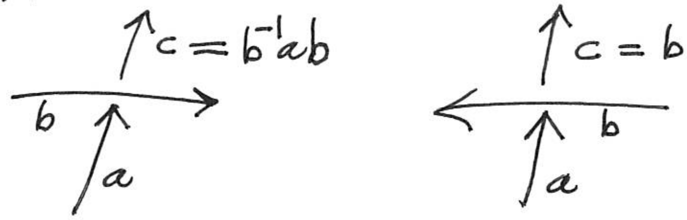
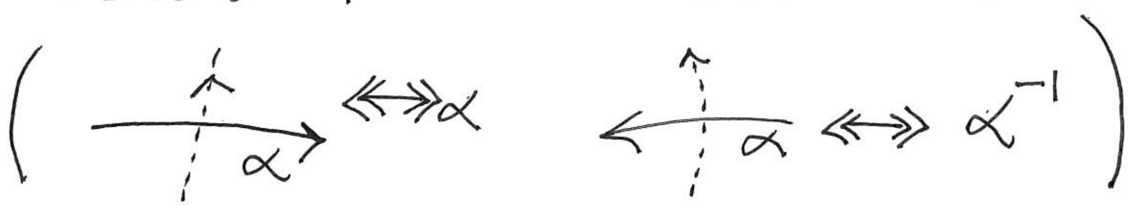
Knot Theory – Spring 2017 – Problem Set #2

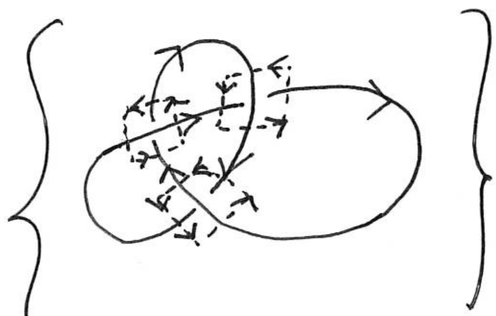
1° In the fundamental group of a knot or link we have the relations:



These can be rewritten as follows:

$cb^{-1}a^{-1}b = 1$ $\qquad\qquad$ $cba^{-1}b^{-1} = 1$





Use this point of view on the relations to prove that in a classical link diagram, any one of the crossing relations is a consequence of the remaining relations.

2. (a) Let M be a module over a commutative ring R with identity element 1.
For $a, b \in M$ define
$$a * b = pa + qb$$
$$a \bar{*} b = p'a + q'b$$
with $p, q \in R$. Detetermine properties for p and q to satisfy so that $(M, *, \bar{*})$ is an oriented quandle. That is, we require that

I. $a * a = a$, $a \bar{*} a = a$, $\forall a \in M$.
II. $(a * b) \bar{*} b = a$, $(a \bar{*} b) * b = a$, $\forall a, b \in M$.
III. $(a*b)*c = (a*c)*(b*c)$ and
$(a \bar{*} b) \bar{*} c = (a \bar{*} c) \bar{*} (b \bar{*} c)$, $\forall a, b, c \in M$.

(b) Show that if $R = \mathbb{Z}[t, t^{-1}]$,
then $a * b = ta + (1-t)b$
$a \bar{*} b = t^{-1}a + (1-t^{-1})b$
gives an oriented quandle.
(Prove I, II, III above.)

(c) The <u>Alexander Polynomial $\Delta_K(t)$</u> for an oriented link K is the "polynomial modulus" for the quandle coloring via part (b) of this exercise. e.g.

K 

$c = a * b$

$\Rightarrow a = b * c$

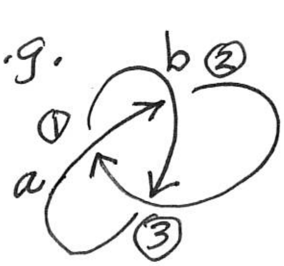
$\boxed{a = b * (a * b)}$

$a = tb + (1-t)(ta + (1-t)b)$
$a = tb + ta - t^2 a + (1 - 2t + t^2)b$
$(t^2 - t + 1)a = (t^2 - t + 1)b$
$(t^2 - t + 1)(a - b) = 0$
$\Rightarrow \boxed{\Delta K(t) \doteq t^2 - t + 1}$

Here $X \doteq Y$ *iff* $X = \pm t^n Y$ for some $n \in \mathbb{Z}$.

> The general algorithm for $\Delta_K(t)$ is to write down the "Alexander Matrix" corresponding to the relations
> $$c = t^{\varepsilon} a + (1 - t^{\varepsilon}) b \qquad (\varepsilon = +1 \text{ or } -1),$$
> remove one column and one row, and compute the determinant.

e.g.



T

$$\left\{\begin{array}{l} 1.\ c * a = b \\ 2.\ a * b = c \\ 3.\ b * c = a \end{array}\right\}$$

1. $tc + (1-t)a = b$
2. $ta + (1-t)b = c$
3. $tb + (1-t)c = a$

	a	b	c
1.	$1-t$	-1	t
2	t	$1-t$	-1
3	-1	t	$1-t$

$$\mathrm{Det}\begin{pmatrix} -1 & t \\ 1-t & -1 \end{pmatrix} = 1 - t(1-t) = t^2 - t + 1.$$

$$\Delta_T \doteq t^2 - t + 1.$$

<u>Problem</u>. Calculate Δ_T for 4_1

and 6_2

.

3.° John Horton Conway

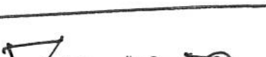
defined a knot polynomial by the axioms:

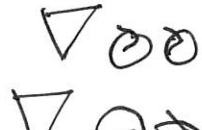
(a) Given an oriented knot or link L there is a polynomial $\nabla_K(z) \in \mathbb{Z}[z]$ (polynomials in z with integer coefficients) such that $K \underset{\text{Reid Moves}}{\sim} K' \Longrightarrow \nabla_K(z) = \nabla_{K'}(z)$.
(Note this is exact equality.)

(b) If K_+, K_-, K_0 are 3 diagrams differing at the site of one crossing by K_+ K_- K_0

), then $\nabla_{K_+} - \nabla_{K_-} = z\nabla_{K_0}$

(c) ∇  $= 1$

Show that ∇  $= \nabla$  $= \cdots = \phi$.

Show that ∇  $= 1 + z^2$, ∇  $= z$.

∇  $= 1 - z^2$

Show (in these examples) that $\boxed{\nabla_K\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) \doteq \Delta_K(t)}$.

4° The Conway polynomial $\nabla_K(z)$ was generalized in the 1980's to the <u>Homflypt polynomial</u> $P_K(a, z)$ with axioms:

(a) $P_K(a, z) \in \mathbb{Z}[a, a^{-1}, z, z^{-1}]$ is an invariant of K (all 3 RM's).

(b) $a^{-1}P$ $- \, aP$ $= zP$.

(c) $P_{\bigcirc} = 1$.

(i) Show $P_{\bigcirc K} = \delta P_K$ where $\delta = (a^{-1} - a)/z$.

(ii) Show that if K^* = mirror image of K (switch all crossings), then $P_{K^*}(a, z) = P_K(a^{-1}, z)$.

(iii) Compute P_K for

H, E, T

You should find that $P_T(a, z) \neq P_T(a^{-1}, z)$ (whence $T \neq T^*$) and that $P_E(a, z) = P_E(a^{-1}, z)$.