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MAP COLORING, q-DEFORMED SPIN NETWORKS, AND TURAEV-VIRO INVARIANTS FOR 3-MANIFOLDS

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1. Introduction

This paper explores a range of ideas that interconnect map coloring, knots, links and 3-manifolds, statistical mechanics, Temperley-Lieb algebra, and generalizations of angular momentum theory to the study of q-deformed spin networks.

In Sec. 2 we reformulate the Four Color Theorem in terms of an algebraic problem about the vector cross product algebra in three-dimensional space. The structure of this reformulation relates the quaternions to an example of spin network theory.

In Sec. 3 we review the bracket polynomial and its relation to the Potts model, chromatic polynomial, Jones polynomial and the Temperley-Lieb algebra. We discuss how the general bracket can be expanded in terms of the Temperley-Lieb algebra. It is shown how the Potts partition function (hence the chromatic polynomial) of any plane graph can be expressed in terms of the Temperley-Lieb algebra. By using this formulation we describe joint work with H. Saleur that reformulates the Four Color Theorem as part of a larger conjecture about the Temperley-Lieb algebra.

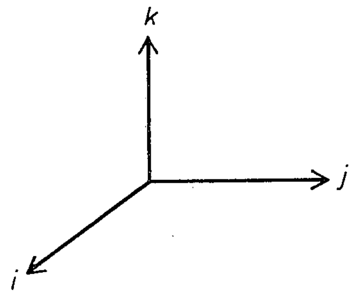
Section 4 discusses the relationship of the bracket polynomial with the SL(2) quantum group.

Section 5 recalls the framework of Penrose spin network theory and shows how this viewpoint correlates the coloring reformulations of Secs. 2 and 3. Section 6 details our generalization of spin networks to q-deformed spin networks. We discuss q-spin network recoupling theory and state joint results with Sostenes Lins (Propositions A and B) about the structure of this theory when q is a root of unity.

Finally, Sec. 7 describes joint work of the author and Sostenes Lins in constructing 3-manifold invariants of the Turaev-Viro type by using the recoupling theory of q-spin networks. The epilogue discusses connections and applications.

2. Map Coloring

Let's begin with a combinatorial problem about the vector cross product algebra. Recall that the vector cross product algebra for three-dimensional space is generated by the three principal (orthogonal) vector directions, denoted by **i** and **j** and **k**.



The algebra obeys the rules

$$\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = 0,$$
$$\mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j},$$
$$\mathbf{ji} = -\mathbf{ij}, \mathbf{jk} = -\mathbf{kj},$$
$$\mathbf{ki} = -\mathbf{kj}.$$

As a consequence, this is a non-associative algebra, with the simplest instance of non-associativity being

$$(\mathbf{ii})\mathbf{j} = 0\mathbf{j} = 0$$
$$\mathbf{i}(\mathbf{ij}) = \mathbf{ik} = -\mathbf{j}.$$

(The product in the vector cross product algebra is written without the usual x sign. Let this cause no confusion.)

Since this algebra is non-associative, it is natural to consider the following proposition.

Proposition P. Let L and R denote two associations of the product $X_1X_2 \ldots X_n$ for any positive integer n. Then there exist values for $X_1, X_2, \cdots, X_n$ taken from the set $\{i, j, k\}$ such that neither L nor R are zero, and $L = R$.

For example, if $n = 4$, then we can consider the equation

$$(\mathbf{xy})(\mathbf{zw}) = (\mathbf{x}(\mathbf{yz}))\mathbf{w},$$

and we find that

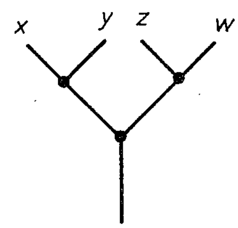
$$(\mathbf{ij})(\mathbf{ik}) = (\mathbf{i}(\mathbf{ji}))\mathbf{k}$$

gives a non-zero solution.

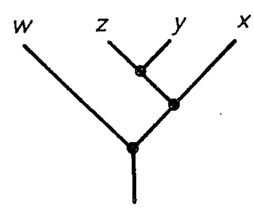
Theorem 2.1 [18]. *Proposition P is equivalent to the Four Color Theorem.*

The Four Color Theorem states that any map drawn in the plane so that no region borders on itself can be colored in four colors so that no two regions, sharing a border, are colored with the same color. This theorem has a notorious history, and there is — to date — no simple proof of its veracity.

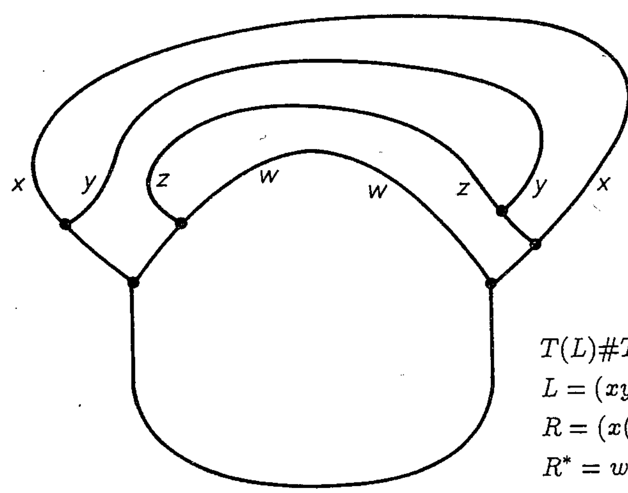
A hint of the proof of 2.1 will remove the mystery: Any associated product can be regarded as encoding a tree where the simplest branching indicates a single multiplication. Thus the tree corresponding to $(xy)(zw)$ is



and the tree corresponding to $w((zy)x)$ is

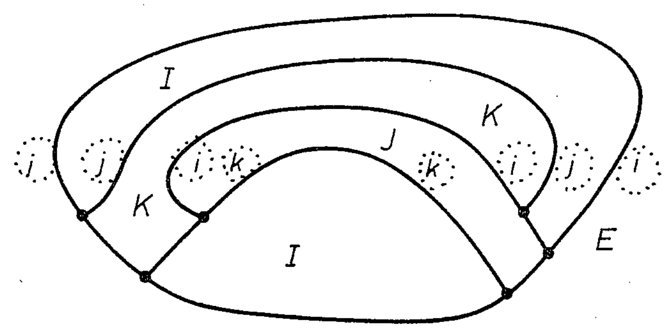


In these trees an assignment of values to the branches gives rise to the parenthesized product value at the trunk. An equation of the form $L = R$ can be encoded by connecting the trees $T(L)$ and $T^*(R)$ where $T^*(R)$ is the mirror image of the tree $T(R)$. That is, $T^*(R)$ is the tree $T(R^*)$ where R^* is the result of writing R in reverse order. For example, $[(x(yz))w]^* = w((zy)x)$. The connection of the two trees is denoted by $T(L)\#T(R^*)$; it is obtained by connecting the trunks and the corresponding branches as shown below:



$T(L)\#T(R^*)$
$L = (xy)(zw)$
$R = (x(yz))w$
$R^* = w((zy)x)$

We see that $T(L)\#T(R^*)$ is a planar map, and that coloring the map gives rise to a coloring of the edges with three distinct edge colors at each vertex. If the region colors are $\mathbf{I}, \mathbf{J}, \mathbf{K}, \mathbf{E}$, then we take $\mathbf{E}$ as an identity element , and for purposes of labelling associate $\mathbf{i}$ to any edge that is bordered by $\mathbf{E}/\mathbf{I}$ or $\mathbf{J}/\mathbf{K}$. In general, associate the product xy of region colors to the common edge where we take the products $\mathbf{IJ} = \mathbf{JI} = \mathbf{k}$, $\mathbf{JK} = \mathbf{KJ} = \mathbf{i}$, $\mathbf{IK} = \mathbf{KI} = \mathbf{j}$. (This method of translating region colorings to edge colorings for cubic maps is due to Heawood [7]. Kempe [30] was the first to observe , by replacing higher order vertices by regions, that it is sufficient to four color cubic plane maps in order to derive the colorability of arbitrary pane maps.) Read off the associations of edge colors as assignments to the variables in L and R. This gives rise to the solution to the equation $L = R$.



$$(ij)(ik) = [k((ij)i)]^* = (i(ji))k \ .$$

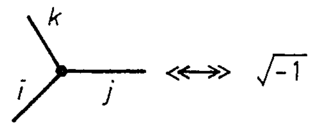
This prescription for solving $L = R$ clearly works up to the sign. That the sign also works can be seen as follows (This short argument is due to George Bergmann (private communication)): If $L = R$ is solved in i, j, k and both sides are non-zero, then each product can be regarded as a non-zero product in the quaternions. Since the quaternions are associative, the two products must be equal. This argument shows that it is sufficient to find values for the X_i $(i = 1, 2, \ldots, n)$ such that L and R are each non-zero. Then they are necessarily equal.

The equivalence of the solvability of the vector cross product problem with the Four Color Theorem depends upon a theorem of Hassler Whitney . See [18] for a full discussion of the equivalence.

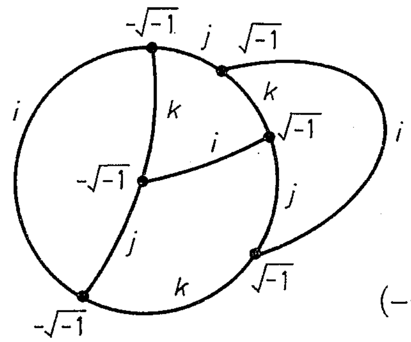
It is this matter of the sign (of L and of R) that leads into state models and chromatic sums. Rather than using quaternions the proof in [18] (of the validity of the sign) uses a Lemma about colorings due to Penrose [38].

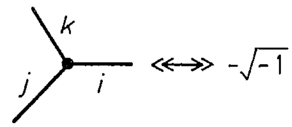
Coloring Lemma. If one colors a plane cubic (three edges per vertex) graph with three colors (say i, j, k) at each vertex, and associates to each vertex $+\sqrt{-1}$ or $-\sqrt{-1}$ according to the cyclic order of the vertex as shown below, then the product of these imaginary values (product taken over all the vertices in the graph) is equal to 1.

See [18] for a proof of this lemma.



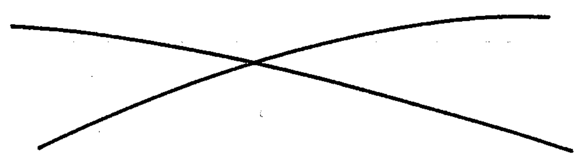
Example:





$$(-\sqrt{-1})^3\,(\sqrt{-1})^3 = +1\ .$$

The Coloring Lemma can be used to give a chromatic recursion for cubic maps. Let G be a cubic graph with an immersion in the plane all of whose singularities are of the type shown below (transversal crossing segments).

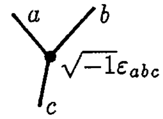


Let C be any coloring of the edges of G with three colors and three distinct colors per vertex. Call such a coloring an admissible coloring of G. Let $\pi(G|C)$ denote the product of the imaginary values associated (via the conventions described above) to the edges of G by the coloring C. If G is embedded in the plane, then this product is equal to 1, by the Lemma. Now define the state summation

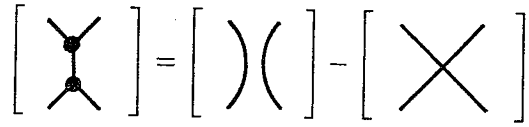
$$[G] = \sum \pi(G|C)$$

where the summation runs over all the admissible colorings of G.

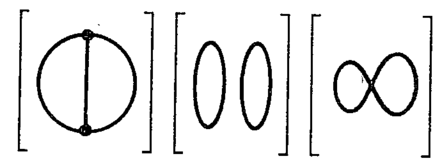
Remark: Another way to view the state summation $[G]$ is to assign the tensor $\sqrt{-1}\varepsilon_{abc}$ to each vertex of the graph G. (ε_{abc} denotes the matrix that assigns the value 1 to ε_{123}, changes sign under transpositions of indices, and is zero if any index appears more than once.) Since G has a (singular) embedding in the plane, there is an assigned cyclic order to each vertex of G, and this gives a unique assignment according to the convention that abc refers to a clockwise cyclic order at the vertex. Each assignment of the numbers 1,2,3 to the edges of G gives rise to a set of (scalar) values for $\sqrt{-1}\varepsilon_{abc}$, one value for each vertex of G. Then $[G]$ is the sum, over all assignments of 1,2,3 to the edges of G, of the products of these vertex weights $\sqrt{-1}\varepsilon_{abc}$.



The summation, $[G]$, satisfies the recursion formula

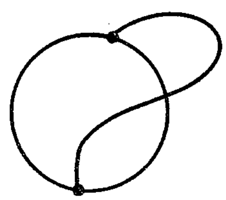


where the value of any closed loop (possibly self-intersecting) is 3. For example



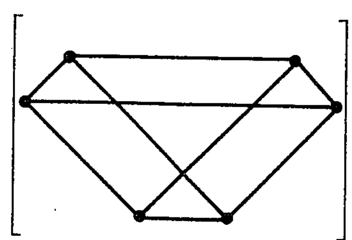
$$= 3^2 - 3 = 6 .$$

The Penrose recursion only counts colors for plane cubic graphs. Let the terminology **planar graph** denote a graph that has an embedding in the plane, and let plane graph mean a graph that is equipped with a given embedding in the plane. The recursion applied to a planar graph with a singular embedding may not count colors, as in the example given below.



$$= 3 - 3^2$$
$$= -6 .$$

Furthermore, there are examples of non-planar graphs that are colorable, but receive 0 as the value of the recursion for a singular embedding. For example,



$$= 0 .$$

Remarks: (1) The Coloring Lemma can be used [18] to give an alternate proof that the signs work in the vector cross product formulation. This provides a relationship between the vector cross product algebra and the partition function $[G]$ that we have associated with the tensor $\sqrt{-1}\varepsilon_{abc}$.

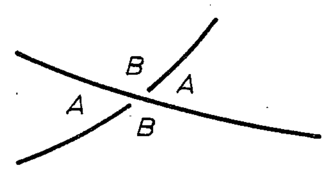
3. Bracket Polynomial and Temperley-Lieb Algebra

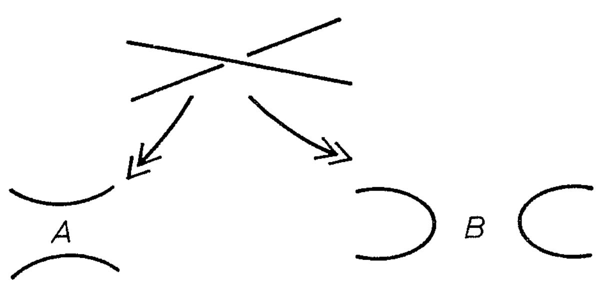
Recall the bracket polynomial and its relation with the Temperley-Lieb algebra [12]. The three-variable bracket polynomial is defined on link diagrams by the following formulas:

$$< \times > = A < \asymp > + B <) \, (>$$
$$< 0K > = d < K >$$
$$< 0 > = d \, .$$

Here the small diagrams stand for otherwise identical parts of larger diagrams, and the second formula means that any Jordan curve disjoint from the rest of the diagram contributes a factor of d to the polynomial. This recursive description of the bracket is well-defined so long as the variables A, B and d commute with one another.

The bracket can be expressed as a state summation where the states are obtained by splicing the link diagram in one of two ways at each crossing. These choices are designated type A and type B, as shown below.





A splice of type A contributes a vertex weight of A to the state sum. A splice of type B contributes a vertex weight of B to the state sum. The norm of a state S, denoted $||S||$, is defined to be the number of Jordan curves in S. It then follows that the bracket is given by the formula

$$< \mathbf{K} > = \sum \mathbf{S} < \mathbf{K}|\mathbf{S} > \mathbf{d}^{||\mathbf{S}||}$$

where the summation is over all states S of the diagram K, and $< K|S >$ denotes the product of the vertex weights for the state S of K.

The formalism of the bracket can be specialized to a number of different situations — including the Potts model for planar graphs (equivalently, to the dichromatic polynomial for planar graphs) and the Jones polynomial for knot and links.

Potts Model

The Potts model is defined for any graph G. A state in the q-state Potts model for G is an assignment of "spins" to the vertices of G from the index set $\{1, 2, 3, \ldots, q\}$. If s is a state of G, then $s(i)$ denotes the spin assigned to the ith vertex of G.

The energy, $E(s)$, of a state s is the summation $E(s) = \sum \delta(s(i), s(j))$ where δ denotes the Kronecker delta ($\delta(x, y) = 1$ if x equals y, $d(x, y) = 0$ if x is not equal to y), and the summation is over all pairs of vertices forming the endpoints of an edge in the graph G.

The Potts partition function is the summation

$$\mathbf{Z(G) = \sum_{s} \exp((-1/(kT))E(s))}$$

where the summation is over all states s of G, k denotes Boltzmann's constant, and T is the temperature.

For G a planar graph, by taking the alternating medial link diagram, $K(G)$, associated with G, we obtain [15] **a formula for the Potts partition function in terms of the bracket polynomial:**

$$\mathbf{Z(G) = q^{N/2} < K(G) > (B = q^{-1/2}v, A = 1, d = q^{1/2})}$$

where N denotes the number of vertices of G and $v = \exp(-1/(kT)) - 1$.

For arbitrary v, $\mathbf{Z(G)(q, v)}$ is the dichromatic polynomial of the graph G. In particular, if $v = -1$, then $Z(G)(q, -1) = C(G)(q)$, the chromatic polynomial for the planar graph G. The value of the chromatic polynomial at a positive integer q is equal to the number of proper vertex colorings of the graph G using q colors. (A coloring is proper if pairs of vertices joined by an edge receive different colors.)

Jones Polynomial

Letting $\mathbf{B = A^{-1}}$, and $\mathbf{d = -A^2 - A^{-2}}$, the bracket becomes an invariant of regular isotopy (Reidemeister moves II and III). With this specialization, bracket behaves multiplicatively on the type-I move, allowing the normalization

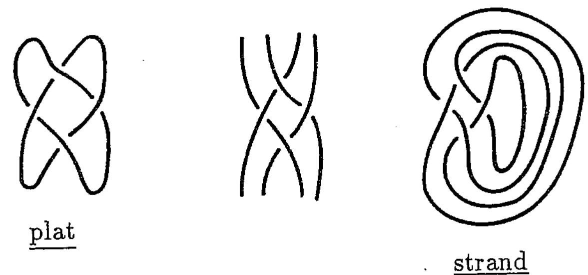
$$\mathbf{f_K = (-A^3)^{-w(K)} < K >}$$

so that $\mathbf{f_K}$ is an invariant of ambient isotopy for links in three space. Here $w(K)$ is the sum of the signs of the crossings of the oriented link K.

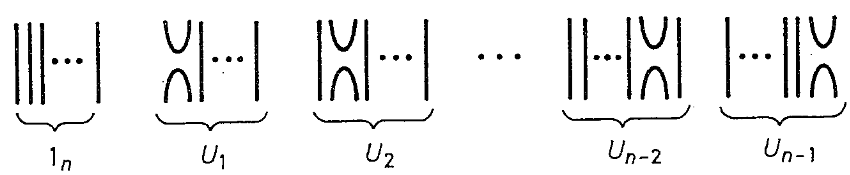
Theorem [12]. $f_K(^{-1/4}) = V_K(t)$ where $V_K(t)$ is the original Jones polynomial [8].

Temperley-Lieb Algebra

First consider calculating the bracket for a braid. We must choose a specific closure for the braid. The two most common closures are indicated below.



They are **strand closure** and **plat closure**. In strand closure, the braid is attached to a trivial braid to form a link that circulates around an axis. In plat closure, the top and bottom of the braid are completed by adding maxima and minima. Plat closure requires an even number of braid strands. In either case, we can consider the state expansion of the bracket of the braid by regarding the **states of the braid** that are obtained by splicing the crossings of the braid. Thus a braid state is a product (in the sense of braids) of elementary states in the forms shown below



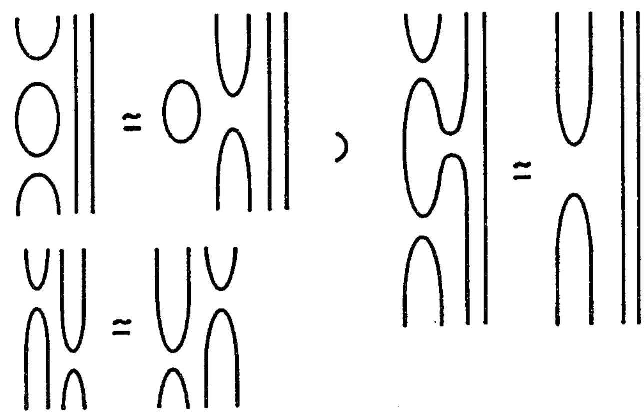
These states — regarded as tangles with tangle multiplication in the form of braid mutiplication — have multiplicative relations as indicated below

$$U_i^2 = dU_i$$

$$U_i U_{i+1} U_i = U_i$$

$$U_i U_{i-1} U_i = U_i$$

$$U_i U_j = U_j U_i \quad \text{if} \quad |i - j| > 1 \;.$$



The **Temperley-Lieb algebra T_n** is the free additive algebra over $Z[A, B, d]$ generated by $1_n, U_1, U_2, \ldots, U_{n-1}$ with multiplicative relations as indicated above. This algebra has its origins in the statistical mechanics of the Potts model, and it also appears crucially in Vaughan Jones' work on von Neumann algebras, and his construction of the original Jones polynomial [7]. The diagrammatic (tangle theoretic) interpretation of the Temperley-Lieb algebra, as shown above first appears in [12].

The states of a braid can be regarded as elements of the Temperley-Lieb algebra, and the bracket of the braid is obtained by converting these Temperley-Lieb elements to powers of d, via the braid closure (strand or plat). This becomes an algebraic algorithm:

1. Replace each braid generator σ_i in the braid word b by $AU_i + B$, and replace the inverse of the generator by $A + BU_i$.
2. Let $< b >$ denote the element of the Temperley-Lieb algebra that is obtained by this replacement.

Let $C(b)$ denote the strand closure of b, and $P(b)$ denote the plat closure of b. Then $<C(b)>$ and $<P(b)>$ are obtained from $<b>$ by replacing the Temperley-Lieb algebra products in the summation $<b>$ by their corresponding closure evaluations (powers of d that count the number of loops in the closure).

We do **not** assert that the bracket with coefficients A, B, d gives a representation of the braid group to the Temperley-Lieb algebra (This is true in the special choice of A, B, d that gives the Jones polynomial.) Nevertheless, the algorithm gives a well-defined method to evaluate the bracket for any braid (and arbitrary A, B, d) via the Temperley-Lieb algebra. This is a parallel of the way the Temperley-Lieb algebra comes into play in evaluating the partition function of the Potts model.

Colors Again

It is shown in [28] (joint work with H. Saleur) that the Potts partition function of any plane graph can be expressed in terms of the Temperley-Lieb Algebra, and that the Four Color Theorem is equivalent to a purely algebraic problem expressed in terms of this algebra. Here is a sketch of this approach.

We shall use the formulation of the Potts model in terms of the alternating medial. Thus

$$\mathbf{Z(G) = q^{N/2}\{K(G)\}}$$

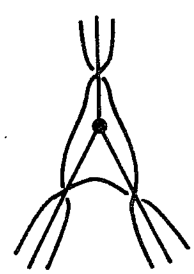
where N denotes the number of vertices in G and

$$\{\mathrm{K(G)}\} = < \mathrm{K(G)} > (B = q^{-\frac{1}{2}}v, A = 1, d = q^{\frac{1}{2}})$$

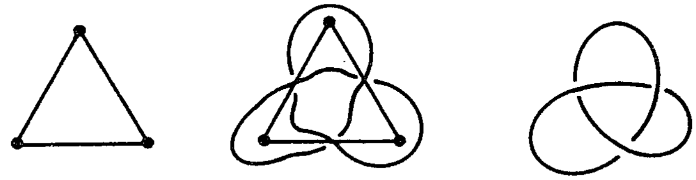
denotes the three-variable bracket, evaluated as shown above. Thus $\{\mathbf{K}\}$ enjoys the formulas

$$\{ \times \} = \{ \supset\subset \} + \mathbf{q^{-1/2}v}\{ \asymp \}$$
$$\{0\} = \mathbf{q^{1/2}}\ .$$

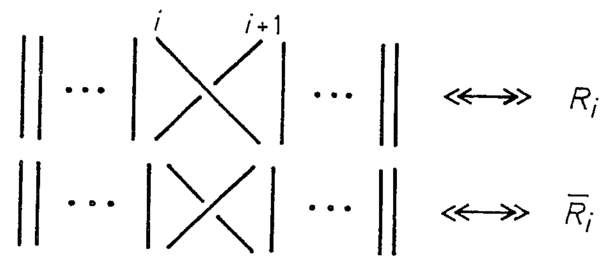
Recall that the alternating medial link diagram is obtained from an arbitrary plane graph by placing a crossing on each edge of the graph, and connecting the crossings as indicated below:



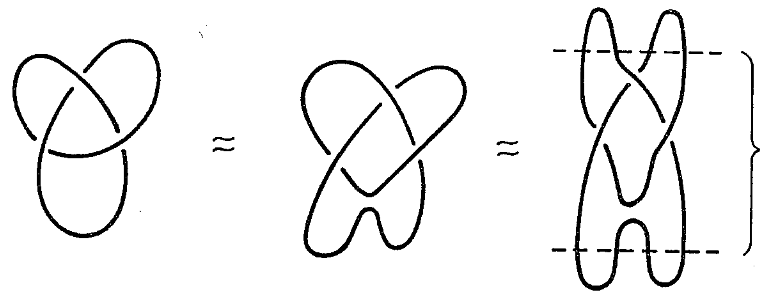
In this construction the edge of the graph goes through the crossing on its A-regions. As a result the link $K(G)$ is necessarily an alternating link. For example, the triangle graph G gives rise to a diagram for a trefoil knot as shown below:



In order to express $\{K(G)\}$ in terms of the Temperley-Lieb algebra, we arrange the diagram $K(G)$ with respect to the vertical direction on the page so that each internal minimum is paired with an internal maximum. The result is a plat closure that can be interpreted as a product of Temperley-Lieb elements and formal braid elements R_i and $\bar{R}_i$ as indicated below:



In the trefoil case we have



We write $K(G) = P(R_2\bar{R}_1\bar{R}_3U_2)$ where P denotes the plat closure.

The expansion formula for $\{K\}$ indicates how to expand R_i and $\bar{R}_i$ in terms of the Temperley-Lieb algebra:

$$R_i = 1 + q^{-1/2}vU_i$$
$$\bar{R}_i = q^{-1/2}v + U_i\ .$$

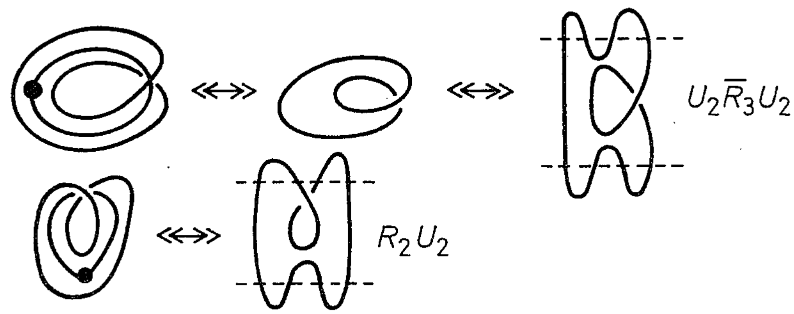
With these assignments, we have an expansion of $\{K\}$ in terms of the Temperley-Lieb algebra for any diagram K. In fact, this approach shows how to expand the three variable brackets in terms of the Temperley-Lieb algebra for any link diagram K. With $W(K)$ denoting the word in R's and U's we have $\{K\} = \{P(W(K))\}$. In the case $q = 4$, $v = -1$ (for the four color problem) it follows from the faithfulness of the Jones trace [9] that $\{K\} = 0$ if and only if $W(K)$ is equal to zero in the Temperley-Lieb algebra.

This is the key to an algebraic reformulation of the Four Color Theorem. For in the case $q = 4$, $v = -1$, we find

$$R_i = 1 - (1/2)U_i$$
$$\bar{R}_i = -(1/2) + U_i$$

Hence $R_iU_i = 0$ and $U_i\bar{R}_{i\pm1}U_i = 0$ since $U_iU_i = 2U_i$.

It is easy to see that each of the words R_iU_i and $U_i\bar{R}_{i+1}U_i$ corresponds to a plat formation of a graphical loop. For example:



The fact that these words are zero in the algebra corresponds to the uncolorability of a loop. (A graph containing a loop is not vertex-colorable since the vertex at a loop is asked to be colored differently from itself.) Call a word W in the R's and U's **loop free** if it cannot be transformed to a word containing R_iU_i or $U_i\bar{R}_{i+1}U_i$ by Temperley-Lieb relations among the U_i, augmented by the external relations $R_iU_j = U_jR_i$, $\bar{R}_iU_j = U_j\bar{R}_i$ for $|i-j| > 1$.

Conjecture 1. If a word W (as described above) is loop free, then W is non-zero as an element of the Temperley-Lieb algebra at loop value 2.

Conjecture 2. If a word W (as described above) is loop free and such that all indices i of the form R_i have the same parity and all indices j of the form R_j have the opposite parity, then W is non-zero as an element of the Temperley-Lieb algebra at loop value 2.

In [29] we show that Conjecture 2 is equivalent to the Four Color Theorem. The parity conditions refer to what is the case for words arising from an alternating medial. To us, Conjecture 1 appears just as plausible as Conjecture 2, but Conjecture 1 is a considerable generalization of the original coloring problem.

4. The Bracket Polynomial and SL(2)q

First recall the specialization of the bracket polynomial [12] , that gives an elementary picture of the Jones polynomial as a state summation. This bracket is defined by the equations:

1. $<\asymp> = A <\asymp> + A-1 <)(>$
2. $< 0K > = d < K >$ with $d = -A^2 - A^{-2}.$
3. $< O > = d.$

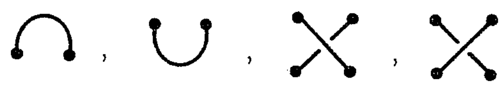
A Yang-Baxter model for the bracket is obtained as follows [19] : Let M denote the matrix

$$M = \sqrt{-1}\tilde{\varepsilon}$$

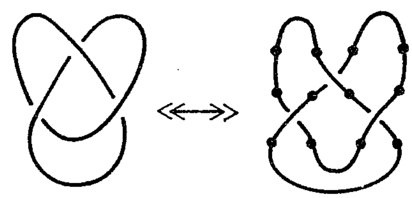
where

$$\tilde{\varepsilon} = \begin{bmatrix} 0 & A \\ -\bar{A}^1 & 0 \end{bmatrix} .$$

Associate to each diagram a height function so that the diagram is decomposed into crossings, minima and maxima. Delineate each of these critical points by placing nodes on the diagram as shown below:



For example,

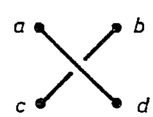


A **state** *(note that the states to follow are of a somewhat different character from the general combinatorial states for the bracket)* of the diagram is a choice of assignments of the indices $\{1, 2\}$ to the nodes of the diagram. With a given state, associate matrix elements to each maxima, minima or crossing as indicated below

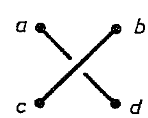
$\Longleftrightarrow M_{ab}$,


$\Longleftrightarrow M^{ab}$,


$\Longleftrightarrow \delta^a_b$



$\Longleftrightarrow R^{ab}_{cd}$



$\Longleftrightarrow \bar{R}^{ab}_{cd}$

Here the lower and upper index entries of M denote the same elements of M — its ab entry, and the matrices R and $\bar{R}$ are defined by the formulas

$R^{ab}_{cd} = A$


$+ A^{-1}$

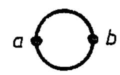

$= AM^{ab}M_{cd} + A^{-1}\delta^a_c\delta^b_d$.

$$\bar{R}^{ab}_{cd} = A^{-1}M^{ab}M_{cd} + A\delta^a_c\delta^b_d .$$

The Kronecker delta corresponds to an arc that is free of critical points with respect to this height function.

To each state $S : \mathbf{Nodes(K)} \longrightarrow \{\mathbf{1}, \mathbf{2}\}$, let $< \mathbf{K}|\mathbf{S} >$ denote the product of the matrix entries that are produced by S at the critical points of the diagram. Then $< \mathbf{K} >= \mathbf{S} < \mathbf{K}|\mathbf{S} >$ where the summation is taken over all possible states.

For example,

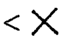
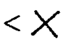


$$\Longleftrightarrow \sum_{a,b} M_{ab}M^{ab} = \sum_{a,b}(M_{ab})^2$$
$$= (\sqrt{-1}A)^2 + (-\sqrt{-1}A^{-1})^2 = -A^2 - A^{-2} \, .$$

It is easy to see from the definition of the matrices that $< K >$ satisfies the identities 1.,2., 3. above. Abstract properties of the bracket make it easy to deduce that R satisfies the Yang-Baxter equation (see [23] and [19] for a discussion of this point.).

Special Value A$= -1$

With this model of the bracket in front of us, note how it behaves at the special value $A = -1$. Here we have

0. <  >=<  >= (by def) <  > .
1. <  > + <  > + <  >= 0.
2. $< 0 >= -2$.
3. <  >=<  > .

In this special case, the matrix $M = \sqrt{-1}\varepsilon$ where ε is the matrix

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This matrix , epsilon, is the defining invariant for the group SL(2) (read SL(2,C) if you like). That is, SL(2) is the set of matrices of determinant 1, and the epsilon has the property that for any matrix with commuting entries

$$\mathbf{P}\varepsilon\mathbf{P^T} = \mathbf{DET(P)}\varepsilon \, .$$

In this context, we can interpret Eq. (1) above as a matrix identity where the crossed arcs are Kronecker deltas:

$$\varepsilon^{\mathbf{ab}}\varepsilon_{\mathbf{cd}} = \delta^{\mathbf{a}}_{\mathbf{c}}\delta^{\mathbf{b}}_{\mathbf{d}} - \delta^{\mathbf{a}}_{\mathbf{d}}\delta^{\mathbf{b}}_{\mathbf{c}} \, .$$

The network calculus associated with the value $A = -1$ corresponds to a diagrammatic calculus of tensor identities for SL(2) invariant tensors. This calculus was studied by Roger Penrose [38] in order to investigate the foundations of spin, angular momentum and the structure of space-time.

The Quantum Group SL(2)q

The calculus of link evaluations via the bracket polynomial provides a significant generalization of the Penrose spin networks. This is obtained by shifting from SL(2) at $A = -1$ to SL(2)q at $q = \sqrt{A}$ for arbitrary A, where SL(2)q denotes the quantum group in the sense of Drinfeld. Here is how SL(2)q arises in this context:
Since SL(2) is characterized by matrices P (with commutative entries) such that $P\varepsilon P^T = \varepsilon$, it is natural to ask what sorts of matrices will satisfy the equations of invariance with ε replaced by $\hat{\varepsilon}$:

$$\hat{\varepsilon} = \begin{pmatrix} 0 & \mathbf{A} \\ -\mathbf{A}^{-1} & 0 \end{pmatrix}$$

$$(*) \quad \begin{aligned} \mathbf{P}\hat{\varepsilon}\mathbf{P}^{\mathbf{T}} &= \hat{\mathbf{e}} \\ \mathbf{P}^{\mathbf{T}}\hat{\varepsilon}\mathbf{P} &= \hat{\mathbf{e}} \end{aligned}$$

Attempting to generalize invariance in this way leads to well-known difficulties. It is necessary to assume that the entries of P do not necessarily commute with one another. Assume that P has the form

$$\mathbf{P} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$$

where a, b, c, d belong to an associative (not necessarily commutative) ring. It is then an exercise in elementary algebra to see that the equations $(*)$ are equivalent to the system of relations shown below.

$$\begin{aligned} &\mathbf{ba} = \mathbf{qab} \quad \mathbf{ca} = \mathbf{qac} \\ &\mathbf{dc} = \mathbf{qcd} \quad \mathbf{db} = \mathbf{qbd} \\ &\mathbf{bc} = \mathbf{cb} \\ &\quad \mathbf{ad} - \mathbf{da} = (\mathbf{q}^{-1} - \mathbf{q})\mathbf{bc} \\ &\quad \mathbf{ad} - \mathbf{q}^{-1}\mathbf{bc} = \mathbf{1} \end{aligned}$$

where $\mathbf{q} = \sqrt{\mathbf{A}}$.

The entries of the matrix P form a non-commutative algebra, A. The algebra A is a Hopf algebra with coproduct

$$\Delta : A \longrightarrow A \otimes A$$

given by the formula

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c \\ \Delta(b) &= a \otimes b + b \otimes d \\ \Delta(c) &= c \otimes a + d \otimes c \\ \Delta(d) &= c \otimes b + d \otimes d \ . \end{aligned}$$

The antipode is determined by the fact that P is invertible with respect to the algebra of its own elements. We have $s : A \longrightarrow A$, the antipode with

$$\mathrm{s}\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{pmatrix} = \begin{pmatrix} \mathrm{d} & -\mathrm{qb} \\ -\mathrm{q}^{-1}\mathrm{c} & \mathrm{a} \end{pmatrix} .$$

This Hopf algebra is the quantum group $\mathrm{SL}(2)_q$. Thus $\mathrm{SL}(2)_q$ arises quite naturally from the bracket model of the Jones polynomial. See [19] for a more complete discussion of this point of view.

5. Spin Networks

Classical Penrose spin networks are based on the binor calculus (see below), and they are designed to facilitate calculations about angular momentum and SL(2). A spinor is a vector in two complex variables, denoted by Ψ^A, $A = 1, 2$. The spinor space is acted on by elements U in SL(2) so that

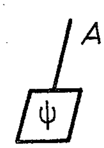
$$(U\Psi)^A = U^A_B \Psi^B$$

(Einstein summation convention). A natural SL(2) invariant inner product on spinors is given by the formula $\Psi\Psi^*$ where

$$\Psi^* = \varepsilon_{AB}\Psi^B$$

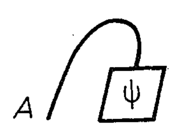
so that $\Psi\Psi^* = \Psi^A \varepsilon_{AB} \Psi^B$ (sum on A and B). If we wish to diagram this inner product, then we let

$$\Psi^A \quad \longleftrightarrow$$



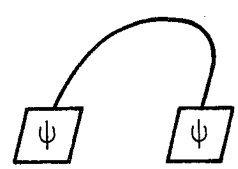
It is natural to lower the index via

$$\Psi^*_A \quad \longleftrightarrow$$

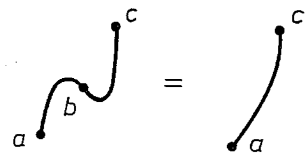


Then

$$\Psi\Psi^* \quad \longleftrightarrow$$

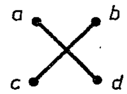


and the fragment  is interpreted as the epsilon, ε_{AB}. Penrose made special conventions for maxima or minima (a minus sign for the minima) in order to insure planar topological invariance. These conventions are equivalent to choosing to replace ε by $\sqrt{-1}\varepsilon$, as we have done in Sec. 2. Thus in the classical spin nets



$$\Longleftrightarrow \quad (\sqrt{-1}\varepsilon_{ab})\,(\sqrt{-1}\varepsilon^{bc}) = \delta_a^c$$

and

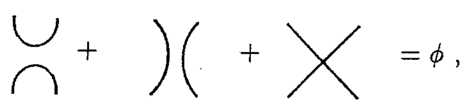


$$\Longleftrightarrow \quad -\,\delta_d^a\delta_c^b\ .$$

Note that the loop value is

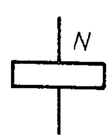
$$= (\sqrt{-1})^2 + (-\sqrt{-1})^2 = -2\ .$$

This calculus entails the binor identity



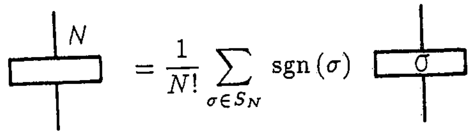
and (directly or from the fact that we are looking at a special case of the bracket) it is invariant under the projections of the Reidemeister moves. Therefore, any loop, even with self-crossings, has value -2. The binor calculus is a unique planar calculus associated with both the bracket polynomial and the representations of SL(2).

The next important spin network ingredient is the **antisymmetrizer**. This is a diagram sum associated to a bundle of lines, and is denoted by

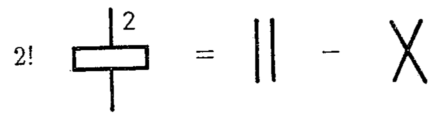


where the N denotes a bundle of parallel strands of multiplicity N.

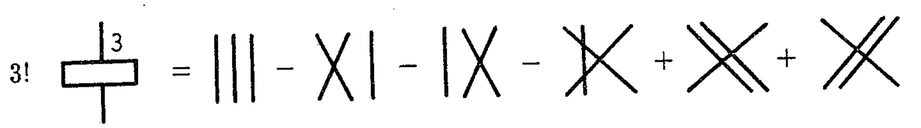
The antisymmetrizer is defined by the formula



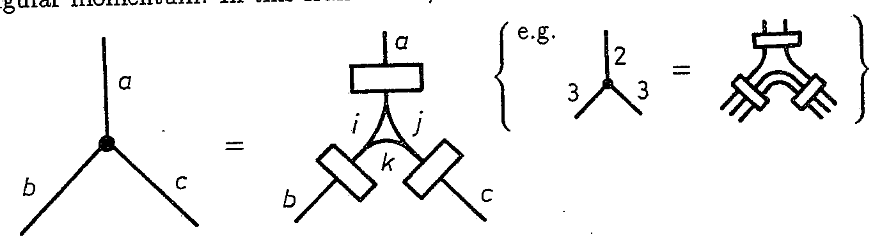
where σ runs over all permutations in S_n, sgn(σ) is the sign of the permutation, and the σ in the box denotes the diagrammatic representation of this permutation as a braid projection. Thus



and

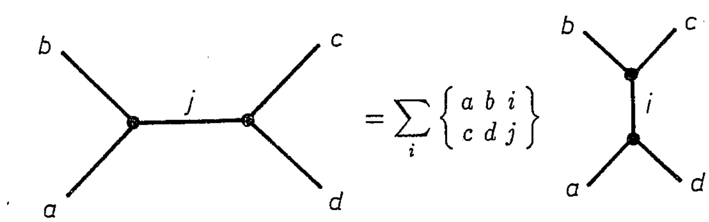


These antisymmetrizers are the basic ingredients for making the spin network calculations of Clebsch-Gordon coefficients, 3_j and 6_j symbols, and other apparatus for angular momentum. In this framework, the 3-vertex is defined as follows:



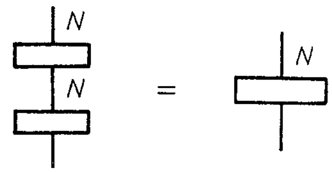
Here a, b, c are positive integers satisfying the condition that the equations $i+j=a$, $i+k=b$, $j+k=c$ can be solved in non-negative integers. Think of spins being apportioned in this way in interactions along the lines. This vertex gives the spin-network analog of the quantum mechanics of particles of spins b and c interacting to produce spin a.

Formally, the $6j$ symbols are defined in terms of these three-vertices by the formula

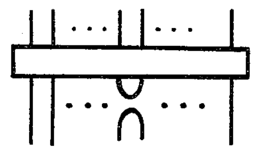


From this it is clear why the $6j$ symbol is called a recoupling coefficient.

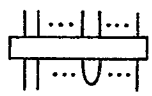
Now, with this sketch of recoupling theory in mind, let us return to the antisymmetrizers. They are projection operators in the sense that



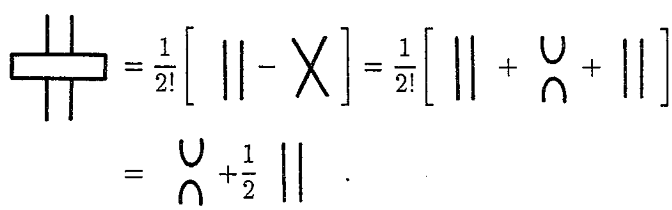
and that they kill off the generators of the Temperley-Lieb algebra $U_1, U_2, \dots, U_{n-1}$:



$= \phi$

since  $= \phi$ by dint of antisymmetry.

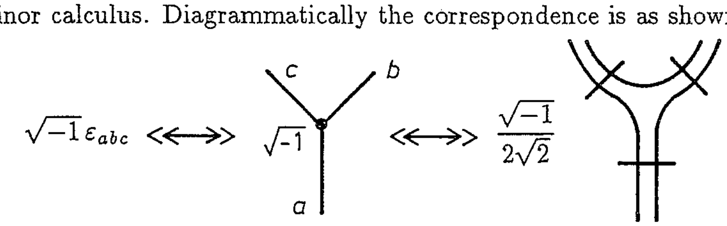
These remarks mean that the expanded forms of the antisymmetrizers are special projection operators in the Temperley -Lieb algebra. For example

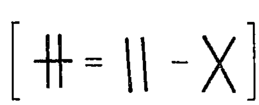


We see in the next section, that a suitable generalization of spin-nets produces the analogs of these operators in the full Temperley-Lieb algebra with arbitrary loop value.

Colors One More Time

With the binor calculus in hand, we can return to the coloring problem from a slightly different angle. The tensor object $i\varepsilon_{abc}$ $(i = \sqrt{-1})$ can be expressed inside the binor calculus. Diagrammatically the correspondence is as shown below [38]



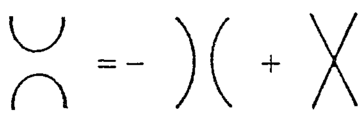


where the bars denote 2-line anti-symmetrizers in the spin nets associated with the binor calculus. The algebraic properties of the $i\varepsilon_{abc}$ object on the left and the binor object on the right are identical. The upshot of this remark is that the number of edge three-colorings of a cubic plane graph can be expressed in terms of the binor calculus. In our terms this becomes the following formula: Let $[G]$ denote the number of edge three-colorings of a cubic plane graph G. Let $K(G)$ be the alternating medial of G, described in Sec. 2. Let $\|K\|$ denote the bracket expansion defined by the equations

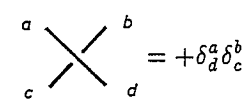
$$\| 0 \| = -2 \, .$$

Then $[G] = (-1)^{N/2} 2^{-N} \|K(G)\|$ where N is the number of vertices of the graph G.

It is of interest to note that the underlying bracket expansion, $\|K\|$, for this coloring formula is directly tied with the representation of the Temperley-Lieb algebra associated to the binors. Thus we can write



$$0 = -2,$$



where the crossed lines denote Kronecker deltas. This corresponds to a representation of the Temperley-Lieb algebra into a quotient of the group ring of the symmmetric group. In this way the results of Sec. 3, reformulating the coloring problem in terms of the Temperley-Lieb algebra can be reformulated in terms of this quotient of the symmetric group. This SL(2) governed model for the coloring problem interconnects the points of view of Secs. 2 and 3. In this way we have drawn an admittedly indirect connection between the combinatorics of the vector cross product algebra and the combinatorics of the Temperley-Lieb algbra. The relationship deserves further study.

6. q-Deformed Spin Networks

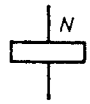
We now construct the generalized antisymmetrizers (see Sec. 5). I shall refer to the generalization as a q-symmetrizer. A q-spin network is nothing more than a link diagram with special nodes that are interpreted as these q-symmetrizers.

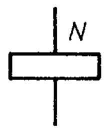
We take $q = \sqrt{A}$, and use the bracket identity

in place of the binor identity, with loop value

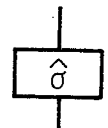
$$-A^2 - A^{-2} \, .$$

Thus any q-spin network computes its own bracket polynomial.

Now define the q-symmetrizer  by the formula

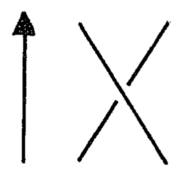


$$= \frac{1}{[N]!} \sum_{\sigma \in S_N} (A^{-3})^{T(\sigma)}$$



$$[N]! = \sum_{\sigma \in S_N} (A^{-4})^{T(\sigma)} = \prod_{k=1}^{N} \left(\frac{1 - A^{-4k}}{1 - A^{-4}} \right)$$

where $T(\sigma)$ is the minimal number of transpositions needed to return σ to the identity, and σ is a minimal braid representing σ with all negative crossings, i.e. with all crossings in the form shown below with respect to the braid direction



Example 1.

$$= (1+A^{-4})^{-1}\left[|| + A^{-3} \; X \right] = (1+A^{-4})^{-1}\left[|| + A^{-3}\left[A \; X + A^{-1} \;)(\right]\right]$$

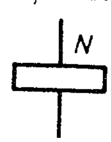
$$= || + (A^2 + A^{-2})^{-1} \; \begin{smallmatrix} \cup \\ \cap \end{smallmatrix} \, .$$

Note that $f_1 = 1 - d^{-1}U_1$ is the first of a sequence of Jones–Temperley–Lieb projectors f_n defined inductively (see e.g. [9] and [34]) via

$$\begin{aligned} f_0 &= 1 \\ f_{n+1} &= f_n - \mu_{n+1} f_n U_{n+1} f_n \\ \mu_1 &= d^{-1}\,, \quad \mu_{n+1} = (d - \mu_n)^{-1} \\ d &= -A^2 - A^{-2} \end{aligned}$$

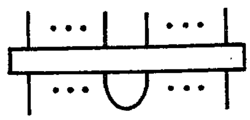
Theorem. The Temperley–Lieb elements f_n (for loop value $-A^2 - A^{-2}$) are equivalent to the symmetrizers. In particular , we have the formula

$$f_{N-1} =$$



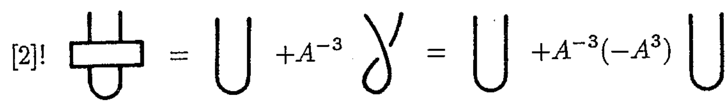
.

Proof. The proper generalization of the antisymmetric property is that



$$= \phi \ ,$$

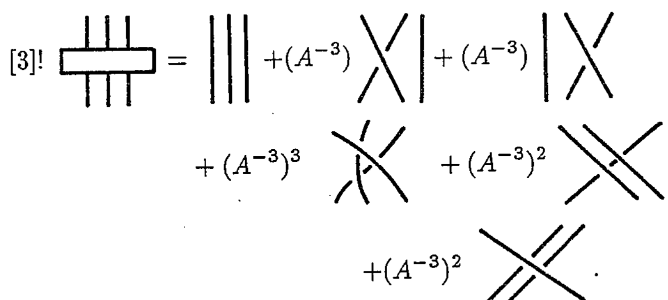
and we see that this is a curl compensation.



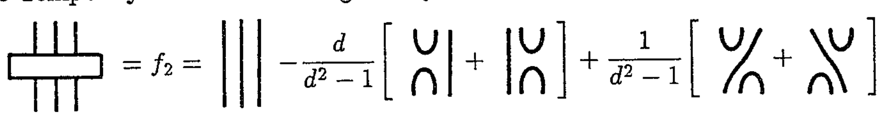
$$= \phi \ .$$

The rest follows easily from the uniqueness of the projectors. //

Example.



The Temperley–Lieb element is given by

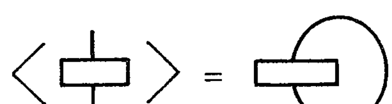


$$(d = -A^2 - A^{-2}) \ .$$

We now see that the q-symmetrizers form a special structure for direct generalization of the methods outlined in Sec. 4 to q-angular momentum. Thus spin network techniques can be used as a foundation for the theory of $q - 6j$ recoupling and other intricacies of the SL(2)q quantum group. In particular , this applies directly to the 3-manifold invariants of Viro, and Turaev [45]. Thus these invariants can also be given a basis in terms of the Jones polynomial and the Temperley–Lieb Algebra. In order to explain this connection, I outline the our results about q-spin network recoupling theory below. The 3-manifold invariant is discussed in the next section.

q-Spin Recoupling Theory

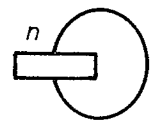
The (Jones) trace of a tangle is calculated via the bracket polynomial by closing the tangle



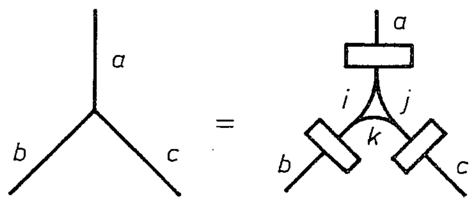
One then has that $\mathbf{tr}(\mathbf{f_{n-1}}) = \Delta_\mathbf{n}$ where Δ_n is the Chebyshev polynomial

$$\Delta_\mathbf{n} = (\mathbf{t}^{\mathbf{n+1}} - \mathbf{t}^{\mathbf{-n-1}})/(\mathbf{t} - \mathbf{t}^{\mathbf{-1}}), \quad \mathbf{t} = -\mathbf{A^2}\ .$$

Hence

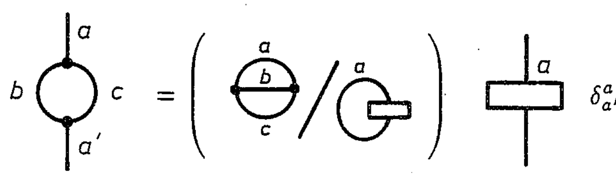
$\Delta_n =$ 

The 3-vertex is now defined as before, with q-symmetrizers replacing the anti-symmetrizers.

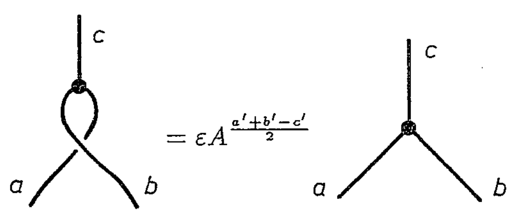


Here a, b, c are positive integers satisfying the condition that the equations $i+j = a$, $i+k=b$, $j+k=c$ can be solved in non-negative integers.

By using this strand-description of the vertices, it is easy to verify that

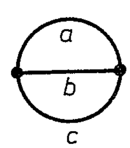


and



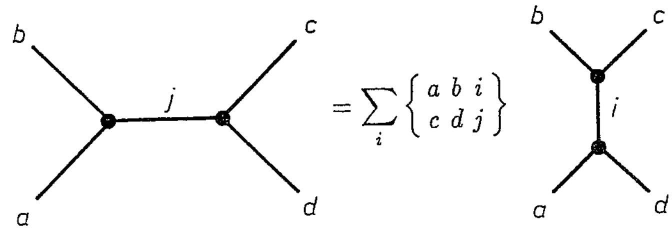
where $\varepsilon = (-1)^{(a+b-c)/2}$ and $x' = x(x+2)$.

We call the evaluation of the θ–shaped network

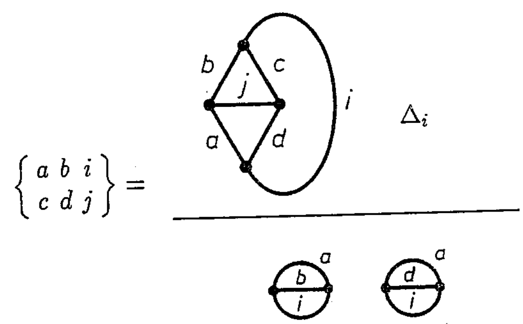


the θ–symbol, and denote its value by $\theta(a,b,c)$.

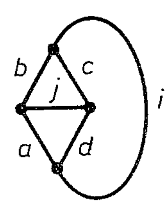
The $\mathbf{q-6j}$ symbols are defined in terms of the three-vertices by the recoupling formula (again as before).



To obtain a formula for these recoupling coefficients, proceed diagrammatically using properties of the 3-vertex and find:



Here we call the factor



the **tetrahedron**. In calculating the tetrahedron, we expand its vertices to get a network of q-symmetrizers.

The formula shown above determines the $q-6j$ symbol in terms of spin network evaluations of some small nets. These can be handled by combinatorial means. This same approach can be used to prove various properties of these objects such as orthogonality relations, and pentagon (or Elliot-Biedenharn) identities (See [24], [26], [27]).

All of these evaluations work as advertised for generic q. When q is a root of unity, then the story is a bit different. We say that a triple is **r-admissible** if $a+b+c <= 2r-4$ where r is an integer greater than 2.

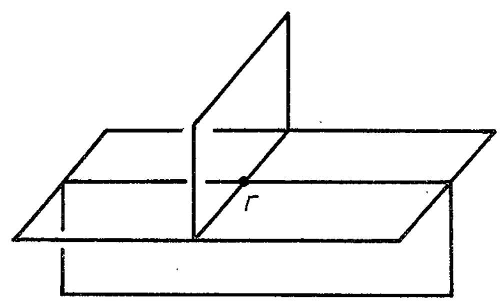
Proposition A [27]. Let q be a primitive $2r$-root of unity, and $\theta(a,b,c)$ be the θ-symbol for an admissible triple (a,b,c). Then (a,b,c) is r-admissible iff $\theta(a,b,c)$ is non-zero.

Proposition B [27]. For q a primitive $2r$-root of unity the recoupling formula for $q-6j$ symbols exists in the sense that a vertex triple in the formula is present iff it is r-admissible.

Under these conditions of r-admissibility, the orthogonality and Elliot-Biedenharn identities continue to hold, and the specific formula for the $q-6j$ symbol works since the denominators are non-zero.

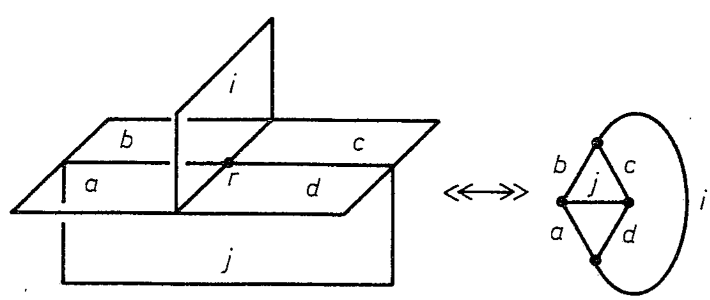
7. The Turaev-Viro Invariant of 3-Manifolds

We use the Matveev representation (see [35]) of three-manifolds in terms of special spines. In such a spine , a typical vertex appears as shown below with four adjacent one-cells , and six adjacent two-cells. Each one-cell abuts to three two cells.

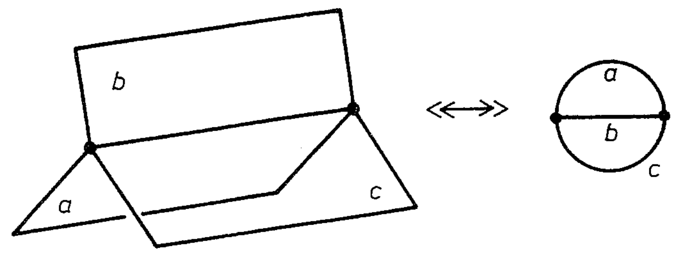


For an integer $r >= 3$ the color set is $\mathbf{C(r)} = \{\mathbf{0,1,2,\ldots,r-2}\}$. **A state at level r** of the three-manifold M is an assignment of colors from $C(r)$ to each of the two-dimensional faces of the spine of M. Let q be a primitive 2r-root of unity.

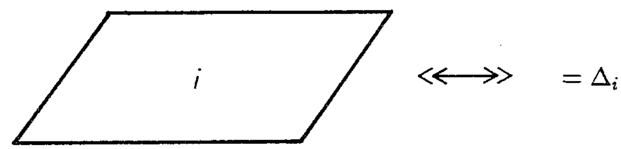
Given a state S of M, assign to each vertex the tetrahedral symbol whose edge colors are the face colors at that vertex. The form of this assignment is shown below with the standard orientation at the vertex .



Assign to each edge the θ-symbol associated with its triple of colors.



Assign to each face the Chebyshev polynomial



whose index is the color of that face.

Let $\mathbf{I(M,r)}$ denote the sum over all states for level r, of the products of the vertex evaluations and the face evaluations divided by the edge evaluations **for those evaluations that are r-admissible (see Sec. 5)**.

$$I(M,n) = \sum_S \frac{\Pi\,\mathrm{TET}\,(v,s)\Pi_f \Delta_{s(f)}^{x(f)}}{\Pi_e \theta(S_a(e), S_b(e), S_c(e))^{-x(e)}}$$

where $x(f)$ and $x(e)$ are the Euler characteristics of f and of e.

Here $\mathbf{TET(v,S)}$ denotes the tetrahedral evaluation associated with a vertex v, for the state S. $S(f)$ is the color assigned to a face f, and Sa, Sb, Sc are the triplet of colors associated with an edge in the spine.

It follows via the orthogonality and Elliot-Biedenharn identities for the $q-6j$ symbols, that $\mathbf{I(M,r)}$ is invariant under the Matveev moves. Hence, by Matveev's work [35]. $\mathbf{I(M,r)}$ is a topological invariant of the 3-manifold M. This is our version of the Turaev-Viro invariant.

$\mathbf{I(M,r)}$ does not depend upon the orientation of the 3-manifold M. This follows easily from the symmetries in the evaluation of the tetrahedron. It is an open question whether a state sum of this sort can be constructed to produce an invariant of 3-manifolds that detects orientation. This question is particularly interesting in this context, since the Lickorish approach [34] to the (oriented) Reshetikhin-Turaev invariant also uses the Temperley-Lieb algebra and the properties of the projection operators. The same underlying formalisms apply to both invariants. Perhaps a subtle combinatorial insight will produce the full Reshetikhin-Turaev invariant as a state summation on the 3-manifold.

8. Epilogue

This paper has spanned a number of themes: coloring problems, statistical physics, low dimensional topology and quantum groups. The combinatorial structure of the bracket polynomial and its relationship with the Temperley–Lieb algebra motivates these connections. The coloring problem is a central theme in these considerations. It is a test case for subtle questions and reformulations. The q-spin networks are of interest in themselves. In particular, one would like to know if the large scale limit of q-spin nets leads to models for spacetime (in analogy with ideas of Penrose), and if the known relationships of spin nets and recoupling theory to quantum gravity ([6], [37]) have a new life in the q-deformed context. The nature of the 3-manifold invariants (of Turaev and Viro) deserves further study. The combinatorial approach via spin nets, laying the recoupling theory out on the table, may help in generalizing these invariants to handle orientation, and to fit them into the contexts of Reshetikhin, Turaev and Witten.

Acknowledgments

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References

[1] M. F. Atiyah, *The Geometry and Physics of Knots* (Cambridge University Press, 1990).

[2] L. Crane,"Topology of three manifolds and conformal field theories", preprint, 1989.

[3] L. Crane, "2-D physics and 3-D topology", *Commun. Math. Phys.* **135** (1991) 615.

[4] L. Crane, "Conformal field theory, spin geometry and quantum gravity", preprint, 1990.

[5] I. Fary, "On straight line representations of planar graphs", *Acta. Sci. Math.* **11** (1948), 229.

[6] B. Hasslacher and M. J. Perry, "Spin networks are simplicial quantum gravity", *Phys. Lett.* **103B** (1981).

[7] P. J. Heawood, "On the four-colour map theorem", *Quart. J. Math.* **29** (1898) 270.

[8] V. F. R. Jones, "A polynomial invariant for links via von Neumann algebras", *Bull. Amer. Math. Soc.* **129** (1985) 103.

[9] V. F. R.Jones, "Index for subfactors", *Invent. Math.* **72** (1983) 1.

[10] L. H. Kauffman, *Formal Knot Theory,* Lecture Notes No. 130 (Princeton University Press, 1983).

[11] L. H. Kauffman, *On Knots,* Annals Study Number 115 (Princeton University Press, 1987).

[12] L. H. Kauffman, "State models and the Jones polynomial", *Topology* **26** (1987) 395.

[13] L. H. Kauffman, "State models for link polynomials", *L'Enseignement Mathematique* **36** (1990) 1.

[14] L. H. Kauffman, "New invariants in the theory of knots", *Amer. Math. Monthly* **95** (1988) 195.

[15] L. H. Kauffman, "Statistical mechanics and the Jones polynomial", *Contemp. Math.* **78** (1988) 263.

[16] L. H. Kauffman, "Knots abstract tensors and the Yang-Baxter equation", in *Knots, Topology and Quantum Field Theories*, ed. L. Lusanna (World Scientific, 1990), pp. 179-334.

[17] L. H. Kauffman, "A Tutte polynomial for signed graphs", *Discrete Applied Math.* **25** (1989) 105.

[18] L. H. Kauffman, "Map coloring and the vector cross product", *J. Comb. Theory*, **48** (1990) 145.

[19] L. H. Kauffman, "Spin networks and knot polynomials", *Int. J. Theor. Phys.* **A5** (1990) 93.

[20] L. H. Kauffman, "Knot polynomials and Yang-Baxter models", in *IXth International Congress on Math. Phys.*, ed. Simon, Truman, Davies (Adam-Hilger, 1989), pp. 438–441.

[21] L. H. Kauffman, "An invariant of regular isotopy", *Trans. Amer. Math. Soc.* **318** (1990) 417.

[22] L. H. Kauffman, "Problems in Knot Theory", *Open Problems in Topology*, ed. J. van Mill (North-Holland, 1990).

[23] L. H. Kauffman, "From knots to quantum groups and back", in *Proceedings of the Fall 1989 Montreal Conference on Hamiltonian Systems*, ed. Harnad and Marsden (CRM 1990), pp. 161-176. In expanded form in *Quantum Groups*, ed. T. Curtright, D. Fairlie and C. Zachos (World Scientific, 1991), pp. 1-32.

[24] L. H. Kauffman, "Knots, spin networks and invariants of three manifolds", to appear in the proceedings of *Knots 90*, August 1990, Osaka, Japan.

[25] L. H. Kauffman, *Knots and Physics* (World Scientific, 1991).

[26] L. H. Kauffman and Sostenes Lins, "Computing Turaev-Viro Invariants for 3-Manifolds", *Manuscripta Math.* **72** (1991) 81.

[27] L. H. Kauffman and Sostenes Lins, "A 3-manifold invariant by state summation", preprint, 1990.

[28] L. H. Kauffman and H. Saleur, "Free fermions and the Alexander-Conway polynomial", *Comm. Math. Phys.* **141** (1991) 293.

[29] L. H. Kauffman and H. Saleur, "Map coloring and the Temperley–Lieb algebra", preprint, 1992.

[30] A. B. Kempe, "On the geographical problem of four colors", *Amer. J. Math.* **2** (1879) 193.

[31] R. Kirby and P. Melvin, "On the 3-manifold invariants of Witten and Reshetikhin-Turaev for sl(2,C)", *Invent. Math.* **105** (1991) 473.

[32] T. Kohno, "Topological invariants for 3-manifolds using representations of mapping class groups I", preprint, 1990.

[33] M. Kontsevich, "Rational conformal field theory and invariants of three-dimensional manifolds", preprint, 1989.

[34] W. B. R. Lickorish, "Invariants for 3-manifolds derived from the combinatorics of the Jones polynomial", to appear in *Pacific J. Math.*

[35] S. V. Matveev, "Transformations of special spines and the Zeeman conjecture", *Math. USSR Izvestia* **31:2** (1988).

[36] G. Moore and N. Seiberg, "Classical and quantum conformal field theory", *Comm. Math. Phys.* **123** (1989).

[37] J. P. Moussouris, "Quantum models of space-time based on recoupling theory", Mathematics Thesis, Oxford University, 1983.

[38] R. Penrose, "Applications of negative dimensional tensors", in *Combinatorial Mathematics and its Applications*, ed. D. J. A. Welsh (Academic Press, 1971).

[39] G. Ponzano and T. Regge, "Semiclassical limit of Racah coefficients", in *Spectroscopic and Group Theoretical Methods in Physics* (North-Holland, Amsterdam, 1968).

[40] N. Yu Reshetikhin, "Quantized universal enveloping algebras, the Yang-Baxter equation and invariants of links I and II", LOMI preprints, 1988.

[41] N. Yu. Reshetikhin and V. Turaev, "Invariants of three manifolds via link polynomials and quantum groups", preprint, 1989.

[42] H. Saleur, "Virasoro and Temperley–Lieb algebras", in *Knots, Topology and Quantum Field Theories*, ed. L. Lusannna (World Scientific, 1989), pp. 485-496.
[43] H. Saleur, "Zeroes of chromatic polynomials: a new approach to Beraha conjecture using quantum groups", *Commun. Math. Phys.* **132** (1990) 657.
[44] V. G. Turaev, "The Yang-Baxter equations and invariants of links", LOMI preprint E-3-87, Steklov Institute, Leningrad, USSR, *Inventiones Math.* **92** Fasc.3, 527-553.
[45] V. G. Turaev and O. Viro, "State sum invariants of 3-manifolds and quantum $6j$ symbols", preprint, 1990.
[46] V. G. Turaev, "Quantum invariants of links and 3-valent graphs in 3-manifolds", preprint, 1990.
[47] V. G. Turaev, "Shadow links and face models of statistical mechanics", Publ. Inst. Recherche Math. Avance, Strasbourg, 1990.
[48] V. G. Turaev, "Quantum invariants of 3 manifolds and a glimpse of shadow topology", preprint, 1990.
[49] V. G. Turaev, "The Yang-Baxter equation and invariants of links", *Invent. Math* **92**, **3** (1988) 527.
[50] W. T. Tutte, *Graph Theory. Encyclopedia of Mathematics and its Applications 21* (Cambridge University Press, 1984).
[51] E. Witten, "Quantum field theory and the Jones polynomial", *Comm. Math. Phys.* **121**, (1989) 351.