Knot Theory Notes - Detecting Links With The Jones Polynonmial
by Louis H. Kauffman

1 Some Elementary Calculations

The formula for the bracket model of the Jones polynomial can be indicated as follows: The letter chi, χ, denotes a crossing in a link diagram. The barred letter denotes the mirror image of this first crossing. A crossing in a diagram for the knot or link is expanded into two possible states by either smoothing (reconnecting) the crossing horizontally, $\asymp$, or vertically $><$. Any closed loop (without crossings) in the plane has value $\delta = -A^2 - A^{-2}$.

$$\chi = A\asymp + A^{-1} ><$$

$$\overline{\chi} = A^{-1}\asymp + A >< .$$

One useful consequence of these formulas is the following *switching formula*

$$A\chi - A^{-1}\overline{\chi} = (A^2 - A^{-2})\asymp.$$

Note that in these conventions the A-smoothing of χ is $\asymp$, while the A-smoothing of $\overline{\chi}$ is $><$. Properly interpreted, the switching formula above says that you can switch a crossing and smooth it either way and obtain a three diagram relation. This is useful since some computations will simplify quite quickly with the proper choices of switching and smoothing. Remember that it is necessary to keep track of the diagrams up to regular isotopy (the equivalence relation generated by the second and third Reidemeister moves). Here is an example. View Figure 1.

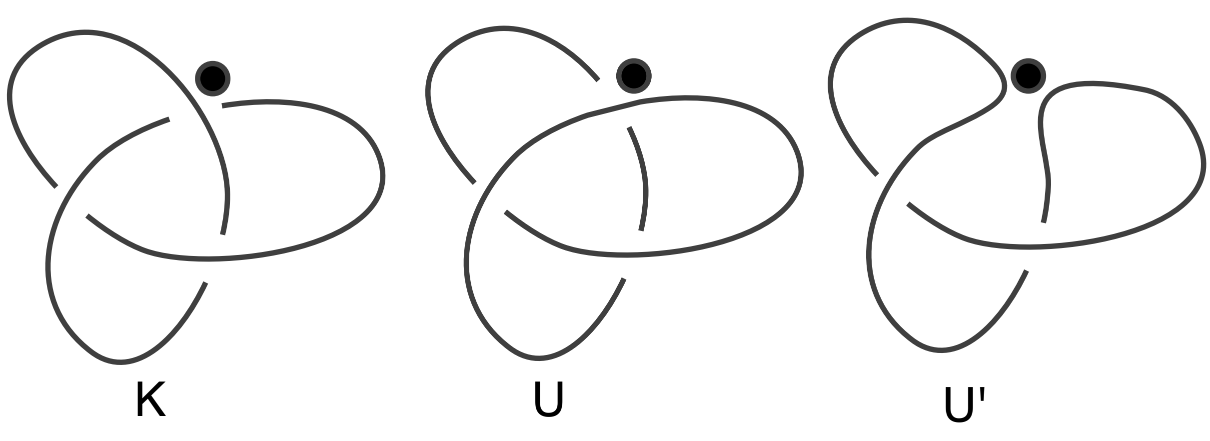

Figure 1 – Trefoil and Two Relatives

You see in Figure 1, a trefoil diagram K, an unknot diagram U and another unknot diagram U'. Applying the switching formula, we have

$$A^{-1}K - AU = (A^{-2} - A^{2})U'$$

and $U = -A^3$ and $U' = (-A^{-3})^2 = A^{-6}$. Thus

$$A^{-1}K - A(-A^3) = (A^{-2} - A^{2})A^{-6}.$$

Hence

$$A^{-1}K = -A^4 + A^{-8} - A^{-4}.$$

Thus

$$K = -A^5 - A^{-3} + A^{-7}.$$

This is the bracket polynomial of the trefoil diagram K. We have used to same symbol for the diagram and for its polynomial.

Since the trefoil diagram K has writhe $w(K) = 3$, we have the normalized polynomial

$$f_K(A) = (-A^3)^{-3} < K >= -A^{-9}(-A^5 - A^{-3} + A^{-7}) = A^{-4} + A^{-12} - A^{-16}.$$

The asymmetry of this polynomial under the interchange of A and A^{-1} proves that the trefoil knot is not ambient isotopic to its mirror image.

In Figure 2 you see the knot $K = N42 = 9_{42}$ (the latter being its standard name in the knot tables) and a skein tree for it via switching and smoothing. In Figure 3 we show simplified (via regular isotopy) representatives for the end diagrams in the skein tree.

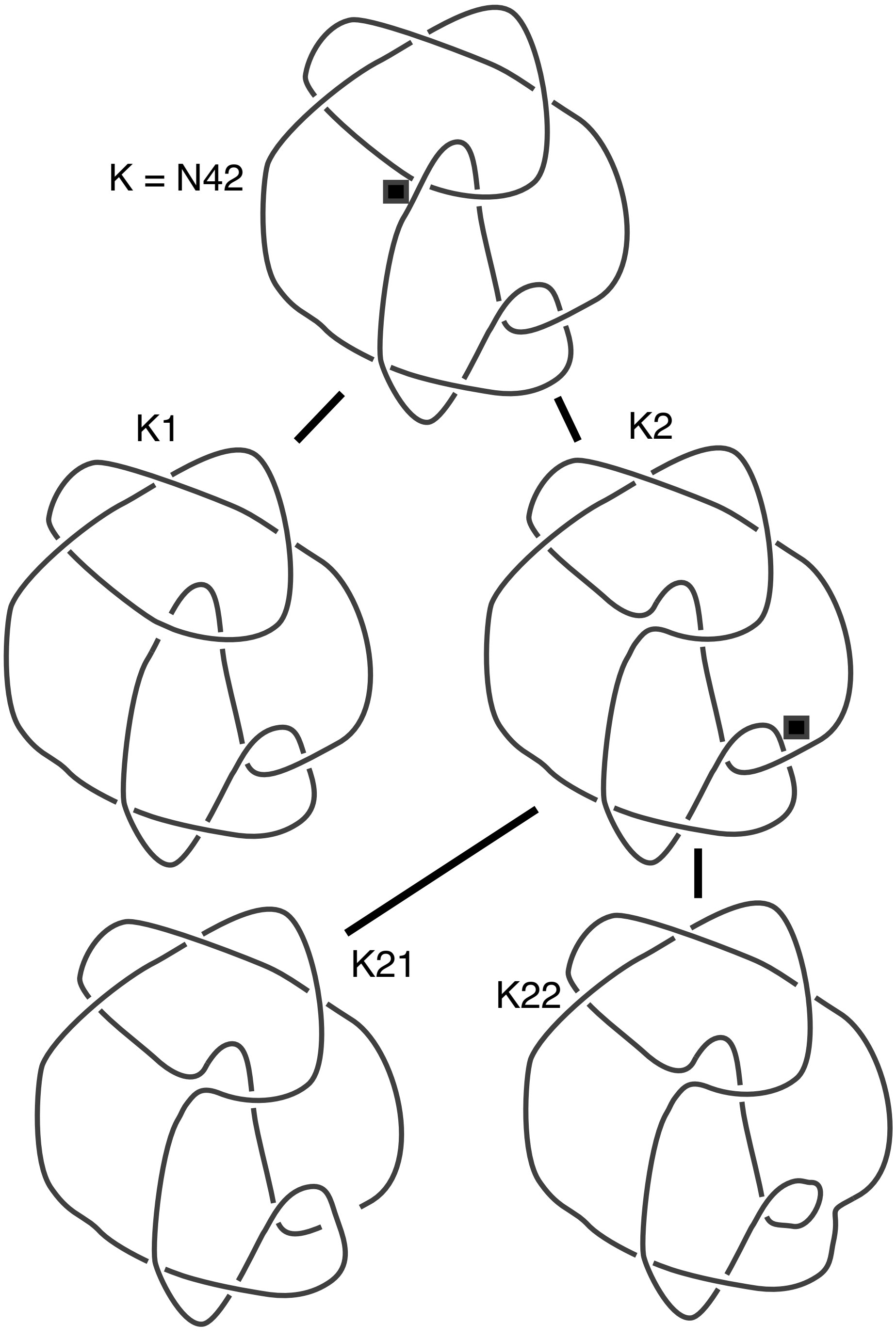

Figure 2 – Skein Tree for 9_{42}

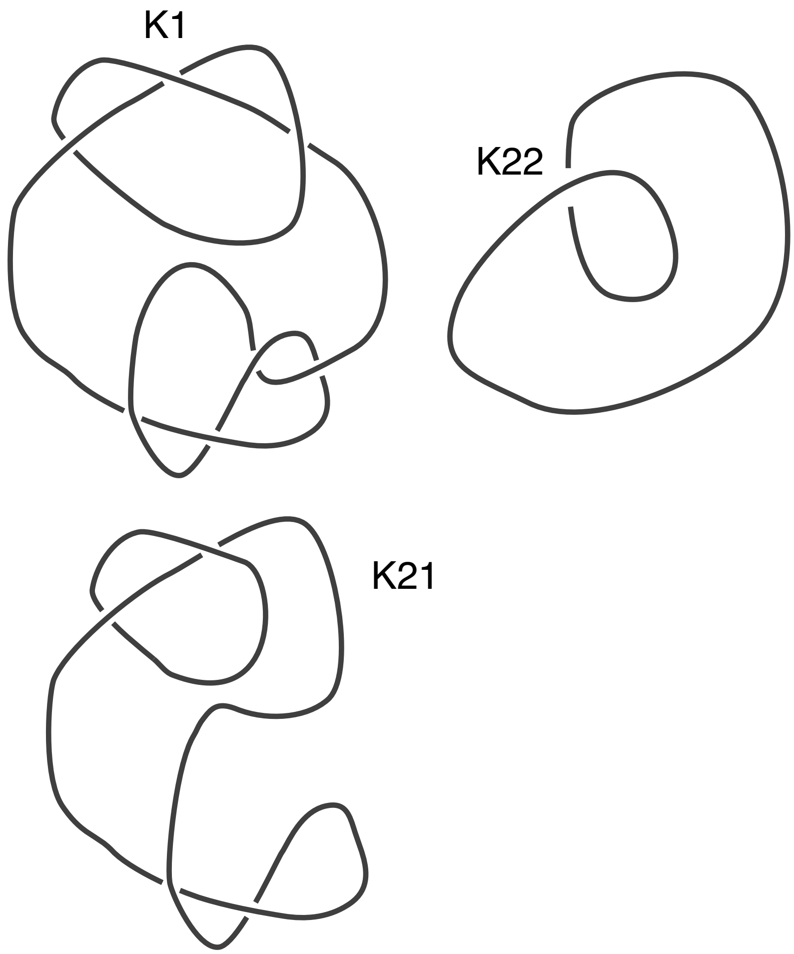

Figure 3 – Regular Isotopy Versions of Bottom of Skein Tree for 9_{42}

It follows from the switching formula that for $K = 9_{42}$,

$$A^{-1}K - AK_1 = (A^{-2} - A^2)K_2$$

$$AK_2 - A^{-1}K_{21} = (A^2 - A^{-2})K_{22}$$

and that K_1 is a connected sum of a right-handed trefoil diagram and a figure eight knot diagram, while K_{21} is a Hopf link (simple link of linking number one) with extra writhe of -2 while K_{22} is an unknot with writhe of 1. These formulas combine to give

$$< K_1 >= -A^{-9} + A^{-5} - A^{-1} + A^3 - A^7 + A^{11} - A^{15}.$$

Since K has writhe one, we get

$$f_K = A^{-12} - A^{-8} + A^{-4} - 1 + A^4 - A^8 + A^{12}.$$

This shows that the normalized bracket polynomial does not distinguish 9_{42} from its mirror image. This knot is, in fact chiral (inequivalent to its mirror image), a fact that can be verified by other means. The knot 9_{42} is the first chiral knot whose chirality is undetected by the Jones polynomial.

2 Using a Computer Program

In a separate set of notes [Mathematica] we have recorded a Mathematica worksheet showing how to use the computer to do bracket polynomial calculations. The strategy for this computer program is to record the process of translating the diagram into its state expansion into symbols that can be handled by the computer language of Mathematica. Mathematica is very good at symbol manipulation and substitution, making this a good strategy. In this section I will explain the method in a way that is independent of any particular computer language.

View Figure 4. Here we indicate a trefoil diagram with labels on each of the edges of its underlying shadow graph. The crossings are indicated by the numbers $1, 2, 3$ and each corresponds to a string of symbols from the edge labelings. The translation from crossing to symbol string is effected by *reading the labels in counterclockwise order around the crossing such that the first label is on an overcrossing line.*

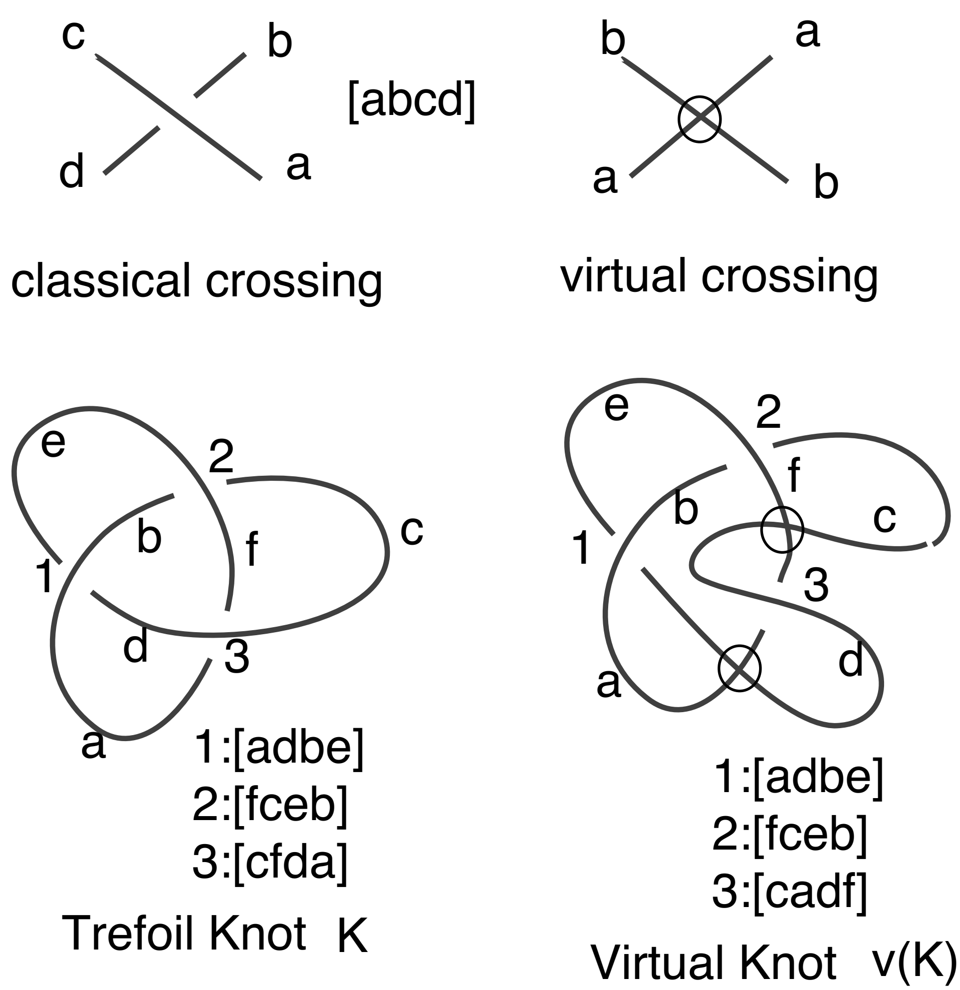

Figure 4 – Coding a Diagram

In the case of the trefoil knot shown in Figure 4, we have that the crossings correspond to the following sequences

1. $[adbe]$
2. $[fceb]$
3. $[cfda]$.

Each sequence is well-defined up to a cyclic permutation by two symbols (since one could have started the sequence with the other over-crossing label. Thus $[adbe]$ and $[bead]$ represent the same crossing and the same information. This list of crossing sequences completely determines the knot or link diagram. Note that each letter occurs twice in the list of sequences and no letter occurs twice at a given crossing unless a curl in the diagram is being encoded. Note also that a one-place cyclic permutation has the effect of switching the crossing. Thus $[abde]$ and $[bdea]$ represent a crossing and its mirror image.

In Figure 4 we also illustrate a code set that has no planar realization without extra self-crossings.

1. $[adbe]$

2. $[fceb]$

3. $[cadf]$.

Such extra self-crossings are called *virtual crossings*. A diagram with two virtual crossings and this code is illustrated in the Figure and labeled $v(K)$. There is an entire subject here of virtual knot theory where the knots are specified by such codes, and have representative drawings in the plane sucn as the one shown. One can speak of Reidemeister moves for virtual diagrams and much of knot theory generalizes in this way. For example, the algorithm explained below for calculating the bracket polynomial from a code list works perfectly well for virtual codes. In making a virtual code from a diagram with virtual crossings one keeps the labels the same across the virtual crossings just as though they were not there (and they are not there with respect to the code, just there for the purpose of drawing a representative plane diagram). The example $v(K)$ has unit Jones polynomial and yet is knotted as a virtual knot. This shows how the virtual world has counterexamples to problems that are indeed very difficult in the classical world of ordinary knots and diagrams. To this date there is no known example of a classical knotted knot diagram that has unit Jones polynomial. We will return to this topic of virtual knot theory later in the course.

In the Mathematica program we use the notation $X[a,b,c,d]$ for $[abcd]$, and we write the code itself as a formal product as in $X[a,d,b,e]X[f,c,e,b]X[c,f,d,a]$ for the Trefoil diagram in Figure 4.

Now how do we set up the bracket algorithm? We need a symbol for a segment with two labels that are supposed to be the same. For this we take $\delta[ab]$. Then we can write

1. $[abcd] \longrightarrow A\delta[ad]\delta[bc] + B\delta[ab]\delta[cd]$

2. $\delta[ab]\delta[bc] \longrightarrow \delta[ac]$

3. $\delta[aa] \longrightarrow d$

and successive applications of these replacements will turn the codelist from the diagram, written in the form of a product (as in $[adbe][fceb][cfda]$) into an algebraic product that is exactly the three variable (A, B, d) bracket polynomial of the original knot or link. Note that in the sequences for the δ we have $\delta[ab] = \delta[ba]$. Further substitution of A^{-1} for B and $-A^2 - A^{-2}$ for d gives the topological bracket polynomial.

Here is how it looks in Mathematica where I use J for d and del for δ. In rendering the Mathematica syntax in LaTeX I have eliminated certain aspects about dummy variables that are indicated by an underscore. Therefore, if you want to actually use this code, please consult the Matematica file for the exact syntax. For example, in rule 4, Mathematica does not use a dummy variable x, but rather an underscore symbol.

1. $Trefoil = X[a,d,b,e]X[e,b,f,c]X[c,f,d,a]$

2. $rule1 = X[a,b,c,d] :> Adel[ad]del[bc] + Bdel[ab]del[cd]$

3. $rule2 = del[ab]del[bc] :> del[ac]$

4. $rule3 = (del[x])^2 :> J, del[x^2] :> J$

5. $RawBracket[t] := Simplify[(t/.rule1//Expand)//.rule2/.rule3]$

6. $RawBracket[Trefoil]/J$

7. $3A^2B + A^3J + 3AB^2J + B^3J^2$

Here is an example. View Figure 5. Here we have the labeled version of a link L discovered by Morewen Thistlethwaite in December 2000 [Morwen]. We discuss some theory behind this link in the next section. It is a link that is linked but whose linking is not detectable by the Jones polynomial. One can verify such properties by using a computer program, and in fact the Mathematica file uses the coding shown in Figure 5 to do the computation.

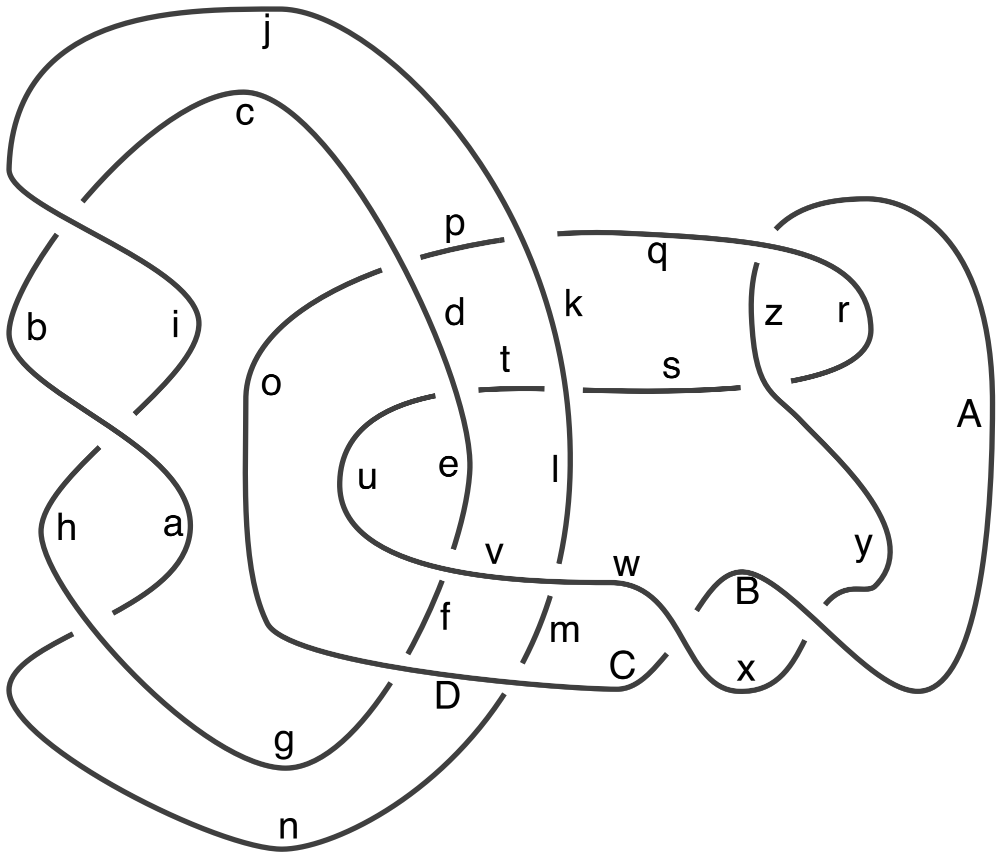

Figure 5 – Morwen's Link

3 Present Status of Links Not Detectable by the Jones Polynomial

In this section we give a quick review of the status of our work [EKT] producing infinite families of distinct links all evaluating as unlinks by the Jones polynomial.

A tangle (2-tangle) consists in an embedding of two arcs in a three-ball (and possibly some circles embedded in the interior of the three-ball) such that the endpoints of the arcs are on the boundary of the three ball. One usually depicts the arcs as crossing the boundary transversely so that the tangle is seen as the embedding in the three-ball augmented by four segments emanating from the ball, each from the intersection of the arcs with the boundary. These four segments are the *exterior edges* of the tangle, and are used for operations that form new tangles and new knots and links from given tangles. Two tangles in a given three-ball are said to be *topologically equivalent* if there is an ambient isotopy from one to the other in the given three-ball, fixing the intersections of the tangles with the boundary.

It is customary to illustrate tangles with a diagram that consists in a box (within which are the arcs of the tangle) and with the exterior edges emanating from the box in the NorthWest (NW), NorthEast (NE), SouthWest (SW) and SouthEast (SE) directions. Given tangles T and S, one defines the *sum*, denoted $T+S$ by placing the diagram for S to the right of the diagram for T and attaching the NE edge of T to the NW edge of S, and the SE edge of T to the SW edge of S. The resulting tangle $T+S$ has exterior edges correponding to the NW and SW edges of T and the NE and SE edges of S. There are two ways to create links associated to a tangle T. The *numerator* T^N is obtained by attaching the (top) NW and NE edges of T together and attaching the (bottom) SW and SE edges together. The denominator T^D is obtained by attaching the (left side) NW and SW edges together and attaching the (right side) NE and SE edges together. We denote by [0] the tangle with only unknotted arcs (no embedded circles) with one arc connecting, within the three-ball, the (top points) NW intersection point with the NE intersection point, and the other arc connnecting the (bottom points) SW intersection point with the SE intersection point. A ninety degree turn of the tangle [0] produces the tangle $[\infty]$ with connections between NW and SW and between NE and SE. One then can prove the basic formula for any tangle T

$$< T >= \alpha_T < [0] > + \beta_T < [1] >$$

where α_T and β_T are well-defined polynomial invariants (of regular isotopy) of the tangle T. From this formula one can deduce that

$$< T^N >= \alpha_T d + \beta_T$$

and

$$< T^D >= \alpha_T + \beta_T d.$$

We define the *bracket vector* of T to be the ordered pair (α_T, β_T) and denote it by $br(T)$, viewing it as a column vector so that $br(T)^t = (\alpha_T, \beta_T)$ where v^t denotes the transpose of the vector v. With this notation the two formulas above for the evaluation for numerator and denominator of a tangle become the single matrix equation

$$\begin{bmatrix} < T^N > \\ < T^D > \end{bmatrix} = \begin{bmatrix} d & 1 \\ 1 & d \end{bmatrix} br(T).$$

We then use this formalism to express the bracket polynomial for our examples. The class of examples that we considered are each denoted by $H(T,U)$ where T and U are each tangles and $H(T,U)$ is a satellite of the Hopf link that conforms to the pattern shown in Figure 6, formed by clasping together the numerators of the tangles T and U. Our method is based on a transformation $H(T,U) \longrightarrow H(T,U)^\omega$, whereby the tangles T and U are cut out and reglued by certain specific homeomorphisms of the tangle boundaries. This transformation can be specified by a modification described by a specific rational tangle and its mirror image. Like mutation, the transformation ω preserves the bracket polynomial. However, it is more effective than mutation in generating examples, as a trivial link can be transformed to a prime link, and repeated application yields an infinite sequence of inequivalent links.

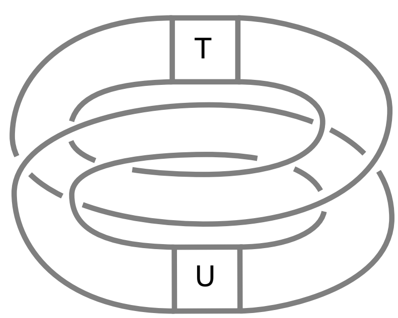

Figure 6 – Hopf Link Satellite H(T,U)

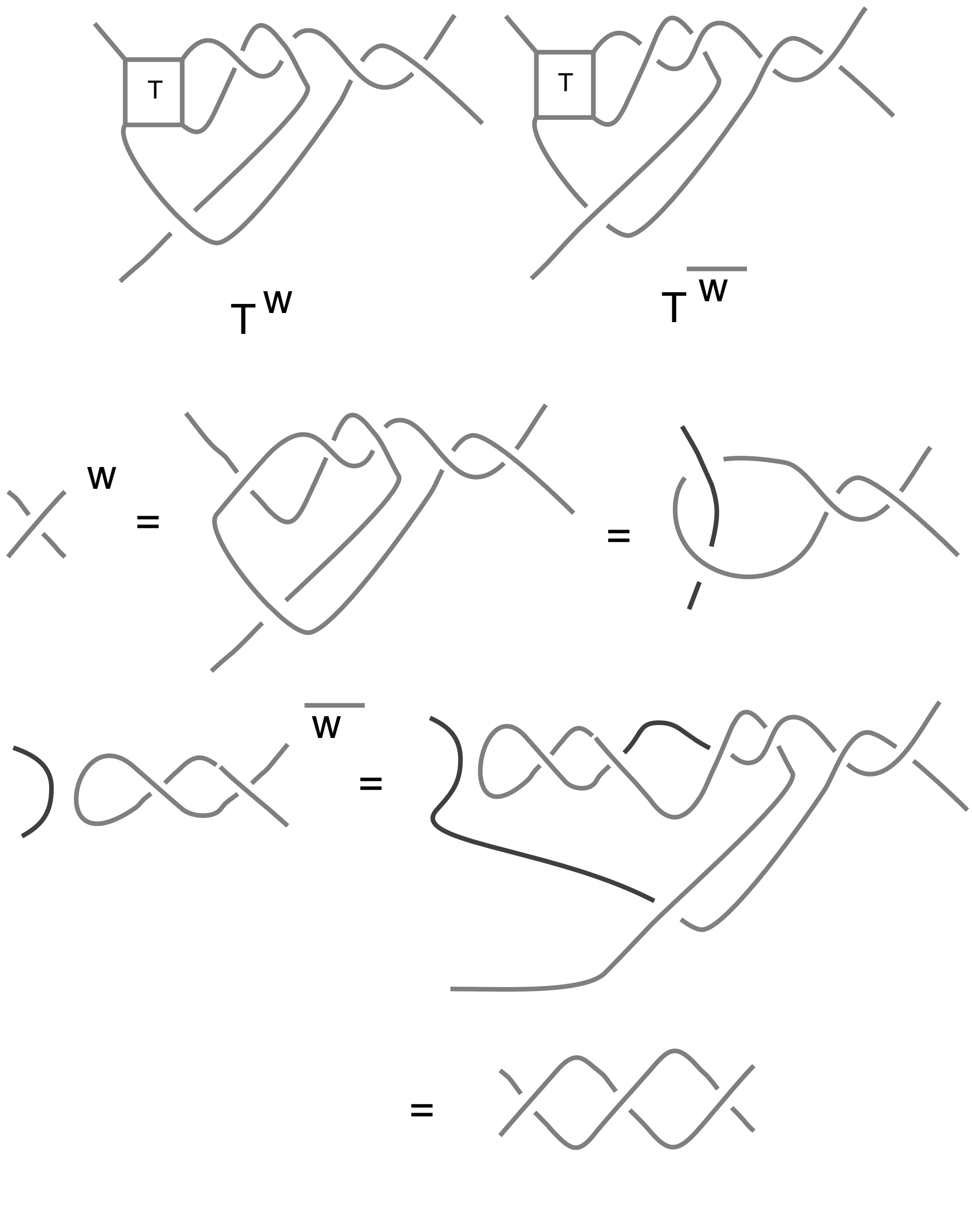

Figure 7 – The Omega Operations

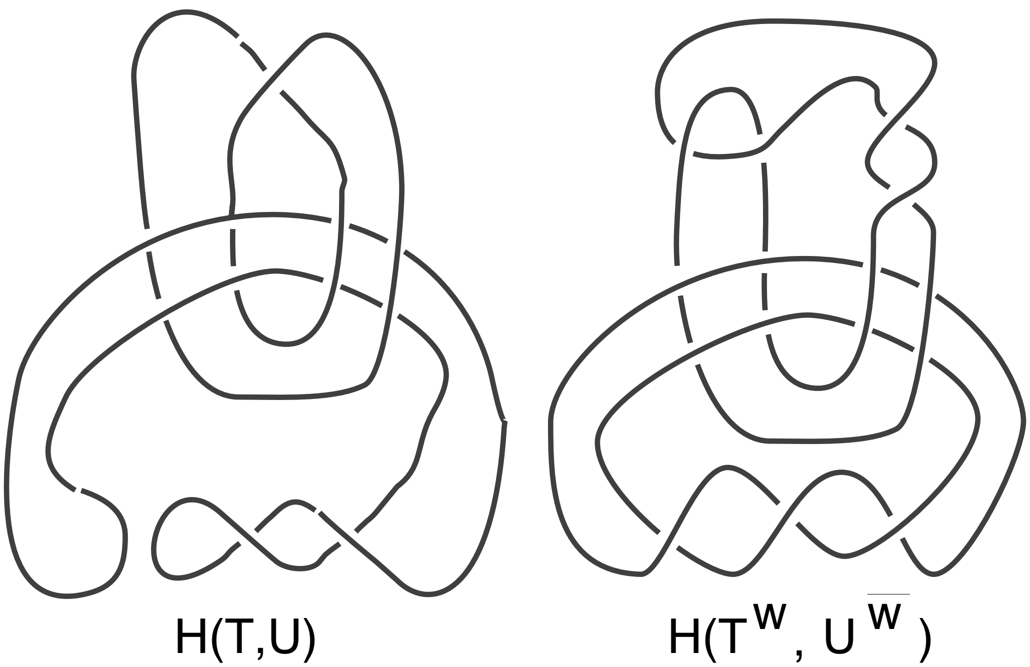

Figure 8 – Applying Omega Operations to an Unlink

Specifically, the transformation $H(T,U)^{\omega}$ is given by the formula

$$H(T,U)^{\omega} = H(T^{\omega}, U^{\bar{\omega}})$$

where the tangle operations T^{ω} and $U^{\bar{\omega}}$) are as shown in Figure 7. By direct calculation, there is a matrix M such that

$$< H(T,U) >= br(T)^{t} M br(U)$$

and there is a matrix Ω such that

$$br(T^{\omega}) = \Omega br(T)$$

and

$$br(T^{\bar{\omega}}) = \Omega^{-1} br(T).$$

One verifies the identity

$$\Omega^{t} M \Omega^{-1} = M$$

from which it follows that $< H(T,U) >^{\omega} = < H(T,U) >$. This completes the sketch of our method for obtaining links that whose linking cannot be seen by the Jones polynomial. Note that the link constructed as $H(T^{\omega}, U^{\bar{\omega}})$ in Figure 8 has the same Jones polynomial as an unlink of two components. This shows how the first example found by Thistlethwaite fits into our construction.

3.1 Switching a Crossing

If in Figure 8, we start with T replaced by $Flip(T)$, switching the crossing, the resulting link $L = H(Flip(T)^{\omega}, U^{\bar{\omega}})$ will still have Jones polynomial the same as the unlink, but the link L will be distinct from the link $H(T^{\omega}, U^{\bar{\omega}})$ of Figure 8. We illustrate this process in Figure 9.

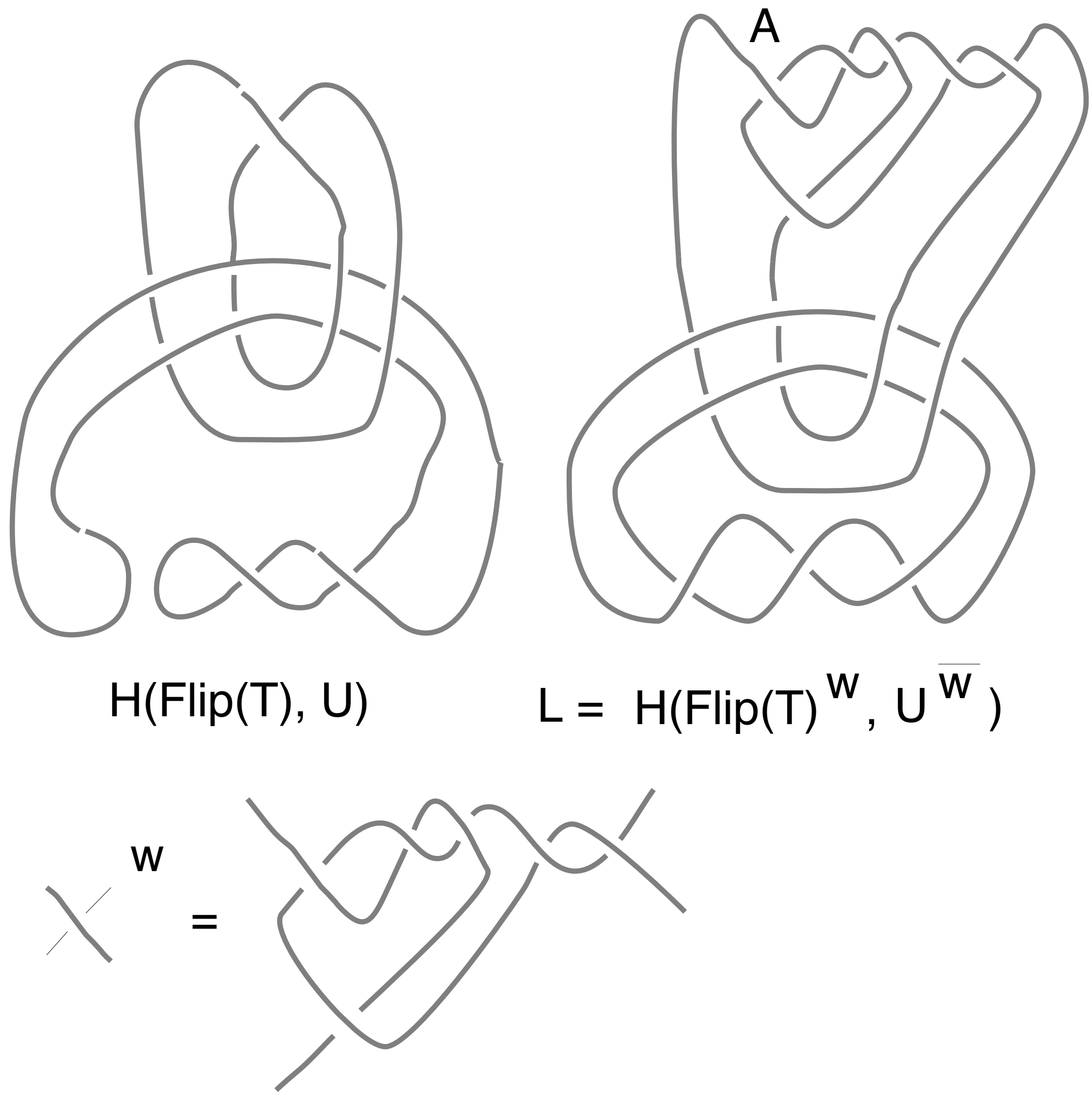

Figure 9 – Applying Omega Operations to an Unlink with Flipped Crossing

The link L has the remarkable property that both it and the link obtained from it by flipping the crossing labeled A in Figure 9 have Jones polynomial equal to the Jones polynomial of the unlink of two components. (We thank Alexander Stoimenow for pointing out the possibility of this sort of construction.)

Now lets think about a link L with the property that it has the same Jones polynomial as a link L' obtained from L by switching a single crossing. We can isolate the rest of the link that is not this crossing into a tangle S so that (without any loss of generality) $L = N(S + [1])$ and $L' = N(S + [-1])$. Lets assume that orientation assignment to L and L' is as shown in Figure 10.

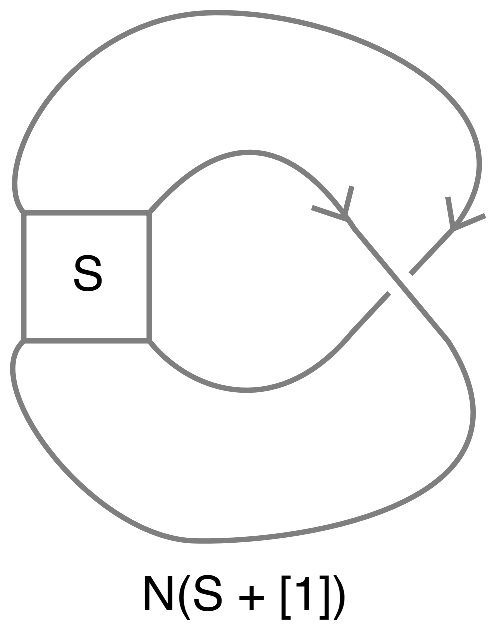

Figure 10 – N(S + [1])

Since we are told that L and L' have the same Jones polynomial, it follows that $< L >= -A^{-3}\kappa$ and $< L' >= -A^{3}\kappa$ for some non-zero Laurent polynomial κ. Now suppose that $< S >= \alpha < [0] > +\beta < [\infty] >$. Then

$$< L >= \alpha(-A^3) + \beta(-A^{-3})$$

and

$$< L' >= \alpha(-A^{-3}) + \beta(-A^3).$$

From this it follows that

$$\kappa = \alpha A^6 + \beta$$

and

$$\kappa = \alpha A^{-6} + \beta.$$

Thus

$$\alpha = 0$$

and

$$\beta = \kappa.$$

We have shown that

$$< S >= \kappa < [\infty] > .$$

This means that we will can, by using the example described above, produce a tangle S that is not splittable and yet has the above property of having one of its bracket coefficients equal to zero. The example is shown in Figure 11.

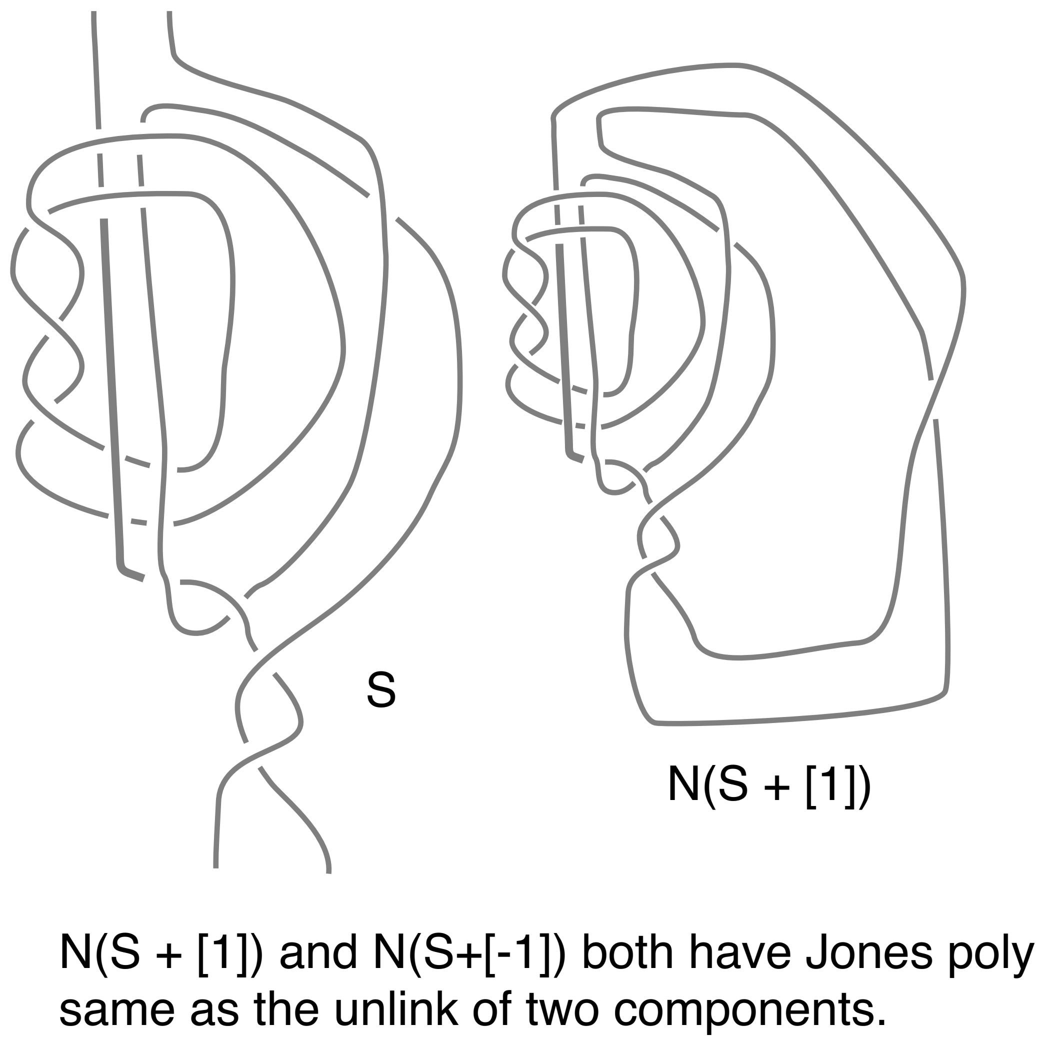

Figure 11 – The Tangle S

Finally in Figure 12 we show L and the link $v(L)$ obtained by virtualizing the crossing corresponding to [1] in the decomposition $L = N(S + [1])$. The virtualized link $v(L)$ has the property that it also has Jones polynonial the same as an unlink of two components. We wish to prove that $v(L)$ is not isotopic to a classical link. The example has been designed so that surface bracket techniques will be difficult to apply. (We will discuss such techniques later in these notes.)

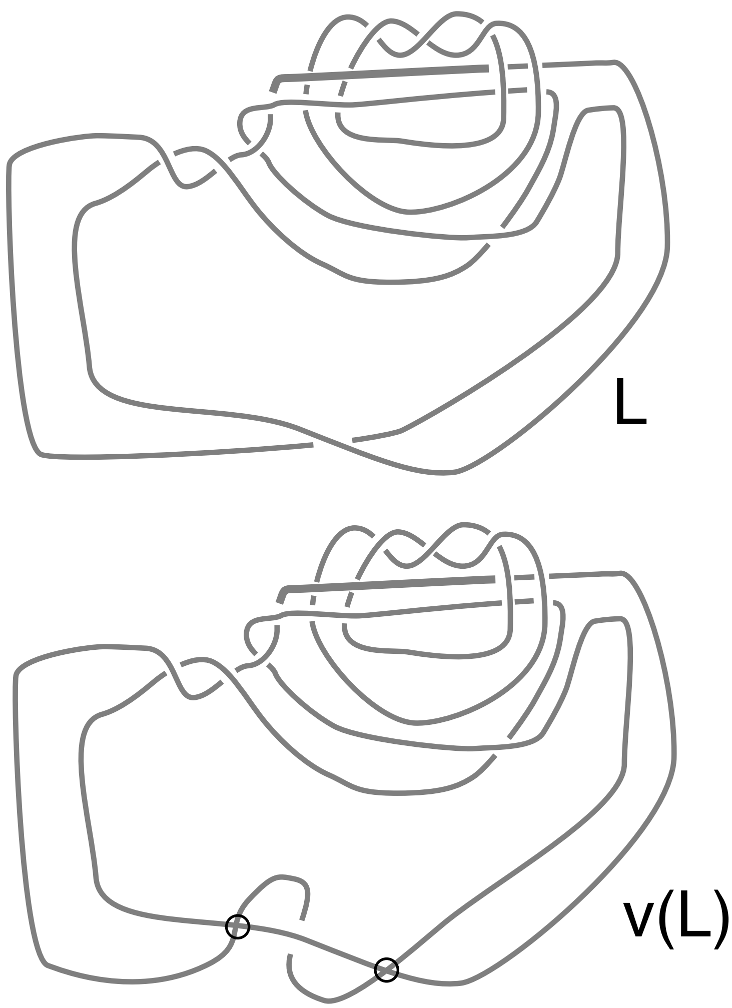

Figure 12 – The Virtual Link v(L)

References

[EKT] S. Eliahou, L. Kauffman and M. Thistlethwaite, Infinite families of links with trivial Jones polynomial, Topology (42) 2003, 155-169.

[Mathematica] www.math.uic.edu/ kauffman/BracketDemo.pdf

[Morwen] M. Thistlethwaite, Links with trivial Jones polynomial, JKTR, Vol. 10, No. 4 (2001), 641-643.