoted to giving a cut and paste description of $\mathcal{K} \otimes \mathcal{L}$.

Let $\mathcal{K} = (S^n, K)$ be a knot. By Lemma 2.3 we may write $S^k = E_K \cup (K \times D^2)$ pasted along boundaries, such that there exists a map $\alpha: E_K \to S^1$ whose restriction $\alpha|\partial E_K: K \times S^1 \to S^1$ is the projection on the second factor. If $\mathcal{L} = (S^l, L, b)$ is a fibered knot we may write $S^l = E_L \cup (L \times D^2)$ and choose $\beta: E_L \to S^1$ satisfying identical conditions and so that β is a smooth fiber bundle giving the fibered structure.

Thus we may form the pull-back $E_K \times_{S^1} E_L$:

$$\begin{tikzcd} E_K \times_{S^1} E_L \arrow[r] \arrow[d] & E_L \arrow[d, "\beta"] \\ E_K \arrow[r, "\alpha"] & S^1. \end{tikzcd}$$

This is a well-defined manifold, and

$$\partial(E_K \times_{S^1} E_L) \cong (K \times E_L) \cup (L \times E_K).$$

PROPOSITION 3.7. *Let $\mathcal{K}$ and $\mathcal{L}$ be as above. Then $K \otimes L \cong (K \times D^{l+1}) \cup (E_K \times_{S^1} E_L) \cup (D^{k+1} \times L)$ where these pieces are pasted along their boundaries via the following natural identifications:*

$$\partial(K \times D^{l+1}) \cong (K \times E_L) \cup (K \times D^2 \times L)$$
$$\partial(D^{k+1} \times L) \cong (E_K \times L) \cup (K \times D^2 \times L).$$

Thus the second factors are identified with each other, while the first factors are glued to $\partial(E_K \times_{S^1} E_L)$.

The embedding $j: K \otimes L \hookrightarrow S^{k+l+1}$ may be described as follows. Define the following maps and view

$$S^{k+l+1} \cong (S^k \times D^{l+1}) \cup (D^{k+1} \times S^l).$$
$$j_1: K \times D^{l+1} \longrightarrow S^k \times D^{l+1}, \quad j_1(w, y) = ((w, \tau(y)), y), \quad (w, \tau(y)) \in K \times D^2 \subset S^k.$$
$$j_2: E_K \times_{S^1} E_L \subset E_K \times E_L \subset S^k \times S^l.$$
$$j_3: D^{k+1} \times L \longrightarrow D^{k+1} \times S^l, \quad j_3(x, w) = (x, (w, \alpha(x))), \quad (w, \alpha(x)) \in L \times D^2 \subset S^l.$$

The mapping j is obtained by applying j_1, j_2 and j_3 to the three pieces of the decomposition of

$K \otimes L$ given above. Here $\tau\colon D^{l+1} \to D^2$ is the branched fibration for cb, and $\alpha\colon D^{k+1} \to D^2$ is the map transversal to 0 so that $\alpha^{-1}(0) = F$, $\partial F = K$.

Proof. Recall the construction for $K \otimes L$. Let $M = D^{k+1} \supset F$ with $\partial F = K$. Then

$$K \otimes L = \partial(b(M, F)) = b(\partial M, K) \cup (D^{k+1} \times L).$$

But

$$b(\partial M, K) = \pi^{-1}(E_K) \cup \pi^{-1}(K \times D^2) = \pi^{-1}(E_K) \cup (K \times D^{l+1})$$

where $\pi\colon b(\partial M, K) \to S^k$ is the pull-back branched fibration. Now $\pi|\colon \pi^{-1}(E_K) \to E_K$ and $\alpha(E_K) \subset S^1$. Hence we have the diagram

$$\begin{array}{ccccc} \pi^{-1}(E_K) & \longrightarrow & E_L & \subset & D^{l+1} \\ \downarrow{\scriptstyle \pi|} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \tau} \\ E_K & \xrightarrow{\alpha} & S^1 & \subset & D^2. \end{array}$$

Hence $\pi^{-1}(E_K) = E_K \times_{S^1} E_L$. Therefore

$$K \otimes L = (K \times D^{l+1}) \cup (E_K \times_{S^1} E_L) \cup (D^{k+1} \times L).$$

This proves the first part.

The second part follows in the same manner by breaking up the diagram

$$\begin{array}{ccc} b(M, F) & \longrightarrow & D^{l+1} \\ \downarrow & & \downarrow \\ D^{k+1} & \xrightarrow{\alpha} & D^2 \end{array}$$

and examining the embedding

$$K \otimes L = \partial(b(M, F)) \subset \partial(D^{l+1} \times D^{k+1}) = S^{k+l+1}.$$

Remark. This a good place to remark on what happens if we try to define the product of two non-fibered knots. Suppose that one tries to use the description given in Proposition 3.7 as a tentative definition. Then $E_K \times_{S^1} E_L$ is no longer necessarily a manifold. If we choose the maps α and β so that they have disjoint sets of critical values then this space can be given a manifold structure. Thus, up to certain choices, arbitrary products can be constructed. While there is no canonical product in this general case, it might be interesting to study the entire collection of products so obtained.

Remark. Some remarks about smoothing and differentiable structure are in order at this point. Note that given (M, V) as in Lemma 2.3, $b(M, V)$ does have a "corner" in its boundary along $\partial M \times L$ (b is the fibered structure for (S^l, L, b)), which however is uniquely smoothable up to diffeomorphism by standard angle straightening. Note that at some points we have a pair $A \subset B$ of manifolds with corners in their boundaries (for example in Definition 3.3) and want to straighten corners simultaneously, for which one should really check that A meets B's corner transversally in A's corner (as is the case for 3.3). Only smoothing of the above type is needed and we leave it to the reader to check these details. For example in 3.4 and 3.5 one should multiply the bottom arrow in a diagram of the form

$$\begin{array}{ccc} X & \longrightarrow & D^{n+1} \\ \downarrow & & \downarrow{\scriptstyle \tau} \\ D^{k+1} & \xrightarrow{\gamma} & D^2 \end{array}$$

by some $r < 1$ to arrange that $\gamma(D^{k+1}) \subset \mathring{D}^2$ before taking the pull-back. This arranges that ∂X is transverse to the corner in $\partial(D^{k+1} \times D^{n+1})$ so there are no problems with differentiable structure.

§4. LINKS OF SINGULARITIES

Let $f: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a polynomial mapping with an isolated singularity at the origin. Recall that one defines the *link* of the singularity, $L(f) \subset S^{2n+1}$, by the formula $L(f) = f^{-1}(0) \cap S^{2n+1}$. Here S^{2n+1} denotes a sphere of small radius about the origin. We write $\mathscr{L}(f) = (S^{2n+1}, L(f))$.

Milnor proved that the fibered structure associated with such a singularity occurs naturally on S^{2n+1}. The fibering of the complement is given by the mapping $\bar{f}: S^{2n+1} - L(f) \to S^1$, $\bar{f}(x) = f(x)/\|f(x)\|$. Thus we shall let $\mathscr{L}(f)$ denote this knot with its fibered structure.

Given another such singularity $g: (\mathbf{C}^{m+1}, 0) \to (\mathbf{C}, 0)$, we may form $f+g: \mathbf{C}^{n+1} \times \mathbf{C}^{m+1} \to \mathbf{C}$ by $(f+g)(x, y) = f(x) + g(y)$. In this section we show that $\mathscr{L}(f+g) \cong \mathscr{L}(f) \otimes \mathscr{L}(g)$. In fact, these knots are ambient isotopic. In referring to isotopy of knots in this section we shall often write $\cong$ as a sign of equivalence of pairs and append the word isotopy in parentheses. Thus $\cong$ may be used with dual meaning, but no confusion should arise.

The same remarks hold more generally for polynomial mappings $f: (\mathbf{R}^{k+1}, 0) \to (\mathbf{R}^2, 0)$ with an isolated singularity at the origin, except that the fibration $S^k - L(f) \to S^1$ can in general no longer be given by $\bar{f} = f/\|f\|$. We shall show that the formula $\mathscr{L}(f+g) \cong \mathscr{L}(f) \otimes \mathscr{L}(g)$ still holds in this more general situation. However, the proof is less transparent, so for reasons of clarity we shall first give a detailed proof of the complex case and then indicate the necessary alteration for the real case.

Finally, we shall describe to what extent the result is still true for arbitrary tame singularities.

The factor which makes the complex case easier is the following useful lemma, which is a slight sharpening of Lemma 5.9 of Milnor[17].

LEMMA 4.1. *Let $f: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ be a complex polynomial with an isolated singularity at the origin. For sufficiently small ϵ there exists a smooth vector field v on $D_\epsilon - \{0\}$ lying over the radial vector field $w(x) = x$ on $\mathbf{C} \cong \mathbf{R}^2$ and satisfying: $|z|$ increases along trajectories of v.*

Proof. We construct v locally. It can then be pasted together by a smooth partition of unity. Lemma 5.9 of [17] says one can find a vector field v on $D_\epsilon - f^{-1}(0)$ such that $|z|$ increases along v-trajectories and such that $f(z)$ has constant argument and increasing norm along v-trajectories. Thus after adjusting the length of this v by a positive real function, it is suitable on $D_\epsilon - f^{-1}(0)$. For $z_0 \in f^{-1}(0) \cap (D_\epsilon - \{0\})$, transversality of $f^{-1}(0)$ with small spheres implies that the function $(f(z), |z| - |z_0|): C^{n+1} \to C \times \mathbf{R} \cong \mathbf{R}^3$ is regular in a neighborhood of z_0, so it is projection onto the first three coordinates in suitable local coordinates $z_0 \in U \cong V \subset \mathbf{R}^3 \times \mathbf{R}^{2n-1}$ about z_0. In these coordinates $v(a, b, c, \ldots) = (a, b, c + |z_0|, 0, \ldots, 0)$ is a suitable vector field on the neighborhood U of z_0. This completes the proof.

LEMMA 4.2. *Let $f: (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ and $g: (\mathbf{C}^{m+1}, 0) \to (\mathbf{C}, 0)$ be complex polynomial mappings, each having an isolated singularity at the origin. Choose $0 < \epsilon \ll 1$ and $0 < \delta_1 \ll \epsilon$, $0 < \delta_2 \ll \epsilon$. Let neighborhoods N_f and N_g be given as follows*

$$N_f = \{x \in \mathbf{C}^{n+1} | \; \|x\| \leq \epsilon, \quad \|f(x)\| \leq \delta_1\}$$
$$N_g = \{y \in \mathbf{C}^{m+1} | \; \|y\| \leq \epsilon, \quad \|g(y)\| \leq \delta_2\}.$$

Then

$$\mathscr{L}(f+g) \cong (\partial(N_f \times N_g), (f+g)^{-1}(0) \cap \partial(N_f \times N_g))$$

after smoothing corners.

Remark. To avoid problems with smoothing corners one should choose $\delta_1 \neq \delta_2$ (see the remark at the end of §3).

Proof. Suppose we have a vector field v for $f+g$ as in Lemma 4.1 (in particular v is tangential to $(f+g)^{-1}(0)$) but with the additional properties: along v-trajectories each of the functions $\lambda(x, y) = \|x\|, \|y\|, \|f(x)\|, \|f(y)\|$ is non-decreasing, and actually strictly increasing outside a small neighborhood of $\lambda(x, y) = 0$. Then by pushing points of $\partial(N_f \times N_g)$ out along v-trajectories we obtain a diffeomorphism (except for the corners) $h: (\partial(N_f \times N_g), (f+g)^{-1}(0) \cap \partial(N_f \times N_g)) \to (S^{2(n+m)+3}, L(f+g))$, which can be interpreted as a smoothing of corners of the first pair. By uniqueness up to diffeomorphism of such smoothing the lemma is thus proven.

To find a suitable v choose vector fields v_f and v_g for f and g as in Lemma 4.1. Then $v = 1/2(v_f, v_g)$ has the required properties on $(\mathbf{C}^{n+1}-\{0\})\times(\mathbf{C}^{m+1}-\{0\})$ and $v=(v_f,0)$ respectively $v=(0,v_g)$ do so on neighborhoods of $(\mathbf{C}^{n+1}-\{0\}\times\{0\}$ and $\{0\}\times(\mathbf{C}^{m+1}-\{0\})$ respectively. Pasting by a partition of unity gives the required v, completing the proof.

PROPOSITION 4.3. *Let f and g be complex polynomial mappings as in* 4.2. *Then* $\mathscr{L}(f+g)\cong\mathscr{L}(f)\otimes\mathscr{L}(g)$ *(isotopic).*

Proof. Since $\mathscr{L}(g)\cong\mathscr{L}(-g)$, we can look instead at $\mathscr{L}(f-g)$. By 4.2 this is isomorphic to the pair $(\partial(N_f\times N_g),(f-g)^{-1}(0)\cap(\partial N_f\times N_g))=(\partial(N_f\times N_g),\partial X)$ where X is the pull-back

$$\begin{tikzcd} X \arrow[r] \arrow[d] & N_g \arrow[d, "g"] \\ N_f \arrow[r, "f"] & D^2. \end{tikzcd}$$

But this has already been identified as $\mathscr{L}(f)\otimes\mathscr{L}(g)$ in Lemma 3.5, since the maps $N_f\to D^2$ and $N_g\to D^2$ are branched fibrations corresponding to $\mathscr{L}(f)$ and $\mathscr{L}(g)$ respectively.

The real case

In the real case problems are caused by the fact that Lemma 4.1 is no longer true (a counter-example is given by Milnor[17, p. 99]). The following weaker version is still true and will be useful.

LEMMA 4.4. *Let $f:(\mathbf{R}^{n+1},0)\to(\mathbf{R}^2,0)$ be a real polynomial with isolated singularity at the origin. For sufficiently small ϵ there exists a smooth vector field v on $D_\epsilon-\{0\}$ such that $|z|$ increases along v-trajectories and $|f(z)|$ increases or is constant zero along any v-trajectory.*

Proof. Such a vector field is constructed on $D_\epsilon-f^{-1}(0)$ in the proof of 11.3 in [17], and can be constructed in a neighborhood of $(D_\epsilon-\{0\})\cap f^{-1}(0)$ as in Lemma 4.1 above.

We shall show that Lemma 4.2 holds without change in the real case, whence Proposition 4.3 also follows. Suppose therefore we have $f:(\mathbf{R}^{n+1},0)\to(\mathbf{R}^2,0)$, $g:(\mathbf{R}^{m+1},0)\to(\mathbf{R}^2,0)$ polynomial mappings with isolated singularities at zero. Let

$$N_f=\{x\in\mathbf{R}^{n+1}|\,\|x\|\le\epsilon,\|f(x)\|\le\delta_1\}$$
$$N_g=\{y\in\mathbf{R}^{m+1}|\,\|y\|\le\epsilon,\|g(y)\|\le\delta_2\}$$

where ϵ, δ_1, δ_2 will be chosen sufficiently small. To prove Lemma 4.2 we shall, as in the complex case, use a vector field v which points in a direction of increasing $\|(x,y)\|$ and is tangent to $V=(f+g)^{-1}(0)$. That such a vector field exists is clear by transversality of V with small spheres, but we must check that we can choose v transversal outwards on the boundary of $N_f\times N_g$. So this first choice so far only works outside $N_f\times N_g$. If v_f and v_g are vector fields as in Lemma 4.4 for f and g then, as in the proof of 4.2, they can be used to construct a vector field v which points in a direction of increasing $\|(x,y)\|$ and is transversal outwards on $\partial(N_f\times N_g)$; however, this v may not be tangent to V, so it can only be used in the complement of V. It thus remains to construct a suitable vector field v on a neighborhood of $V\cap\partial(N_f\times N_g)$; the three local candidates for v can then be put together by a partition of unity.

The boundary of $N_f\times N_g$ is the union of the pieces

$$A=\{x|\,\|x\|=\epsilon,\|f(x)\|\le\delta_1\}\times N_g,$$
$$B=N_f\times\{y|\,\|y\|=\epsilon,\|g(y)\|\le\delta_2\},$$
$$C=\{x|\,\|x\|\le\epsilon,\|f(x)\|=\delta_1\}\times N_g,$$
$$D=N_f\times\{y|\,\|y\|\le\epsilon,\|g(y)\|=\delta_2\}.$$

A suitable vector field is easily found in a neighborhood of $V\cap A$ and $V\cap B$, so it remains only to consider $V\cap(C\cup D)$. Let k and l be the restrictions to V of the polynomial functions $\|f(x)\|^2$ and $\|(x,y)\|^2$. An application of the curve selection lemma as in Corollary 3.4 of Milnor[17] shows that near zero on V the differentials $\mathrm{d}k$ and $\mathrm{d}l$ cannot point in opposite directions unless one of them vanishes (see Remark below). But $\mathrm{d}l$ only vanishes at zero, and $\mathrm{d}k$ only along $f(x)=0$ since $f(x)$ (which equals $-g(y)$ on V) is regular on V near the origin except when $f(x)=-g(y)=0$. Since $V\cap(C\cup D)$ contains no point with $f(x)=0$, there exists a vector field on V in a neighborhood of $V\cap(C\cup D)$ pointing in a direction of increasing $\|(x,y)\|$ and increasing $\|f(x)\|=\|g(y)\|$. If v is an extension of this to a neighborhood of $V\cap(C\cup D)$ in $\mathbf{R}^{n+1}\times\mathbf{R}^{m+1}$, then v has the desired properties in a neighborhood of any point of $V\cap(C\cup D)-(A\cup B)$. This completes the proof.

Remark. To extend the proof of Milnor's Corollary 3.4 to the situation mentioned above, one must check that $\mathrm{d}k$ and $\mathrm{d}l$ can be described by polynomials. This is in fact not true, but with the natural identification of the cotangent space of V with a subspace of $\mathbf{R}^{n+1}\times\mathbf{R}^{m+1}$, one has

$$\|\mathrm{grad}\,(f+g)\|^2\,\mathrm{d}k=\|\mathrm{grad}\,(f+g)\|^2(\mathrm{grad}\,k)-\langle\mathrm{grad}\,(f+g),\mathrm{grad}\,k\rangle\,\mathrm{grad}\,(f+g)$$

so one can use $\|\mathrm{grad}\,(f+g)\|^2\,\mathrm{d}k$ instead of $\mathrm{d}k$ and similarly for $\mathrm{d}l$. The proof in question then extends without difficulty.

We have thus proved

PROPOSITION 4.5. *Proposition 4.3 holds for isolated real polynomial singularities.*

For arbitrary tame singularities f and g the above argument fails. In fact it is even not clear if $f+g$ is necessarily tame, so $\mathscr{L}(f+g)$ may not even be defined.† Suppose however that $f+g$ is also tame. Then the neighborhood pair $(N_{f+g},(f+g)^{-1}(0)\cap N_{f+g})$ is isomorphic to the cone over its boundary, $\mathscr{L}(f+g)=(\partial N_{f+g},L(f+g))$, and the neighborhood pair $(N_f\times N_g,(f+g)^{-1}(0)\cap(N_f\times N_g))$ is isomorphic to the cone over its boundary which we have seen is $\mathscr{L}(f)\otimes\mathscr{L}(g)$. Thus the cones on $\mathscr{L}(f+g)$ and $\mathscr{L}(f)\otimes\mathscr{L}(g)$ are locally isomorphic, so by a standard argument (see Lemma 1.5) we have.

PROPOSITION 4.6. *If $f:(\mathbf{R}^{n+1},0)\to(\mathbf{R}^2,0)$, $g:(\mathbf{R}^{m+1},0)\to(\mathbf{R}^2,0)$ are smooth maps with tame isolated singularities at the origin and $f+g$ is also tame, then $\mathscr{L}(f+g)$ and $\mathscr{L}(f)\otimes\mathscr{L}(g)$ are h-cobordant (in the strongest sense).*

COROLLARY 4.7. *In the situation of 4.6, if $n\geq 3$ and $m\geq 3$ then $\mathscr{L}(f+g)\cong\mathscr{L}(f)\otimes\mathscr{L}(g)$ (isotopic).*

Proof. A van Kampen argument shows $\pi_1(L(f)\otimes L(g))=\{1\}$. In the next section we will show that $\mathscr{L}(f)\otimes\mathscr{L}(g)$ has a fibered structure with simply connected fiber, so $\pi_1(S^{n+m+1}-L(f)\otimes L(g))=\mathbf{Z}$. Given an h-cobordism (M^{n+m+2},N^{n+m}) between $\mathscr{L}(f+g)$ and $\mathscr{L}(f)\otimes\mathscr{L}(g)$, apply the h-cobordism theorem first to N and then the relative h-cobordism theorem to the complement of an open tubular neighborhood of N. This complement has fundamental group $\mathbf{Z}$, so the h-cobordism theorem applies $(\mathrm{Wh}\,(\mathbf{Z})=\{1\})$. This proves $(M,N)\cong(S^{n+m+1},L(f)\otimes L(g)\times I)$ and hence the corollary.

Once we have the explicit description of fibered structure of a product of fibered knots, it will become clear that the isotopy $\mathscr{L}(f+g)\cong\mathscr{L}(f)\otimes\mathscr{L}(g)$ for complex polynomial singularities preserves fibered structure. In the real polynomial case this is however not clear, and remains a problem in low dimensions. In higher dimensions, namely in the situation of Corollary 4.7, there is no problem, since by a theorem of Durfee[5] and Kato[11], two fibered knots in this dimension which are isotopic as knots are isotopic as fibered knots. This theorem needs simple connectivity of both knot and fiber, which however holds for $\mathscr{L}(f)\otimes\mathscr{L}(g)$, and hence also for $\mathscr{L}(f+g)$ since the fiber of a fibered knot is unique up to h-cobordism.

We can thus state (slightly prematurely).

PROPOSITION 4.8. *The isotopies $\mathscr{L}(f+g)\cong\mathscr{L}(f)\otimes\mathscr{L}(g)$ of Proposition 4.3 and Corollary 4.7 preserve fibered structure. This is also true of the h-cobordism in Proposition 4.6.*

†For a tame isolated singularity $f:\mathbf{R}^{n+1}\to\mathbf{R}^2$ we define $\mathscr{L}(f)=(\partial N_f,L(f))$, which has a natural fibered structure (§1), rather than $\mathscr{L}(f)=(S_\epsilon^n,L(f))$, which doesn't. For polynomial singularities these definitions are compatible, since we may push one out into the other along a vector field as in Lemma 4.4.

Corollary 4.7 and Proposition 4.8 also hold under the modified assumption: $n+m \geq 6$ and $\pi_1(L(f) \otimes L(g)) = \{1\}$.

§5. FIBERED STRUCTURE AND EXTERIORS OF PRODUCTS

In this section we examine the knot complement of a product of knots. It will then be apparent that the product of fibered knots is again fibered. Other structural details will also emerge and to clarify these we preface our discussion with some remarks on arbitrary knots.

If $\mathcal{K} = (S^k, K)$ is any knot then by Lemma 2.3 there exists a map $a\colon S^k \to D^2$ with $0 \in D^2$ as a regular value and $K = a^{-1}(0)$ as an oriented submanifold. The map $\bar{a} = a/\|a\|\colon S^k - K \to S^1$ represents $S^k - K$ as a "fibration with singular fibers" over S^1. The non-singular fibers of this map are spanning surfaces for K and are of great importance for the calculation of homological invariants of $\mathcal{K}$. One can always arrange that the critical points of $\bar{a}$ are non-degenerate, hence isolated, hence finite in number, since they occur in a compact subset of $S^k - K$.

Now suppose $\mathcal{L} = (S^l, L, b)$ is a fibered knot. We shall show that the knot complement $S^{k+l+1} - K \otimes L$ of the product admits a fibration with singular fibers over S^1 as above, which is a fiberwise $\mathcal{L}$-branched fibration of $\mathring{D}^{k+1} \times S^1$ branched along a fiberwise embedding $S^k - K \hookrightarrow \mathring{D}^{k+1} \times S^1$. In particular a fiber of $S^{k+l+1} - K \otimes L$ is singular only if the corresponding fiber of $S^k - K$ is.

In order to describe this fibering, we need to be a bit more precise about the mapping $\alpha\colon D^{k+1} \to D^2$ associated with a knot $K \subset S^k$. When $K \subset S^k$ is fibered, we have taken this map to be the associated branched fibration. Even when the knot is not fibered, there is still a smooth map α such that $\alpha^{-1}(0)$ is homeomorphic to CK. The next lemma articulates a useful vector field associated with such a map. This vector field will play the same role as the vector fields used in studying polynomial singularities (compare with Lemma 4.1).

Lemma 5.1. *Let $\mathcal{K} = (S^k, K)$ be a knot and $a\colon S^k \to D^2$ a smooth map with $0 \in D^2$ as a regular value and $K = a^{-1}(0)$. Then there exists a smooth extension $\alpha\colon D^{k+1} \to D^2$ of a and a vector field v on $D^{k+1} - \{0\}$ such that*

(i) *v lifts the radial vector field $w(y) = y$ on D^2.*

(ii) *$\|x\|$ increases along v-trajectories, $x \in D^{k+1}$.*

(iii) *the critical set of α is the cone consisting of all v-trajectories through critical points of $\bar{a} = a/\|a\|$.*

Proof. (iii) is a consequence of (i) and (ii). To satisfy (i) and (ii), smooth the cone on a as in §1. Then the radial vector field on D^{k+1} is suitable after adjusting by the positive real valued factor $|\alpha(x)|/|D\alpha_x(x)|$, $x \in D^{k+1}$, to satisfy (i). Note that $|D\alpha_x(x)|$ is non-zero on $D^{k+1} - \{0\}$. QED.

Using the notation of Lemma 5.1, we may assume that $1 \in S^1$ is a regular value of $\bar{a}$, so $\bar{a}^{-1}(1) \cup K = F$ is a smooth spanning surface for K. Then, for sufficiently small ϵ, we have that $\alpha^{-1}(\epsilon) = F_\epsilon$ is a properly embedded smooth submanifold of D^{k+1} isomorphic to F and with boundary isotopic to K, as can be seen by pushing F_ϵ out to F along the vector field v. We may further assume that $\alpha(D^{k+1})$ is contained in the disk of radius $1-\epsilon$, so that the map $\alpha - \epsilon\colon D^{k+1} \to D^2$ is defined. Note that $\alpha - \epsilon$ has zero as a regular value and $(\alpha - \epsilon)^{-1}(0) = F_\epsilon$.

Now if $\beta\colon D^{l+1} \to D^2$ is the branched fibration corresponding to the fibered knot $\mathcal{L} = (S^l, L, b)$ then we have the pull-back branched fibration

$$\begin{array}{ccc} X_\epsilon & \longrightarrow & D^{l+1} \\ \big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ D^{k+1} & \xrightarrow{\alpha-\epsilon} & D^2 \end{array}$$

and by definition, $K \otimes L = \partial X_\epsilon$. As in the proof of 3.5, as ϵ goes to zero a singularity appears in the interior of X_ϵ but not in the boundary, so

$$K \otimes L = \partial X_0 = \partial\{(x, y) \in D^{k+1} \times D^{l+1} \mid \alpha(x) - \beta(y) = 0\}.$$

For sufficiently small ϵ write

$$E_K = \{x \in S^k |\ \|\alpha(x)\| \geq \epsilon\} = S^k - K \times \mathring{D}^2$$

$$E'_K = \{x \in D^{k+1} |\ \|\alpha(x)\| = \epsilon\}.$$

There is an isomorphism $E_K \cong E'_K$, $x \mapsto x'$, obtained by pushing in along the vector field of Lemma 5.1.

PROPOSITION 5.2. *With notation as above, let*

$$E = \{(x, y) \in D^{k+1} \times D^{l+1} |\ \|\alpha(x) - \beta(y)\| = \epsilon\}.$$

Then

(i) $E \cong E_{K \otimes L} = S^{k+l+1} - (K \otimes L \times \mathring{D}^2)$.

(ii) $E \xrightarrow{\pi} D^{k+1} \times S^1$, *where* π *is the mapping* $\pi(x, y) = (x, (\alpha(x) - \beta(y))/\epsilon)$, *exhibits* E *as a branched fibration of* $D^{k+1} \times S^1$ *along* $\{(x, \lambda) \in D^{k+1} \times S^1 | \alpha(x) = \lambda\epsilon\} = j(E_K)$, *where* $j: E_K \to D^{k+1} \times S^1$ *by* $j(x) = (x', \bar{a}(x))$.

(iii) *Hence if both* $(\mathcal{K}, a)$ *and* $(\mathcal{L}, b)$ *are fibered knots, then* $\mathcal{K} \otimes \mathcal{L}$ *has fibered structure given by* $E_{K \otimes L} \cong E \xrightarrow{pr \circ \pi = P} S^1$ *where* $pr: D^{k+1} \times S^1 \to S^1$ *is projection on the second factor.*

(iv) *With* $P: E \to S^1$ *as in* (iii), *the fiber* $P^{-1}(\lambda)$ *is describable as follows: Take the fiber* $\bar{a}^{-1}(\lambda) \subset S^k$ *of* $\mathcal{K}$ *and push it (keeping its boundary fixed) into* D^{k+1} *to get a proper embedding* $F' \subset D^{k+1}$ *and then take the* $\mathcal{L}$*-branched fibration of* D^{k+1} *branched along* F'. *This holds even if* $\mathcal{K}$ *is not a fibered knot if* λ *is a regular value of* $\bar{a}$.

Proof. (i) Take vector fields v_a and v_b on D^{k+1} and D^{l+1} as in Lemma 5.1. The vector field $v = (v_a, v_b)$ on $D^{k+1} \times D^{l+1}$ is transverse to E, and pushing E out along v yields an isomorphism

$$E \cong \{(x, y) \in \partial(D^{k+1} \times D^{l+1}) = S^{k+l+1} |\ \|\alpha(x) - \beta(y)\| \geq \epsilon\}.$$

Since

$$\{(x, y) \in S^{k+l+1} |\ \|\alpha(x) - \beta(y)\| \leq \epsilon\} \cong (K \otimes L) \times D^2,$$

we have $E \cong E_{K \otimes L}$.

(ii) is clear. In fact the map $D^{k+1} \times S^1 \xrightarrow{\varphi} D^2$, where $\varphi(x, \lambda) = \alpha(x) - \epsilon\lambda$, has zero as a regular value and $\varphi^{-1}(0) = j(E_K)$, so the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{(x,y)\mapsto x} & D^l \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \beta} \\ D^k \times S^1 & \xrightarrow{\varphi} & D^2 \end{array}$$

identifies E as the $\mathcal{L}$-branched fibration in question.

(iii) The map $P: E \to S^1$ is given by $P(x, y) = (\alpha(x) - \beta(y))/\epsilon$. Thus on $\partial E = \partial((K \otimes L) \times D^2) = (K \otimes L) \times S^1$ it is projection on the second factor as it should be. Now it follows easily from Lemma 5.1 (iii) that $\lambda \in S^1$ is a critical value of P if and only if it is a critical value of $\bar{a}$, so P is a fibration if $\bar{a}$ is.

(iv) We can assume $\lambda = 1$. Then the fiber $P^{-1}(1)$ is the space X_ϵ defined above, which is the $\mathcal{L}$-branched fibration of D^{k+1} along $F_\epsilon = \alpha^{-1}(\epsilon)$. As remarked above, F_ϵ is obtainable by pushing $F = \bar{a}^{-1}(1)$ into D^{k+1} along the vector field v. This completes the proof.

Remark. In part (i) above we used a vector field to show

$$E \cong E_{K \otimes L} = \{(x, y) \in \partial(D^{k+1} \times D^{l+1}) \|\alpha(x) - \beta(y)\| \geq \epsilon\}.$$

Let $\bar{P}: (D^{k+1} \times D^{l+1}) - X_0 \to S^1$ be defined by the formula $\bar{P}(x, y) = (\alpha(x) - \beta(y))/\|\alpha(x) - \beta(y)\|$. Then $\bar{P}|E = P$. The vector field we used points in the direction of constant $\bar{P}$, so the above isomorphism commutes with the maps $P = \bar{P}|E$ and $\bar{P}|E_{K \otimes L}$ to S^1. A similar remark holds in the proof of 4.2. It hence follows that the isotopy of links of complex polynomial singularities $\mathcal{L}(f + g) \cong \mathcal{L}(f) \otimes \mathcal{L}(g)$ of Proposition 4.3 does preserve fibered structure, as claimed in §4.

Another way of interpreting Proposition 5.2 is as follows: $E_{K \otimes L}$ is the $\mathcal{L}$-branched fibration of $D^{k+1}\times S^1$ along $j(E_K)$ where $j\colon E_K\to D^{k+1}\times S^1$ is essentially the embedding of Lemma 2.4. That is, $S^{k+2}=(D^{k+1}\times S^1)\cup(S^k\times D^2)$, and by 2.4 the triple $K\subset S^k\subset S^{k+2}$ gives an embedding $j\colon S^k\to S^{k+2}$ with $j(S^k)\cap S^k=K$ transversally. The embedding $j\colon E_K\to D^{k+1}\times S^1$ is just the restriction.

§6. FIBER STRUCTURE AND SEIFERT PAIRING

This section is a continuation of §5. Given $K\otimes L\subset S^{k+l+1}$, we shall examine submanifolds of S^{k+l+1} with boundary $K\otimes L$.

Let $\mathcal{K}=(S^k,K)$ be an arbitrary knot and choose $\alpha\colon D^{k+1}\to D^2$ as in Lemma 5.1. Let $\mathcal{L}=(S^l,L)$ be a fibered knot with $\beta\colon D^{l+1}\to D^2$ a branched fibration for $\mathcal{L}$. By §5, $\mathcal{K}\otimes\mathcal{L}\cong(\partial(D^{k+1}\times D^{l+1}),\partial X_0)$ where X_0 is the pull-back (we are replacing β by $-\beta$ for convenience)

$$\begin{array}{ccc} X_0 & \longrightarrow & D^{l+1} \\ \downarrow & & \downarrow{\scriptstyle -\beta} \\ D^{k+1} & \xrightarrow{\alpha} & D^2. \end{array}$$

That is,

$$X_0=\{(x,y)\in\partial(D^{k+1}\times D^{l+1})\,|\,\alpha(x)+\beta(y)=0\}.$$

Let $\gamma\colon D^{k+1}\times D^{l+1}\to D^2$ be given by the formula $\gamma(x,y)=1/2(\alpha(x)+\beta(y))$. Let $a=\alpha|S^k$, $b=\beta|S^l$ and $c=\gamma|\partial(D^{k+1}\times D^{l+1})=\gamma|S^{k+l+1}$.

For $t\in S^1$ denote the radial line from 0 to t by $[0,t]$ and put $F_t=a^{-1}[0,t]$, $G_t=b^{-1}[0,t]$, $H_t=c^{-1}[0,t]$, so G_t is a fiber of $\mathcal{L}$ and F_t and H_t are (possibly singular) spanning surfaces for $\mathcal{K}$ and $\mathcal{K}\otimes\mathcal{L}$. Choose t_0 so that F_{t_0} is nonsingular. Then by the proof of 5.2, H_{t_0} is also a good spanning surface for $\mathcal{K}\otimes\mathcal{L}$. Also, all nearby values of t_0 will still have this property.

LEMMA 6.1. *There exists an inclusion $F_t*G_t\to H_t$ which is a homotopy equivalence for t near t_0, by a homotopy equivalence which varies continuously with t. Furthermore, the diagram*

$$\begin{array}{ccc} F_t*G_t & \xrightarrow{i_t*j_t} & S^k*S^l \\ \downarrow & & \| \\ H_t & \longrightarrow & S^{k+l+1} \end{array}$$

commutes, where the horizontal arrows are the inclusions.

Proof. Recall that the join of two spaces X and Y can be described as the union $X*Y=(CX\times Y)\cup(X\times CY)$, pasted along $X\times Y$, where CX denotes the cone over X. Using the vector field of 5.1, it is clear that $CF_t\approx\{y\in D^{l+1}|\beta(y)\in[0,t]\}$, so we can express F_t*G_t as a subset of

$$S^k*S^l=(CS^k\times S^l)\cup(S^k\times CS^l)=(D^{k+1}\times S^l)\cup(S^k\times D^{l+1})=\partial(D^{k+1}\times D^{l+1})$$

as follows:

$$\begin{aligned} F_t*G_t&=(CF_t\times G_t)\cup(F_t\times CG_t)\\ &=\{(x,y)\in D^{k+1}\times S^l|\alpha(x)\in[0,t],\beta(y)\in[0,t]\}\\ &\cup\{(x,y)\in S^k\times D^{l+1}|\alpha(x)\in[0,t],\beta(y)\in[0,t]\}\\ &=\{(x,y)\in\partial(D^{k+1}\times D^{l+1})|\alpha(x)\in[0,t]\quad\text{and}\quad\beta(y)\in[0,t]\}. \end{aligned}$$

Since

$$H_t=\{(x,y)\in\partial(D^{k+1}\times D^{l+1})|1/2(\alpha(x)+\beta(y))\in[0,t]\},$$

we have a natural inclusion $F_t*G_t\to H_t$ making the diagram of the lemma commute.

It thus remains to show that this inclusion is a homotopy equivalence for t near t_0, by a homotopy equivalence which varies continuously with t. For this we consider the followin diagram of inclusions:

$$A = \{(x, y) \in (D^{k+1} \times D^{l+1}) - \{0\} | \alpha(x) \in [0, t], \beta(y) \in [0, t]\}$$

$$B = \{(x, y) \in (D^{k+1} \times D^{l+1}) - \{0\} | 1/2(\alpha(x) + \beta(y)) \in [0, t]\}$$

$$\begin{tikzcd} F_t * G_t = \partial A \arrow[r, hook] \arrow[d, hook] & A \arrow[d, hook] \\ H_t = \partial B \arrow[r, hook] & B. \end{tikzcd}$$

The horizontal arrows are homotopy equivalences, since we can slide points out to the relevant boundaries using the vector fields of Lemma 5.1. It thus suffices to show that the inclusion $A \subset B$ is a homotopy equivalence. For this we need two new vector fields.

Using the fact that any t close to t_0 is a regular value of the map $\bar{a} = a/\|a\|: S^k - K \to S^1$, we can easily modify the vector field of 5.1 slightly to get a vector field v on D^{k+1} such that $\|x\|$ is non-decreasing along v-trajectories and v lifts a vector field on D^2 of the form illustrated in Fig. 3(a). Here $[0, t]$ is a line of zeros of the field on D^2.

On the other hand, using the fact that β is a branched fibration, we can find a vector field on D^{l+1} which covers the vector field on D^2 given by Fig. 3(b), and by mixing this in the right proportions with a "radial" vector field, as given by 5.1, we can obtain a vector field w on D^{l+1} which covers the vector field on D^2 shown in Fig. 3(c). Now given any $(x, y) \in B$ we can simultaneously move x along a v-trajectory and y along a w-trajectory in such a way that $1/2(\alpha(x) + \beta(y))$ remains in the interval $[0, t]$. This gives a deformation retraction of B onto A as was to be shown. The required continuity properties are easily seen. This completes the proof of Lemma 6.1.

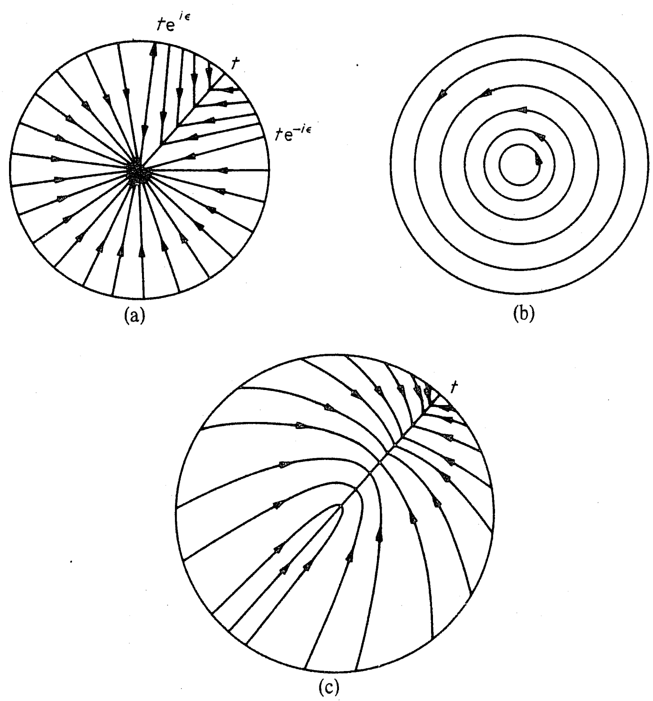


Fig. 3.

Remark. Let $g: G \to G$ be the gluing map for the fibration over S^1 associated with the book structure on $\mathscr{L}$. Suppose $\mathscr{K}$ also has an open book structure with gluing map $f: F \to F$. Then the above lemma implies that the gluing map $h: H \to H$, when restricted to $F * G$, is given by $f * g: F * G \to F * G$.

In this situation the homology map induced by f, g or h is called the *monodromy* of the fibered knot $\mathcal{K}, \mathcal{L}, \mathcal{K} \otimes \mathcal{L}$. Since $\tilde{H}_*(F * G)$ is naturally isomorphic to $(\tilde{H}_*(F) \otimes \tilde{H}_*(G))_{+1}$ (see Proposition 6.2), this shows that the *monodromy of a product of fibered knots is the tensor product of the individual monodromies*. For links $\mathcal{L}(f+g) = \mathcal{L}(f) \otimes \mathcal{L}(g)$ of isolated complex hypersurface singularities this was first shown by Thom and Sebastiani[20].

Lemma 6.1 will allow us to extract algebraic information about $K \otimes L \subset S^{k+l+1}$. Recall that we have Seifert pairings

$$\theta^*_{\mathcal{K}}: \mathcal{H}_p(F) \times \mathcal{H}_{k-p-1}(F) \longrightarrow \mathbf{Z}$$
$$\theta^*_{\mathcal{L}}: \mathcal{H}_q(G) \times \mathcal{H}_{l-q-1}(G) \longrightarrow \mathbf{Z}.$$

Here $\mathcal{H}_*$ denotes the quotient of the reduced homology, $\tilde{H}_*$, by the torsion subgroup.

The Seifert pairing is defined by the formula $\theta^*(x, y) = l(i_*x, y)$ where i_*x is the result of translating x into the complement of the spanning manifold in the positive normal direction. The symbol $l(\,,)$ means linking number in the ambient sphere. Actually, this pairing depends upon the choice of spanning manifold for the knot. When we write $\theta^*_{\mathcal{K}}, \theta^*_{\mathcal{L}}, \theta^*_{\mathcal{K}\otimes\mathcal{L}}$, we understand that these pairings are taken for the specific manifolds F, G, and H respectively.

In order to formulate the next proposition it is convenient to shift the grading on $\mathcal{H}_*$ by putting $\mathcal{H}_p^- = \mathcal{H}_{p-1}$.

PROPOSITION 6.2. *Let $\mathcal{K} = (S^k, K)$ and $\mathcal{L} = (S^l, L)$ be as above with $K = \partial F$, $L = \partial G$, $K \otimes L = \partial H$ as above. Then*

$$\mathcal{H}^-_*(H) \cong \mathcal{H}^-_*(F) \otimes \mathcal{H}^-_*(G)$$

and

$$\theta^*_{\mathcal{K}\otimes\mathcal{L}} \cong \theta^*_{\mathcal{K}} \otimes \theta^*_{\mathcal{L}},$$

where we are taking the standard graded tensor product but using $\mathcal{H}^-_*$-grading. That is, for elements of homogeneous degree:

$$\theta^*_{\mathcal{K}\otimes\mathcal{L}}(a \otimes a', b \otimes b') = (-1)^{|a'||b|}\theta^*_{\mathcal{K}}(a, b)\theta^*_{\mathcal{L}}(a', b'),$$

where $|x|$ denotes the degree of x in the $\mathcal{H}^-_*$-grading (that is, $x \in \mathcal{H}^-_{|x|} = \mathcal{H}_{|x|-1}$).

The following special case of this proposition is particularly important for applications. Suppose $k = 2n+1$, $l = 2m+1$, and denote by $\theta_{\mathcal{K}}: \mathcal{H}_n(F) \times \mathcal{H}_n(F) \to \mathbf{Z}$, $\theta_{\mathcal{L}}: \mathcal{H}_m(G) \times \mathcal{H}_m(G) \to \mathbf{Z}$ the middle-dimensional part of $\theta^*_{\mathcal{K}}, \theta^*_{\mathcal{L}}$.

COROLLARY 6.3. *In the above situation if $\mathcal{L}$ is a simple fibered knot (that is G is $(m-1)$-connected), then*

$$\theta_{\mathcal{K}\otimes\mathcal{L}} = (-1)^{(n+1)(m+1)}\theta_{\mathcal{K}} \otimes \theta_{\mathcal{L}}.$$

Proof of 6.2. The isomorphism

$$(\mathcal{H}^-_*(F) \otimes \mathcal{H}^-_*(G))_n = \bigotimes_{p+q=n} \mathcal{H}_p^-(F) \otimes \mathcal{H}_q^-(G) \longrightarrow \mathcal{H}_n^-(F * G) = \mathcal{H}_n^-(H)$$

is given by $[a] \otimes [b] \to [a * b]$. That this gives an isomorphism follows immediately from the Mayer Vietoris sequence for $F * G = (CF \times G) \cup (F \times CG)$. The rest of the proposition follows from the definition of the Seifert pairing, from Lemma 6.1, and from the following lemma.

LEMMA 6.4. *Let α and β be disjoint cycles of dimension p and $k-p-1$ in S^k and let α', β' be disjoint cycles of dimension q and $l-q-1$ in S^l. Then in $S^k * S^l = S^{k+l+1}$,*

$$l(\alpha * \alpha', \beta * \beta') = (-1)^{(q+1)(k-p)} l(\alpha, \beta) l(\alpha', \beta').$$

Proof. If $\alpha, \beta, \alpha', \beta'$ are embedded spheres, this is proved algebraically by Sakamoto[19]. We describe below how the general case follows geometrically from this, but the lemma can also be seen completely geometrically as follows. By definition $l(x, y) = \langle x, Y \rangle$ where $\langle\,,\rangle$ is the intersection number in the appropriate sphere, and $\partial Y = y$. Note that if $\partial B = \beta$, then $\partial(B * \beta') = \beta * \beta'$, so $l(\alpha * \alpha', \beta * \beta') = \langle \alpha * \alpha', B * \beta' \rangle$. Dividing $\alpha * \alpha'$ up as $\alpha * \alpha' = (\alpha \times C\alpha') \cup (C\alpha \times \alpha') \subset (S^k \times CS^l) \cup (CS^k \times S^l)$ and similarly $B * \beta' = (B \times C\beta') \cup (CB \times \beta')$, we see that the only contribution to $\langle \alpha * \alpha', B * \beta' \rangle$ comes from the intersection of $\alpha \times C\alpha'$ and

$B \times C\beta'$ (since all other intersections are empty), so

$$\begin{aligned}\langle \alpha * \alpha', B * \beta' \rangle &= (-1)^{(q+1)(k-p)} \langle \alpha, B \rangle \langle C\alpha', C\beta' \rangle \\ &= (-1)^{(q+1)(k-p)} l(\alpha, \beta) \langle C\alpha', C\beta' \rangle,\end{aligned}$$

where the sign comes from transposing B and $C\alpha'$ of dimensions $(k-p)$ and $(q+1)$ respectively. It thus only remains to show $\langle C\alpha', C\beta' \rangle = l(\alpha', \beta')$. This is in fact a known alternative definition of linking numbers in the sphere. However, we can also argue as follows: If α' and β' are standardly embedded spheres with linking number ± 1 then the formula $\langle C\alpha', C\beta' \rangle = l(\alpha', \beta')$ is geometrically clear, so Lemma 6.4 is proved in this case. It is clearly also proven if we can choose α and B above disjoint. By symmetry it also holds if we can find B' with $\partial B' = \beta'$ and $\alpha' \cap B' = \emptyset$. By linearity the lemma thus holds if α' and β' are disjoint unions of "nicely positioned" embedded spheres. We now show how to reduce to this case.

For arbitrary cycles α, β, α', β', choose B' with $\partial B' = \beta'$ and assume B' and α' are in general position, so they intersect only in isolated transversal intersections of top dimensional simplices. Let D be the union of small disks in B' around these intersection points and put $\beta'' = \partial D$. If $B'' = B' - \mathring{D}$, then $\partial B'' = \beta' \cup (-\beta'')$ and $B'' \cap \alpha' = \emptyset$. Hence by definition of linking,

$$l(\alpha', \beta') - l(\alpha', \beta'') = l(\alpha', \beta' \cup -\beta'') = \langle \alpha', B'' \rangle = 0.$$

By the same arguments, since $(\beta * B'') \cap (\alpha * \alpha') = \emptyset$, $l(\alpha * \alpha', \beta * \beta') - l(\alpha * \alpha', \beta * \beta'') = 0$. Thus the lemma holds for $\alpha, \beta, \alpha', \beta'$ if it holds for $\alpha, \beta, \alpha', \beta''$. In this way we can successively replace each of $\alpha, \beta, \alpha', \beta'$ by a union of standardly embedded spheres which are "nicely positioned" with respect to each other. This is more than we needed to do.

Example. If we apply Corollary 6.3 to cyclic suspensions we get the following result.

PROPOSITION 6.5. *Let $K \subset S^{2n+1}$ be a knot with spanning manifold F and corresponding Seifert pairing $\theta_{\mathscr{K}}\colon \mathscr{H}_n(F) \times \mathscr{H}_n(F) \to \mathbf{Z}$. Then the a-fold cyclic suspension $K \otimes [a] \subset S^{2n+3}$ spans a manifold $F_a \subset S^{2n+3}$ with F_a the homotopy type of $F * (\mathbf{Z}/a\mathbf{Z})$ and Seifert pairing*

$$\theta_{\mathscr{K} \otimes [a]} \cong (-1)^{n+1} \theta_{\mathscr{K}} \otimes \Lambda_a.$$

Here Λ_a denotes the Seifert pairing of the empty knot of degree a.

Since the empty knot has a spanning manifold consisting of a points in S^1 it is easy to see that Λ_a has an $(a-1) \times (a-1)$ matrix form,

$$\Lambda_a = \begin{bmatrix} 1 & -1 & & \bigcirc \\ & 1 & -1 & \\ & & \ddots & \ddots \; -1 \\ \bigcirc & & & 1 \end{bmatrix}.$$

This result may be used to deduce the signature periodicity theorem of [18] by using the method outlined in ([6], §5).

By iteration we find that the Seifert pairing of $[a_0] \otimes [a_1] \otimes \cdots \otimes [a_n]$ is $(-1)^{n(n+1)/2} \Lambda_{a_0} \otimes \Lambda_{a_1} \otimes \cdots \otimes \Lambda_{a_n}$. As we shall see in the next section this is the (well-known) computation of the Seifert pairing for the Brieskorn knots.

§7. PRODUCTS ARE ASSOCIATIVE AND THEY DISTRIBUTE OVER CONNECTED SUMS

Associativity of the knot product operation follows by essentially the same argument one would use for links of singularities.

THEOREM 7.1. *The product operation is associative. That is, given three knots $\mathscr{K}$, $\mathscr{J}$ and $\mathscr{L}$ so that at least two are fibered, then $(\mathscr{K} \otimes \mathscr{J}) \otimes \mathscr{L} \cong \mathscr{K} \otimes (\mathscr{J} \otimes \mathscr{L})$. When all three knots are fibered, then this is an isomorphism of fibered knots.*

Proof. Letting k, t and l be the ambient dimensions for $\mathscr{K}$, $\mathscr{J}$ and $\mathscr{L}$ respectively, let $\alpha\colon D^{k+1} \to D^2$, $\beta\colon D^{t+1} \to D^2$, $\gamma\colon D^{l+1} \to D^2$, be the associated branched fibrations for the fibered knots, or the representative mapping as constructed in Lemma 5.1 for a non-fibered knot. Let $g = 1/3(\alpha - \beta - \gamma)\colon D^{k+1} \times D^{t+1} \times D^{l+1} = D \to D^2$. Then it follows from the methods of §5 that the triple pull-back $(\partial D, g^{-1}(0) \cap \partial D)$ represents the product of $\mathcal{K}$, $\mathcal{J}$ and $\mathcal{L}$. Thus the theorem follows from this description.

Remark. Note that this associativity property shows that to form the product of a knot $\mathcal{K}$ with a collection of books $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$, it suffices to form the book $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \cdots \mathcal{B}_n$ and then take $\mathcal{K} \otimes \mathcal{B}$. For example, it follows from §4 that $[a_0] \otimes \cdots \otimes [a_n] \cong \mathcal{L}(z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n})$. Hence $\mathcal{K} \otimes [a_0] \otimes \cdots \otimes [a_n]$ may be interpreted as the result of a sequence of cyclic suspensions, or as the product of $\mathcal{K}$ with a single Brieskorn book.

Now recall that for knots $\mathcal{K}$ and $\mathcal{K}'$ of the same dimension k, one has the notion of the *connected sum* $\mathcal{K} \# \mathcal{K}' = (S^k, K \# K')$. This is described as follows: Let $(D^k, D(K))$ denote the manifold pair obtained by removing a small ball from S^k that intersects K transversally in an unknotted $(k-2)$-disk. Then $\mathcal{K} \# \mathcal{K}' = (D^k, D(K)) \cup_H (D'^k, D(K'))$ where one pastes the two k-balls along their boundaries so that $\partial D(K)$ is matched to $\partial D(K')$. That is, one has diffeomorphisms $h_1: (\partial D^k, \partial D(K)) \to (S^{k-1}, S^{k-3})$ and $h_2: (\partial D'^k, \partial D(K')) \to (S^{k-1}, S^{k-3})$, one of which reverses orientation. The pieces for the connected sum are glued together by the map $H = h_2^{-1} \circ h_1$. We shall speak informally, identifying $(\partial D^k, \partial D(K))$ with (S^{k-1}, S^{k-3}).

Remark. If K or K' is not connected, then $\mathcal{K} \# \mathcal{K}'$ is only well defined after choosing a component of K and K'. We assume such components have always been chosen when we talk about $\#$. Connected sum of empty knots is *not* allowed, that is, we are assuming $k > 1$ throughout this discussion.

LEMMA 7.2. *If $\mathcal{K}$ and $\mathcal{K}'$ are fibered knots of the same dimension, then $\mathcal{K} \# \mathcal{K}'$ is also a fibered knot.*

Proof. Cut the ball from S^k so that $(D^k, D(K))$ has a book structure induced from the fibered structure of $\mathcal{K}$ and so that this restricts to the trivial book structure on $(S^{k-1}, S^{k-3}) = (\partial D^k, \partial D(K))$. Do the same for $(D'^k, D(K'))$. Since $\mathcal{K} \# \mathcal{K}' = (D^k \cup D'^k, D(K) \cup D(K'))$ these books glue together, giving a fibered structure for $\mathcal{K} \# \mathcal{K}'$.

Given a closed n-manifold M^n, let $p(M)$, the *puncture of M*, be the manifold with boundary S^{n-1}, obtained by removing an n-ball from M. Thus $M \# M' = p(M) \cup p(M')$ (union along S^{n-1}).

LEMMA 7.3. *Let $\mathcal{K}$ and $\mathcal{L}$ be knots. Assume that $\mathcal{L}$ has fibered structure $b: S^l \to D^2$. Let $(D^k, D(K))$ be the knotted pair associated to $\mathcal{K}$ as described above. Then*

$$p(K \otimes L) \cong b(D^k, D(K)) \cup (D^{k+1} \times L).$$

Here the union sign denotes partial boundary identifications as follows:

$$\partial D^{k+1} = S^k = D^k \cup D_0^{\,k}, \qquad \partial b(D^k, D(K)) = b(S^{k-1}, S^{k-3}) \cup D^k \times L;$$

the two pieces are matched along $D^k \times L$.

Proof.

$$\begin{aligned} K \otimes L &= b(S^k, K) \cup (D^k \times L) \\ &= b(D^k, K) \cup b(D_0^{\,k}, D^{k-2}) \cup (D^k \times L). \end{aligned}$$

However, $b(D_0^{\,k}, D^{k-2}) \cong D^{k-2} \times D^{l+1}$. Hence $p(K \otimes L) = b(D^k, K) \cup (D^k \times L)$. It is easy to see that the identifications are as described in the statement of the lemma.

THEOREM 7.4. *The product operation distributes over connected sums. That is, given knots $\mathcal{K}$ and $\mathcal{K}'$ of the same dimension, and a third knot $\mathcal{L}$ so that either $\mathcal{L}$ is fibered or both $\mathcal{K}$ and $\mathcal{K}'$ are fibered, then*

$$(\mathcal{K} \# \mathcal{K}') \otimes \mathcal{L} \cong (\mathcal{K} \otimes \mathcal{L}) \# (\mathcal{K}' \otimes \mathcal{L}).$$

Connected sums of fibered knots are given a fibered structure as in Lemma 7.2. If all three knots are fibered, then this is an isomorphism of fibered knots.

Proof. We leave most of this proof to the reader. We shall illustrate the argument by showing that $(K \# K') \otimes L \cong (K \otimes L) \# (K' \otimes L)$ when $\mathcal{L}$ is fibered. Let $\mathcal{L}$ have fibered structure $b: S^l \to D^2$. Regard $D^{k+1} = \bar{D}^{k+1} \cup \bar{\bar{D}}^{k+1}$ where $\partial \bar{D}^{k+1} = D^k \cup D_0^k$ and $\partial \bar{\bar{D}}^{k+1} = D'^k \cup D_0^k$. The two balls are glued along D_0^k to form D^{k+1}.

Then

$$\begin{aligned}(K \# K') \otimes L &= b(S^k, K \# K') \cup (D^{k+1} \times L)\\ &= b(D^k, D(K)) \cup b(D'^k, D(K)) \cup (D^{k+1} \times L)\\ &= (b(D^k, D(K)) \cup (\bar{D}^{k+1} \times L)) \cup (b(D'^k, D(K')) \cup (\bar{\bar{D}}^{k+1} \times L))\\ &= p(K \otimes L) \cup p(K' \otimes L) \qquad \text{(by 7.3)}\\ &= (K \otimes L) \# (K' \otimes L).\end{aligned}$$

The rest of the proof proceeds along similar lines. When $\mathscr{L}$ is not fibered the proof is most easily carried out by using the cut and paste description of Proposition 3.7.

§8. GROUP ACTIONS AND GEOMETRIC KNOT PERIODICITY

Let C_l denote the Levine cobordism (concordance) group of spherical knots in S^{2l+1}. It is well-known that for $l \geq 3$, $C_l \simeq C_{l+2}$ ([14]). An explicit construction for this isomorphism has been given by G. Bredon ([2]). In this section we give an exposition of Bredon's construction from the point of view of knot products.

Definition 8.1. Let $L \subset S^3$ denote the link of two unknotted circles with linking number one. This is a fibered link. In fact, $\mathscr{L} = (S^3, L) = [2] \otimes [2] = \mathscr{L}(z_1^2 + z_2^2)$. Thus, with our conventions, it has Seifert matrix of form $(-1)\cdot(1)\otimes(1) = (-1)$ (see Proposition 6.5). Note that the fibered structure for $\mathscr{L}$ is the lift of the trivial fibered structure on S^2 via the Hopf map.

THEOREM 8.2. *Let $\omega: C_l \to C_{l+2}$ be defined by the formula $\omega(\mathscr{K}) = \mathscr{K} \otimes \mathscr{L}$ where $\mathscr{L}$ is the fibered link of* 8.1. *Then ω is an isomorphism for $l \geqslant 3$.*

Proof. By ([14]) it is sufficient to see that $\omega(\mathscr{K})$ and $\mathscr{K}$ have the same Seifert pairing, up to sign. Since $\theta_{\mathscr{L}}$ has matrix (-1), we see from 6.2 that

$$\theta_{\mathscr{K}\otimes\mathscr{L}} = (-1)^l \theta_{\mathscr{K}}.$$

This proves the theorem.

Since $\omega(\mathscr{K}) = \mathscr{K} \otimes \mathscr{L} = \mathscr{K} \otimes ([2] \otimes [2]) \cong (\mathscr{K} \otimes [2]) \otimes [2]$, the isomorphism is also obtained by performing two 2-fold cyclic suspensions. Iteration of ω amounts to tensoring with the Brieskorn books $\mathscr{L}_n = \mathscr{L}(z_1^2 + z_2^2 + \cdots + z_n^2)$.

Now let $G = \mathrm{O}(n)$ be the orthogonal group of real $n \times n$ matrices g such that $g^t = g^{-1}$ (t = transpose). We show that G acts on knots of the form $\mathscr{K} \otimes \mathscr{L}_n$. Thus iteration of the periodicity isomorphism produces knots with more and more symmetry.

The group G acts on $\mathbf{C}^n$ as follows: Regard

$$z = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n) = x + iy \in \mathbf{C}^n$$

with $x, y \in \mathbf{R}^n$. Define $g \cdot z = (gx) + i(gy)$ $(i = \sqrt{-1})$. Next let $f(z) = z_1^2 + z_2^2 + \cdots + z_n^2$. If $z = x + iy$ and $\langle , \rangle$ denotes the standard inner product on $\mathbf{R}^n$, then we have the formula $f(z) = (\langle x, x\rangle - \langle y, y\rangle) + 2i\langle x, y\rangle$. Thus $f(gz) = f(z)$ for all $g \in \mathrm{O}(n)$. This shows that $\mathrm{O}(n)$ acts smoothly on the pair (S^{2n-1}, L_n).

The orthogonal group acts naturally on $S^{2n+k} = \{(y, z) \in \mathbf{R}^{k+1} \times \mathbf{C}^n \mid \|y\|^2 + \|z\|^2 = 1\}$ by $g \cdot (y, z) = (y, g \cdot z)$. If we regard $S^{2n+k} \cong \partial(D^{k+1} \times D^{2n})$ then this action is (after smoothing corners) the same as that obtained from the action on $D^{k+1} \times D^{2n}$. We shall refer to this as the standard $\mathrm{O}(n)$ action on S^{2n+k}.

PROPOSITION 8.3. *Given any knot $\mathscr{K} = (S^k, K)$, the knot $K \otimes L_n \subset S^{2n+k}$ has a smooth $\mathrm{O}(n)$ action obtained by restricting the standard $\mathrm{O}(n)$ action on S^{2n+k}.*

Proof. Let $f(z) = z_1^2 + \cdots + z_n^2$. Then it is easy to verify that $f(D^{2n}) = D^2$. Hence if we regard $f: D^{2n} \to D^2$ and let $E = f^{-1}(D_\epsilon^2)$ for small ϵ, then $f|E: E \to D_\epsilon^2$ is a branched fibration for $\mathscr{L}_n$. Since $f(gz) = f(z)$ for all $y \in \mathrm{O}(n)$, we see that E has an $\mathrm{O}(n)$-action. In fact, after smoothing corners, E is equivariantly diffeomorphic to D^{2n} with the standard action. Thus we obtain a branched fibration $\tau: D^{2n} \to D^2$ so that $\tau(gz) = \tau(z)$ for all $z \in D^{2n}$ and $g \in \mathrm{O}(n)$. Now let $\alpha: D^{k+1} \to D^2$ be a map as in Lemma 5.1, so $\mathcal{K} \otimes \mathcal{L}_n$ can be described as in §5, as $(\partial(D^{k+1} \times D^{2n}), \partial X)$, where X is the pull-back

$$\begin{tikzcd} X \arrow[r] \arrow[d] & D^{2n} \arrow[d, "\tau"] \\ D^{k+1} \arrow[r, "\alpha"] & D^2. \end{tikzcd}$$

By definition, this pull-back is invariant under the standard action of $\mathrm{O}(n)$, so we must only check that the necessary corner smoothing can be done equivariantly. This is in fact true, since the smoothing methods we have sketched earlier only depend on the theorem on uniqueness of tubular neighborhoods, which holds equivariantly (see e.g. [1]). Alternatively, observe that if S_δ^{2n+k} is a sphere of small radius around the origin in $D^{k+1} \times D^{2n}$, then $(S_\delta^{2n+k}, S_\delta^{2n+k} \cap X)$ *is* an equivariantly smoothed version of $(\partial(D^{k+1} \times D^{2n}), \partial X)$, since the former may be pushed out to the latter by the vector field used in §5.

It is easy to verify that $K \otimes L_n$ is the $\mathrm{O}(n)$-manifold corresponding to a knot $\mathcal{K}$ constructed by Bredon (see also Jänich[10], Hirzebruch[9] and Erle[7]). The above construction of $\mathrm{O}(n)$-manifolds can also be done, and generalized, by showing that the whole discussion of knot products is valid in a suitable equivariant category.

§9. PRODUCTS OF KNOTS IN OTHER CODIMENSIONS

The results of this paper all go through in other codimensions, but in codimension $q \geq 3$ we need additional structure (which is "for free" if $q = 1, 2$; see Lemma 2.3). Namely a *well framed knot* of codimension q consists of a smooth oriented manifold pair (S^k, K^{k-q}) plus a smooth map (the *well framing*) $p: S^k \to D^q$ with 0 as a regular value and $K = p^{-1}(0)$ as an oriented submanifold. Well framings are *equivalent* if they are smoothly homotopic through well framings. If the well framing p is such that $\bar{p} = p/\|p\|: S^k - K \to S^{q-1}$ is a fibration, we call (S^k, K^{k-q}, p) a *fibered knot*.

Fibered knots arise in the same way as in codimension 2 as links of tame isolated singularities of maps $(\mathbf{R}^{k+1}, 0) \to (\mathbf{R}^q, 0)$. By Milnor[17, §11], polynomial singularities are tame.

Even if a well framed knot (S^k, K^{k-q}, p) is not fibered, any non-singular fiber of $\bar{p} = p/\|p\|$ is a spanning surface for the knot which can be used to define a Seifert pairing $\theta^*: \mathcal{H}_{r+1}^-(F) \times \mathcal{H}_{k-r}^-(F) \to \mathbf{Z}$ as in §6. In contrast to codimension 2, this pairing is graded symmetric if $q \geq 3$.

The results of this paper (construction of branched fibrations, product of knots, associativity and distributivity of product over sum, behaviour of Seifert form under product, and connection with links of singularities) go through as in codimension 2, but with the following modifications.

9.1. The relationship $\mathcal{L}(f+g) \cong \mathcal{L}(f) \otimes \mathcal{L}(g)$ for links of isolated singularities of polynomial maps $f: (\mathbf{R}^{k+1}, 0) \to (\mathbf{R}^q, 0)$ and $g: (\mathbf{R}^{m+1}, 0) \to (\mathbf{R}^q, 0)$ still holds, but *we do not know if fibered structure is preserved.* If $f, g, f+g$, are just tame this isomorphism only holds up to h-cobordism when $k+m+1 = 3, 4, 3+q$, or $4+q$. For singularities for which a vector field as in Lemma 4.1 exists, everything works, including preservation of fibered structure.

9.2. If $\mathcal{K} = (S^k, S^{k-q}, p)$ is a well framed spherical knot and $\mathcal{L} = (S, L^{l-q}, p')$ is fibered, then in contrast to $q = 2$, $\mathcal{K} \otimes \mathcal{L}$ is a spherical knot if $q \geq 3$. The proof is by observing that the cut and paste description of $K \otimes l$, analogous to Proposition 3.7, looks homologically the same as if $\mathcal{K}$ were the trivial knot.

The following remarks show that the construction does yield interesting knots also for $q \geq 3$. The empty knot $(S^n, \emptyset)$ can be considered to have any codimension. In particular, the Hopf fibration $S^{2k-1} \to S^k$, $k = 2, 4, 8$, gives the empty knot $(S^{2k-1}, \emptyset)$ the structure of a fibered knot of codimension $k+1$, which we denote $\mathcal{L}^{(k)}$. $\mathcal{L}_n^{(2)} = \mathcal{L}^{(2)} \otimes \cdots \otimes \mathcal{L}^{(2)}$ (n times) can be described as Kuiper's link of the polynomial map $f: \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{C} \times \mathbf{R}$, $f(x, y) = (x\bar{y}, \|x\|^2 - \|y\|^2)$ (see Milnor[17, §11]. $\mathrm{U}(n)$ acts on $\mathcal{L}_n^{(2)}$ and we get a similar connection between special $\mathrm{U}(n)$-actions and 3-codimensional knots as for $\mathrm{O}(n)$-actions and 2-codimensional knots in §8. Similar remarks apply to codimension 5, $\mathcal{L}_n^{(4)}$, and $\mathrm{Sp}(n)$.

Signature of a well framed knot is well defined as signature of a spanning surface, since spanning surfaces are unique up to bordism modulo the boundary. In codimension $q = k+1$ as above, $\mathcal{K} \otimes \mathcal{L}^{(k)}$ and $\mathcal{K}$ have the same signature—in fact the intersection form of a spanning surface for $\mathcal{K}$ and the corresponding spanning surface for $\mathcal{K} \otimes \mathcal{L}^{(k)}$ are the same. It follows that if $\mathcal{K} = (S^k, K^{k-q}, p)$ is a well framed spherical knot of non-zero signature then $K \otimes L^{(k)} \otimes \cdots \otimes L^{(k)}$ is always a nonstandard sphere if we take sufficiently many factors.

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