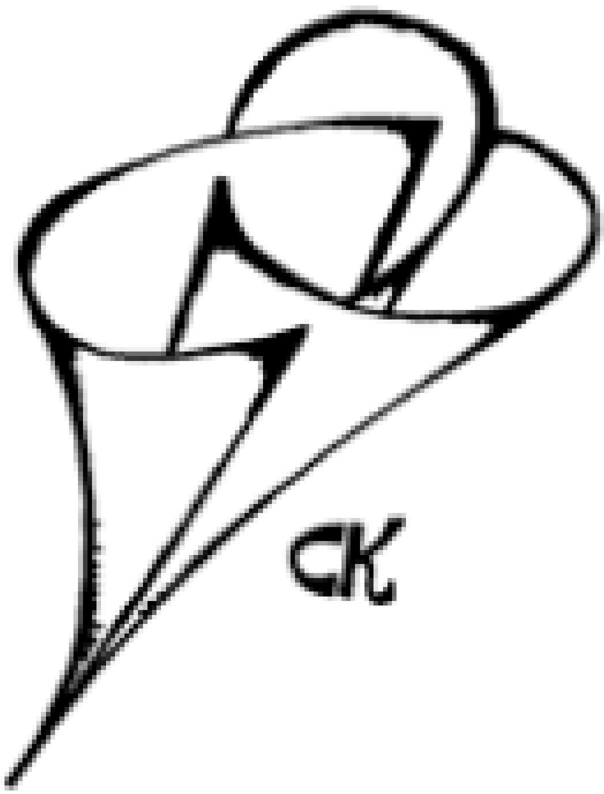
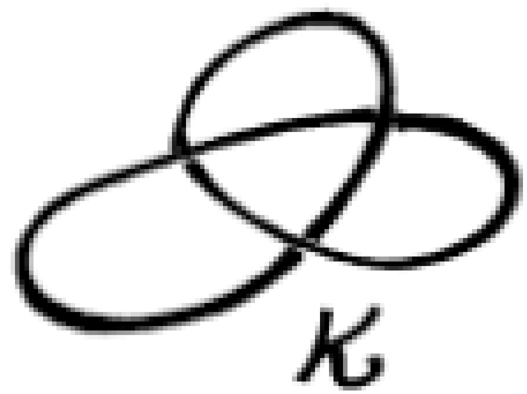
Links of Singularities, Brieskorn Varieties and Products of Knots

From “On Knots”
by Louis H. Kauffman





XIX

SINGULARITIES, KNOTS AND BRIESKORN VARIETIES

A good reference for this section is Milnor's book [M3], *Singular Points of Complex Hypersurfaces*; also the original papers of Pham [PH] and Brieskorn [BK] and the notes by Hirzebruch and Zagier [HZ]. There is a large and continuing literature on this topic. Our intent here is to give a survey of examples and constructions. As we shall see, the subject of the topology of algebraic singularities is intimately related to knot theory and to the structure of branched covering spaces. In the case of the Brieskorn manifolds these ideas come together, so that the link of a Brieskorn singularity may be described completely in terms of knots and branched coverings (Example 19.12 of this chapter). In this sense many constructions of high-dimensional topology, including exotic spheres, may be seen as implicit in, or as generated from the deep three-dimensional knot-work of Alexander and Seifert. Since this early topological work owed much of its impetus to the desire to understand the topology of algebraic varieties, it is fitting that we end our tale of knots and manifolds in this realm.

Let $f(z_0, z_1, z_n) = f$ be a polynomial in $(n+1)$ complex variables. We define the variety of f by $V(f) = \{z \in \mathbb{C}^{n+1} | f(z) = 0\}$. The variety of f is its locus of zeroes.

When $f(0) = 0$ we define the link of f by the equation $L(f) = V(f) \cap S_\epsilon^{2n+1}$ where S_ϵ^{2n+1} is a sphere about $0 \in \mathbb{C}^{n+1}$ of radius $\epsilon > 0$. Usually ϵ is chosen very small so that the topology of $L(f)$ and its embedding in S_ϵ^{2n+1} reflects the nature of the variety $V(f)$ at 0. In the most general case the link $L(f)$ will depend upon the choice of ϵ. However, under special conditions (such as an isolated singularity—see below) $L(f)$ will be independent of ϵ for sufficiently small ϵ.

A point $z \in V(f)$ is said to be a singularity of f if $\nabla_f(z)$ vanishes, where $\nabla_f = (\partial f/\partial z_0, \partial f/\partial z_2, \cdots, \partial f/\partial z_n)$ denotes the complex gradient (not the Conway polynomial!). A singularity is isolated if it has a neighborhood in $\mathbb{C}^{n+1}$ containing no other singularities of f. The polynomials $z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n}$, $(a_0, a_1, \cdots, a_n)$ an $(n+1)$-tuple of positive integers greater than or equal to 2, form a collection having isolated singularities at the origin. They will be referred to as Brieskorn polynomials. It is these polynomials that will occupy our attention in this chapter. The Brieskorn polynomials were first studied by Pham [PH] in relation to problems in particle physics.

Pham's calculations generalized earlier key calculations of Lefschetz [LF] for the behavior of $z_0^2 + z_1^2 + \cdots + z_n^2$. Brieskorn utilized Pham's calculations and recognized that the links of these polynomials comprised an extensive class of manifolds, providing, in particular, realizations of many exotic spheres.

DEFINITION 19.1. *Let* $\Sigma(a_0, \cdots, a_n) = L(z_0^{a_0} + \cdots + z_n^{a_n})$ *denote the link of the Brieskorn singularity defined by* $z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = 0$.

PROPOSITION 19.2. $\Sigma(a_0, a_1) \subset S^3$ *is a torus link of type* (a_0, a_1).

Proof: By definition, $\Sigma(a_0, a_1)$ is the set of points in $\mathbb{C}^2$ satisfying the equations $z_0^{a_0} + z_1^{a_1} = 0$, $|z_1|^2 + |z_2|^2 = 1$. (We use a sphere of radius one for this demonstration. That the link is independent of the radius is easy to verify for Brieskorn manifolds.) Let $z_0 = re^{i\theta}$, $z_1 = se^{i\phi}$. If r and s are real numbers satisfying $r^{a_0} + s^{a_1} = 0$, $r^2 + s^2 = 1$, then we can obtain further complex solutions via the condition

$$r^{a_0} e^{ia_0\theta} = -s^{a_1} e^{ia_1\phi} \Rightarrow e^{ia_0\theta} = e^{ia_1\phi}.$$

This defines a torus link of type (a_0, a_1) on the torus parametrized by $(re^{i\theta}, se^{i\phi})$. That the whole link $\Sigma(a_0,a_1)$ arises in this form is left as an exercise for the reader.

Our next result shows how the higher-dimensional Brieskorn manifolds are cyclic branched coverings along lower-dimensional Brieskorn manifolds. Before proving this fact, we set up some useful notation:

Let $\Sigma = \Sigma(a) = \Sigma(a_0,a_1,\cdots,a_n)$ denote the Brieskorn manifold obtained as the link of the singularity $z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n}$. Here $\underline{a}$ is an abbreviation for the $(n+1)$-tuple $(a_0,\cdots,a_1)$. Let

$$\Sigma_k = \Sigma_k(a) = \Sigma(a_0,a_1,\ \cdots,a_n,k).$$

Thus $\Sigma(a) \subset S^{2n+1}$ while $\Sigma_k(a) \subset S^{2n+3}$.

PROPOSITION 19.3. *There is a map* $\pi : \Sigma_k \longrightarrow S^{2n+1}$ *exhibiting* Σ_k *as a* k*-fold cyclic branched cover of* S^{2n+1}, *with branch set* Σ.

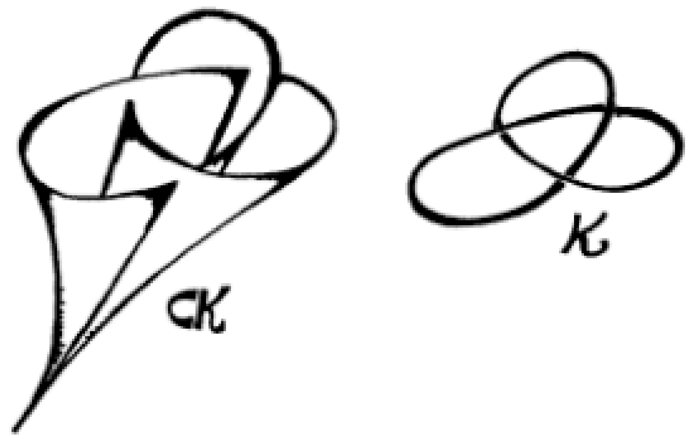
Remark: It follows from this proposition that all of the Brieskorn manifolds are obtained by forming certain towers of branched coverings in the pattern

$$\vdots$$
$$\downarrow$$
$$\Sigma(a_0, a_1, a_2) \subset S^5$$
$$\downarrow$$
$$\Sigma(a_0, a_1) \subset S^3$$

So far, each embedding $\Sigma(a_0, \cdots, a_1) \subset S^{2n+1}$ gives rise to a branched covering manifold, which, by dint of our elementary algebraic geometery, is _itself_ embedded in a sphere so that the construction can continue. In fact, there are topological constructions for the embeddings as well. We will discuss these constructions shortly. Constructions of this type are motivated by the geometry and topology of algebraic singularities.

While we are on the subject of the relation of knot theory and singularities, it is worth remarking that any knot can be regarded as the link of a "singularity" although this is not necessarily algebraic: Given $K \subset S^n$ we have the cone on K, $CK \subset D^{n+1}$. The cone is a topological space with a singularity at the cone point ($CK = \{r\bar{x} \in D^{n+1} | \bar{x} \in K \subset S^n, 0 \le r \le 1\}$). The cone point is, by definition, the origin in D^{n+1}. This apparently simple remark is the key to amalgamating constructions in knot theory with properties of algebraic singularities.

It also is helpful to sketch immersions into $\mathbb{R}^3$ to see the geometry of the singularity. View the following figure.



<u>An Immersion of CK in $\mathbb{R}^3$</u>

Proof of 19.3: Parametrize $\mathbb{C}^{n+2} = \mathbb{C}^{n+1} \times \mathbb{C}$ as $\{(z_0, z_1, \cdots, z_n, x) = (z, x) \mid z_i \in \mathbb{C},\ x \in \mathbb{C}\}$. Let $f(z) = z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n}$. Then let $F : \mathbb{C}^{n+2} \longrightarrow \mathbb{C}$ be the polynomial $F(z,x) = f(z) + x^k$. Let $V(F)$ denote the variety of F. Thus

$$V(F) = \{(z,x) \in \mathbb{C}^{n+2} \mid f(z) + x^k = 0\}.$$

Define $p : V(F) \longrightarrow \mathbb{C}^{n+1}$ by the formula $p(z,x) = z$. The mapping p exhibits $V(F)$ as a branched covering of $\mathbb{C}^{n+1}$ with branching set $V(f) \subset \mathbb{C}^{n+1}$. We wish to modify this mapping to obtain $\pi : \Sigma_k \longrightarrow S^{2n+1}$.

First consider the restriction of p to Σ_k: $\Sigma_k = \{(z,x) \mid f(z) + x^k = 0,\ |z|^2 + |x|^2 = 1\}$, $p : \Sigma_k \longrightarrow \mathcal{S} = p(\Sigma_k) \subset S^{2n+1}$. Since $p(z,x) = z$ we see that for

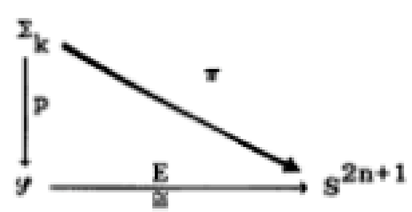
$\Sigma = \{(z,0)\} \subset \Sigma_k$, $p(z,0) = z$ and $p(\Sigma) = \Sigma \subset \mathbb{C}^{n+1}$. Thus $\Sigma \subset \mathcal{S}$, and Σ_k is a k-fold branched covering of $\mathcal{S}$ with branch locus Σ. It remains to show that $\mathcal{S}$ is ambient isotopic to $S^{2n+1} \subset \mathbb{C}^{n+1}$.

To this end, define an operation of the nonnegative real numbers, $\mathbb{R}^+$, on $\mathbb{C}^{n+1}$ via

$$\rho * z = \left(\rho^{1/a_0} z_0, \rho^{1/a_1} z_1, \cdots, \rho^{1/a_n} z_n\right)$$

for $\rho \in \mathbb{R}^+$, $z \in \mathbb{C}^{n+1}$. Note that $f(\rho * z) = \rho f(z)$.

Note also that $0 \notin \mathcal{S}$ since if $p(z,x) = 0$ then $z = 0$, whence $f(0)+x^k = 0$ whence $x^k = 0$, hence $x = 0$. But $(0,0) \notin \Sigma_k$ and $\mathcal{S} = p(\Sigma_k)$. Therefore, define $E : \mathcal{S} \longrightarrow S^{2n+1} = \{z \in \mathbb{C}^{n+1} \mid |z| = 1\}$ by the formula $E(z) = \rho * z$ for that unique $\rho > 0$ such that $|\rho * z| = 1$. We leave it as an exercise to show that $E : \mathcal{S} \longrightarrow S^{2n+1}$ is a diffeomorphism. Thus we have the diagram



and we define $\pi = E \circ p$. Since Σ is invariant under E, we have shown that Σ_k is a k-fold branched covering of S^{2n+1} along $\Sigma \subset S^{2n+1}$. That it is a cyclic branched cover is also left as an exercise. This completes the proof.

Remark: Proposition 19.3 can be considerably generalized by replacing the directly constructed map $E : \mathcal{V} \longrightarrow S^{2n+1}$ by maps obtained through integrating vector fields. See [DK] and [KN]. In [DK] we show that the link $L(f(z)+x^k)$ is always a cyclic branched cover whenever $f(z)$ has an isolated singularity at the origin.

Example 19.4: Proposition 19.3 tells us that $\Sigma(2,2,2)$ is the 2-fold cyclic cover of S^3 branched along $\Sigma(2,2) \subset S^3$. The latter is the $(2,2)$ torus link, also known as the <u>Hopf Link</u> $\Lambda \subset S^3$.

Λ

Here we have said "the" branched cover, by which we mean the branched cover that corresponds to the representation

$$\begin{aligned}\pi_1(S^3-\Lambda) \cong Z \oplus Z &\longrightarrow Z \longrightarrow Z_k\\ (1,0) &\longmapsto 1 \longmapsto 1\\ (0,1) &\longmapsto 1 \longmapsto 1\end{aligned}$$

where $(1,0)$ and $(0,1)$ correspond to meridinal generators oriented positively around respective link components. The link is presented with linking number 1. We leave it as an exercise to show that this representation corresponds to the k-fold cyclic branched covering $\Sigma(2,2,k) \longrightarrow S^3$.

Returning to $\Sigma(2,2,2)$, it is amusing to reformulate this in two related ways:

1. $\Sigma(2,2,2) \cong T$ where T denotes the tangent circle bundle to the two-sphere S^2.

2. $\Sigma(2,2,2) \cong \mathbb{R}P^3$ where $\mathbb{R}P^3$ denotes real projective 3-space.

And also

3. $\mathbb{R}P^3 \cong SO(3)$ the group of orthogonal, orientation preserving linear transformations of $\mathbb{R}^3$.

Thus T, $\mathbb{R}P^3$, $SO(3)$ and $\Sigma(2,2,2)$ are all versions of the same space.

1. We use the algebraic geometry to see that $\Sigma(2,2,2) = T$ as follows:

$\Sigma(2,2,2) = \{(z_0,z_1,z_2) \,\big|\, z_0^2+z_1^2+z_n^2 = 0, |z_1|^2+|z_2|^2+|z_3|^2 = 1\}$.

Let $z_i = X_i + \sqrt{-1}\, Y_i$ for $i = 0,1,2$. Then $z_i^2 = (X_i^2 - Y_i^2) + 2\sqrt{-1}\, X_i Y_i$. Hence, letting $\overline{X} = (X_0,X_1,X_2)$, $\overline{Y} = (Y_0,Y_1,Y_2)$ and $\overline{X}\cdot\overline{Y} = X_0Y_0+X_1Y_1+X_2Y_2$, $\|\overline{X}\|^2 = X_0^2+X_1^2+X_2^2$, we have

$$z_0^2 + z_1^2 + z_2^2 = (\|X\|^2 - \|\overline{Y}\|^2) + 2\sqrt{-1}\,(\overline{X}\cdot\overline{Y}).$$

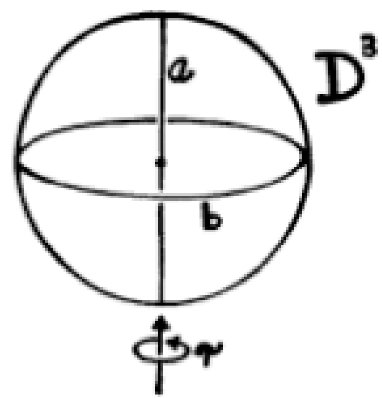
Thus,

$$\Sigma(2,2,2) = \{(\overline{X},\overline{Y}) \in \mathbb{R}^3\times\mathbb{R}^3 \,\big|\, \|\overline{X}\|^2 = \|\overline{Y}\|^2, \overline{X}\cdot\overline{Y} = 0, \|\overline{X}\|^2 + \|\overline{Y}\|^2 = 1\}$$

$$= \{(\overline{X},\overline{Y}) \in \mathbb{R}^3\times\mathbb{R}^3 \,\big|\, \|\overline{X}\| = \tfrac{1}{2} = \|\overline{Y}\|,\ \overline{X}\cdot\overline{Y} = 0\}.$$

This is precisely the set of pairs of points on S^2 $(\overline{X}, \|\overline{X}\| = \frac{1}{2})$ coupled with tangent vectors $\overline{Y}$ $(\overline{X}\cdot\overline{Y} = 0)$ of fixed length. Thus $\Sigma(2,2,2) \cong$ the tangent circle bundle to S^2.

2. To see that $\Sigma(2,2,2) \cong \mathbb{R}P^3$ it will suffice, by 19.3, to show that $\mathbb{R}P^3$ <u>is the 2-fold branched covering of S^3 branched along the Hopf Link</u>. To this end, let D^3 denote the unit 3-ball: $D^3 = \{\overline{X} \in \mathbb{R}^3 \mid \|\overline{X}\| \le 1\}$. Let $\tau : D^3 \longrightarrow D^3$ denote an 180° rotation about an axis of D^3 (straight line through the center).



Let <u>a</u> denote this axis and <u>b</u> denote an equatorial circle on the boundary of D^3.

Now $\mathbb{R}P^3 = D^3/\sim$ where $\overline{x} \sim \overline{x}'$ if and only if $\|\overline{x}\| = \|\overline{x}'\| = 1$ and $\overline{x}' = -\overline{x}$. That is, $\mathbb{R}P^3$ is the 3-ball with antipodal boundary points identified. Since τ preserves antipodal pairs we obtain $\overline{\tau} : \mathbb{R}P^3 \longrightarrow \mathbb{R}P^3$, a map of order two that fixes pointwise $\tilde{\lambda} = \overline{a} \cup \overline{b}$ where $\overline{a}$ and $\overline{b}$ are the images of the axis <u>a</u> and equator <u>b</u> in $\mathbb{R}P^3$.

Note that both $\bar{a}$ and $\bar{b}$ are embedded circles in RP^3. We leave it as an exercise to show that $RP^3/(p \sim \bar{\tau}p)$ is the three sphere S^3 and that $\bar{a} \cup \bar{b}$ projects to the Hopf link in S^3. This completes the proof that $\Sigma(2,2,2) \cong RP^3$.

3. One way to see that $RP^3 \cong SO(3)$ is to prove directly that $SO(3)$ is homeomorphic to $D^3/\sim$. To see this, represent elements of $SO(3)$ by pairs $[\theta,\bar{v}]$ where $0 \leq \theta \leq \pi$ and $\bar{v}$ is a unit vector in R^3. Then $[\theta,\bar{v}]$ represents a rotation of θ about the axis $\bar{v}$ (using the right-hand rule). Note that $[\pi,\bar{v}] = [-\pi,\bar{v}]$, and that otherwise there are no identifications. Then $SO(3) \longrightarrow D_{\pi}/\sim$ via $[\theta,\bar{v}] \longmapsto [\theta\bar{v}]$ shows that $SO(3)$ is homeomorphic to the ball of radius π, modulo antipodal identifications on the boundary.

This completes our tour of points of view on $\Sigma(2,2,2)$.

Example 19.5: Propositions 19.2 and 19.3 taken together show that $\Sigma(a,b,c)$ is

(a) The a-fold branched cover along $\Sigma(b,c)$.

(b) The b-fold branched cover along $\Sigma(a,c)$.

(c) The c-fold branched cover along $\Sigma(a,b)$.

Thus these three spaces are diffeomorphic.

Example 19.6: The Dodecahedral Space. The purpose of this example is to give proof that $\Sigma(2,3,5) = L(Z_0^2+Z_1^3+Z_2^5)$ is the dodecahedral space $\mathcal{D}$. $\mathcal{D}$ is a compact orientable three-dimensional manifold whose fundamental group $\hat{G} = \pi_1(\mathcal{D})$ is the *binary dodecahedral group*. That is, $\hat{G}$ is a subgroup of $SU(2)$ (which double covers the rotation group $SO(3)$). Let $\pi : SU(2) \longrightarrow SO(3)$ be this double covering. Then $\hat{G} = \pi^{-1}(G)$ where $G \subset SO(3)$ is the dodecahedral subgroup of $SO(3)$. That is, G is the group of rotational symmetries of an icosahedron or a dodecahedral (they are dual) in Euclidean three-dimensional space.

The dodecahedral space is an important example in topology. Its history goes all the way back to Poincaré. In fact, it is the first counterexample to a precursor to the Poincaré conjecture. The precursor would state that a three-manifold M with $H_1(M) = \{0\}$ is the 3-sphere. Dodecahedral space $\mathcal{D}$ has a *perfect* but nontrivial fundamental group. Thus $\pi_1(\mathcal{D}) \neq \{1\}$, but $H_1(\mathcal{D}) = \{0\}$. Recall that the *Poincaré conjecture* in dimension three states: *A compact connected three-manifold* M *with* $\pi_1(M) = \{1\}$ *is homeomorphic to the three-sphere* S^3. It remains unproved to this day.

The textbook [ST] by Seifert and Threlfall contains an excellent account of the combinatorial topology of $\mathcal{D}$. We shall show that $\Sigma(2,3,5) \cong S^3/\hat{G}$ with a natural covering space action of $\hat{G}$ on S^3. See also the book [DV] by DuVal, and the papers [M1] by Milnor and [OW] by Orlik and Wagreich.

First recall the definition of the Lie group SU(2). As a space, SU(2) is diffeomorphic to S^3. In fact, it can be defined as the group of unit-length quaternions. We give the definition in terms of complex valued 2×2 matrices:

$$SU(2) = \left\{\begin{bmatrix} z & w \\ -\bar{w} & \bar{z}\end{bmatrix} \,\middle|\, z,w \in \mathbb{C},\ |z|^2+|w|^2 = 1\right\}$$

$$\therefore\ SU(2) = \left\{A \,\middle|\, \begin{matrix} A \text{ is a } 2\times2 \text{ complex matrix, and} \\ AA^* = I,\quad \mathrm{Det}(A) = 1\end{matrix}\right\}.$$

Here A^* denotes the conjugate transpose, and I denotes the identity matrix.

Since $S^3 = \{(z,w) \in \mathbb{C}\times\mathbb{C} \mid |z|^2+|w|^2 = 1\}$, it is manifest that $SU(2) \cong S^3$.

Now let $\mathbb{C}^+$ denote the one-point compactification of $\mathbb{C}$. Thus $\mathbb{C}^+ \cong S^2$. We may also describe S^2 as $\mathbb{CP}^1$ via homogeneous coordinates:

$$S^2 \cong \mathbb{CP}^1 = \{\langle z,w\rangle \mid (z,w) \in \mathbb{C}^2-\{0\}\}.$$

Here $\langle z,w\rangle$ denotes the equivalence class of (z,w) where $(z,w) \sim (\lambda z,\lambda w)$ for any nonzero complex number λ. $\mathbb{CP}^1$ is the set of complex lines through the origin in $\mathbb{C}^2$.

Let $\mathcal{L}$ denote the set of <u>linear fractional transformations of</u> C^+. Thus

$$\mathcal{L} = \{T : \mathbb{C}^+ \longrightarrow \mathbb{C}^+ \mid T(\alpha) = (z\alpha+w)/(-\bar{w}\alpha+\bar{z})\}.$$

Here $\begin{bmatrix} z & w \\ -\bar{w} & \bar{z}\end{bmatrix}$ is an element of SU(2). The linear fractional transformations derive from the natural action of $SU(2)$ on $\mathbb{C}P^1$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU(2),\ (x,y) \in \mathbb{C}P^1 \Longrightarrow$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}(x,y) = (ax+by,\ cx+dy)$. The latter is equal to $(ax+by)/(cx+dy)$ in $\mathbb{C}^+$ where $1/0 = \infty =$ the extra point in the one-point compactification.

Note that $\alpha \in \mathbb{C}^+$ corresponds to $(\alpha,1) \in \mathbb{C}P^1$ for $\alpha \neq \infty$, and that $\infty \in \mathbb{C}^+$ corresponds to $(1,0) \in \mathbb{C}P^1$.

Since A and $-A$ in $SU(2)$ give rise to the same element of $\mathcal{L}$, it is easy to see that the map $\pi : SU(2) \longrightarrow \mathcal{L}$ is 2 to 1 and onto. In fact, $\mathcal{L}$ is isomorphic with $SO(3)$. The isomorphism can be made explicit through a specific choice of stereographic projection $St : S^2 \longrightarrow \mathbb{C}^+$ where

$$S^2 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 \mid \|x\| = 1\}.$$

If $G \subset \mathcal{L}$ is a subgroup, let $\hat{G} \subset SU(2)$ denote $\pi^{-1}(G)$.

With these preliminaries completed, we now turn to the action of $SU(2)$ on the ring $R = \mathbb{C}[X,Y]$ of polynomials over $\mathbb{C}$ in two variables. It is through this action that we shall prove that $\Sigma(2,3,5) \cong S^3/\hat{G}$. $SU(2)$ acts on R as follows: Let $\sigma = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \in SU(2)$, and let $F(X,Y) \in R$. Then F^σ will denote the result of applying σ to F. F^σ is defined by the formula:

$$F^{\sigma}(X,Y) = F(aX+bY,-\bar{b}X+\bar{a}Y).$$

Given a finite subgroup $\hat{G} \subset SU(2)$, we seek $R^{\hat{G}} = \{F \in R \mid F^{\sigma} = F \ \forall\sigma \in \hat{G}\}$, the ring of polynomials invariant under the action of $\hat{G}$. We shall see that for the binary dodecahedral group $\hat{G}$ there are three generators for R: H_1, H_2, H_3 satisfying the relation $H_1^2+H_2^3+H_3^5 = 0$. Thus $R^{\hat{G}} = \mathbb{C}[Z_0,Z_1,Z_2]/(Z_0^2+Z_1^3+Z_2^5)$ and from this it will follow that $\Sigma(2,3,5) \cong S^3/\hat{G}$. The details follow as below.

First we look at the action of $SU(2)$ on R. Note that if F is a homogeneous polynomial, then so is F^{σ}. (The polynomial F is homogeneous if all single terms have the same total degree $d = i+j$.) Since any polynomial is a sum of homogeneous polynomials, it suffices to determine which homogeneous polynomials are invariant under $\hat{G}$.

Now observe that if $F \in R$ is a homogeneous polynomial, then $F = \prod_{i=1}^{k}(a_iX+b_iY)$ where $a_i, b_i \in \mathbb{C}$. Let this correspond to the following "polynomial" with "roots" in $\mathbb{C}^{+} = \mathbb{C}P^1$:

$$F \ \underline{\text{corresponds to}} \ f = \prod_{i=1}^{k}(z-\langle a_i,b_i\rangle).$$

Call f the <u>formal polynomial</u> corresponding to the homogeneous polynomial F. Let $\mathcal{F}$ denote the collection of these formal polynomials, and note that $\mathcal{F}$ is in one-to-one correspondence with the set

$$(\text{homogeneous polynomials in } R)/\sim$$

where $F \sim \lambda F$ for any nonzero complex number λ.

G acts on $\mathscr{F}$ by: Given $g \in G$, let $\sigma \in \hat{G} \subset SU(2)$ be an element projecting to g. Then $f^g = f^\sigma$ where f^σ is the formal polynomial corresponding to F^σ (F corresponds to f). More specifically: If $F = \Pi(a_i X + b_i Y)$ and $\sigma = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$, then

$$F^\sigma(X,Y) = \Pi(a_i(aX+bY) + b_i(-\bar{b}X+\bar{a}Y))$$

$$\therefore \quad F^\sigma(X,Y) = \Pi((aa_i - \bar{b}b_i)X + (a_i b + b_i \bar{a})Y)$$

Consequently, F^σ corresponds to f^σ where

$$f^\sigma = \Pi(z-(ba_i+a\bar{b}_i,\ -aa_i+\bar{b}b_i))$$

$$= \Pi\left[z-\begin{bmatrix} \bar{a} & -b \\ \bar{b} & a \end{bmatrix}(b_i,-a_i)\right]$$

$$\therefore \quad f^\sigma = \Pi(z-\sigma^{-1}(b_i,-a_i))$$

$$(f = \Pi(z-(b_i,-a_i)))$$

Conclusion: f^σ is obtained from f by transforming the "roots" of f via the inverse of the linear fractional transformation corresponding to σ.

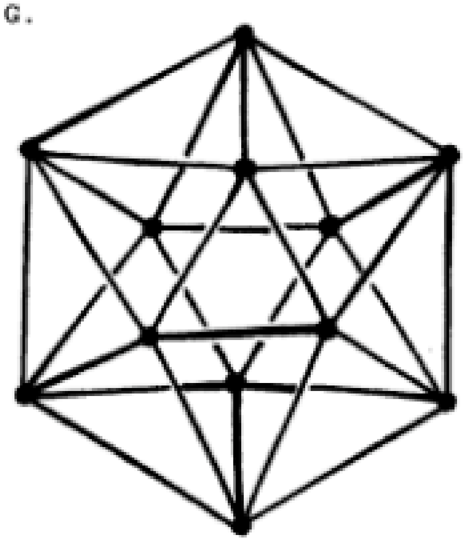
Here is a summary of what we have done so far:

1. If F is homogeneous and invariant under $\hat{G}$, then the corresponding formal polynomial $f \in \mathscr{F}$

is invariant under G.

2. If $f \in \mathcal{F}$ is invariant under G and $F \in R$ corresponds to f, then for any $\sigma \in \hat{G}$, $F^{\sigma} = \lambda F$ for some nonzero complex number λ (depending upon σ). Since we assume G finite, this implies that λ *is a root of unity whose order divides the order of* σ.

The Moral: In order to study $\hat{G}$-invariant polynomials in R, first study G-invariant formal polynomials in $\mathcal{F}$. *The latter correspond (via the roots) to collections of points in* S^2 *(or in* $\mathbb{C}^+$*) that are invariant under the action of* G.



#vertices = V = 12

#edges = E = 20

#faces = FA = 30

The Icosahedron

Let G be the symmetry group of the icosahedron. Then (view the figure above) the icosahedron has V = 12 vertices, $E = 20$ edges, and $FA = 30$ faces. Let $\mathcal{V}$ denote the set of vertices, $\mathcal{E}$ the set of midpoints of edges, and $\mathcal{F}$ the set of midpoints of faces of the icosahedron. Then $\mathcal{V}$, $\mathcal{E}$ and $\mathcal{F}$ are G-invariant subsets (of S^2 via radial outward projection from the rectilinear icosahedral form). Any other G invariant subset noncongruent to $\mathcal{V}$, $\mathcal{E}$, or $\mathcal{F}$ will be a full orbit of 60 points.

Let f_1 denote the polynomial in $\mathcal{Z}$ whose roots are the set $\mathcal{V}$, f_2 the polynomial with roots $\mathcal{E}$, and f_3 the polynomial with roots $\mathcal{F}$. Let F_1, F_2, F_3 be any three corresponding polynomials in R.

Claim: <u>F_1, F_2 and F_3 are each G invariant.</u>

Proof of Claim: We prove the claim for F_1 and leave the rest for the reader. Since the roots of f_1 are the twelve vertices, it is possible that $\sigma \in G$ may multiply factors of F by a 10^{th} root of unity λ. However, a look at the geometry of the situation shows that 10 roots must then be permuted among themselves by σ and two left fixed. This means that F is multiplied by λ^{10}, hence it is left invariant. Similar considerations hold for the other divisors of 120.

We now make the following

Claim: If F is any homogeneous polynomial in R^G of degree 60, then $F = K_1F_1^5 + K_2F_2^3$ for some constants $K_1, K_2 \in \mathbb{C}$.

Proof: Let V correspond to the (formal) polynomial f. Then we may choose $p \in S^2 - \mathcal{V} \cup \mathcal{F} \cup \mathcal{E}$, a point in the complement of the special invariant sets, and constants K_1, K_2 such that

$$f(p) = K_1f_1^5(p) + K_2f_2^3(p).$$

Hence (by invariance)

$$(f-K_1f_1^5 - K_2f_2^3)(\sigma p) = 0$$

for all <u>60</u> points $\{\sigma p | \sigma \in G\}$. Therefore $f = K_1f_1^5 + K_2f_2^3$. Thus $F = K_1F_1^5 + K_2F_2^3$ at least up to a constant. This is sufficient to prove the claim.

As a result of this claim, we may choose constants K_1, K_2, K_3 such that if $H_1 = K_1F_1$, $H_2 = K_2F_2$, $H_3 = K_3F_3$, then

$$\underline{H_1^5 + H_2^3 + H_3^2 = 0}$$

Furthermore, we have shown that R^G is generated by H_1 and H_2.

THEOREM 19.7. *Let* $\mathcal{Y} = \mathbb{C}[A,B,C]/(A^5+B^3+C^2)$ *be the quotient ring of the ring of polynomials in three variables (with complex coefficients) by the relation* $A^5+B^3+C^2$.

Define a map $\psi : \mathcal{Y} \longrightarrow R^G$ *by extending* $\psi(A) = H_1$, $\psi(B) = H_2$, $\psi(C) = H_3$. *Then* ψ *is an isomorphism of rings.*

Proof: ψ is onto, and we know that $\dim_{\mathbb{C}} R^G = 2$ (since there is no relation between H_1 and H_2). Therefore the ideal $(A^5+B^3+C^2)$ must in fact be the kernel of ψ. Otherwise the dimensions would not compare. ∎
(Compare with [KL].)

Now let $V = V(Z_1^5+Z_2^3+Z_3^2) \subset \mathbb{C}^3$ be the Brieskorn Variety $(5,3,2)$. And define $\phi : \mathbb{C}^2 \longrightarrow V$ by the map

$$\phi(\alpha) = (H_1(\alpha), H_2(\alpha), H_3(\alpha)).$$

PROPOSITION 19.8.

(1) ϕ *is surjective.*

(2) *If* $v \in V$, *then* $\phi^{-1}(v)$ *is an orbit under the action of* G *on* $\mathbb{C}^2$.

(3) $V \cong \mathbb{C}^3/G$.

(4) $\Sigma(5,3,2) \cong S^3/G$.

Proof: Using

$$\begin{array}{ccc} \psi : \mathbb{C}[A,B,C]/(A^5+B^3+C^2) & \longrightarrow & \mathbb{C}[X,Y]^G \\ \| & & \| \\ \mathcal{Y} & & R^G \end{array}$$

and the inclusion $R^G \overset{i}{\subset} R$, it suffices to prove that the induced map on ring spectra $\operatorname{Spec} R \longrightarrow \operatorname{Spec} \mathcal{Y}$ is surjective (for 1)). (See [SH] for algebraic geometery background.) However, it is easy to see that R is a finitely generated integral ring extension of $R^{\hat{G}}$. Hence $\mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(R^{\hat{G}})$ is finite to one. Therefore $2 = \dim_{\mathbb{C}} R = \dim_{\mathbb{C}} R^{\hat{G}}$. Since the dimension of $\mathcal{Y}$ is also 2, and $\psi : \mathcal{Y} \longrightarrow R^{\hat{G}}$ is an isomorphism, we see that

$$\mathrm{Spec}\ R \twoheadrightarrow \mathrm{Spec}\ R^{\hat{G}} \xrightarrow{\cong} \mathrm{Spec}\ \mathcal{Y}$$

so that $\mathrm{Spec}\ R \longrightarrow \mathrm{Spec}\ \mathcal{Y}$ is surjective. This translates via the Nullstellensatz [SH] to the statement that $\phi : \mathbb{C} \longrightarrow V$ is surjective.

For the second part it is necessary to show that $\phi(\alpha) = \phi(\alpha') \Longrightarrow \alpha' = \hat{g}\alpha$ for some $\hat{g} \in \hat{G}$. Since we may assume that α, α' are not zero, let $\bar{\alpha}, \bar{\alpha}'$ denote the corresponding elements of $S^2 = \mathbb{C}P^1$. Similarly, let g be the element of $SO(3)$ corresponding to $\hat{g}$. Then from $\phi : \mathbb{C}^2 \longrightarrow V$ we obtain $\bar{\Phi} : \mathbb{C}P^1 \longrightarrow (V-\{0\})/\mathbb{C}^*$ ($\mathbb{C}^*$ = the nonzero complex numbers). Now $\bar{\Phi}(\bar{\alpha}) = \bar{\Phi}(\bar{\alpha}')$ implies that all nonzero formal polynomials in $\mathbb{C}[z]^G$ take the same values on $\bar{\alpha}$ and $\bar{\alpha}'$. Let $f(z) = \prod_{g \in G} (z - g\bar{\alpha})$. Then $f(\bar{\alpha}) = 0$ and hence $f(\bar{\alpha}') = 0$. Thus $\bar{\alpha}' = g\bar{\alpha}$ for some $g \in G$. Transferring to G we conclude that $\lambda\hat{g}\alpha = \alpha'$ for some $\lambda \in \mathbb{C}^*$. Hence $\phi(\lambda\alpha) = \phi(\alpha)$ for some $\lambda \in \mathbb{C}^*$.

Thus we are reduced to showing that $\phi(\lambda\alpha) = \phi(\alpha)$ implies that $\lambda\alpha = \hat{h}\alpha$ for some $\hat{h} \in G$. Now $\phi(\lambda\alpha) = \phi(\alpha)$

means that

$$H_1(\alpha) = H_1(\lambda\alpha) = \lambda^{30}H_1(\alpha)$$

$$H_2(\alpha) = H_2(\lambda\alpha) = \lambda^{20}H_2(\alpha)$$

$$H_3(\alpha) = H_3(\lambda\alpha) = \lambda^{12}H_3(\alpha).$$

Consider the various cases:

Case 1: $H_1(\alpha)$, $H_2(\alpha)$ and $H_3(\alpha)$ all nonzero. Then $\lambda^{30} = \lambda^{20} = \lambda^{12} = 1$. Hence $\lambda^2 = 1$. Thus $\lambda = \pm 1$. Since -1 is an element of $\hat{G}$, we conclude that $\phi(\alpha) = (-\hat{g})\alpha$, as desired.

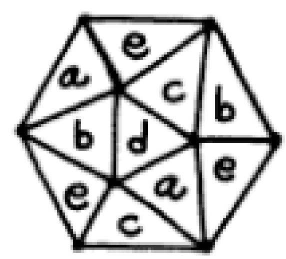
Case 2: If $H_1(\alpha) = 0$, while $H_2(\alpha)$ and $H_3(\alpha)$ are both nonzero, then $\lambda^{10} = \lambda^{12} = 1$, hence $\lambda^4 = 1$. However, $H_1(\alpha) = 0$ implies that α is a midpoint of an edge of the icosahedron. There is an order two symmetry g $(g^2 = 1)$ that rotates by 180° about an axis passing through the midpoints of opposite edges. Therefore $\hat{g}^2 = -1$ and $\hat{g}$ has order four. (That is, $\hat{g}$ exists.) Consequently, we can realize the fourth root of unity with $\hat{g} \in G$ as desired.

The other cases follow by similar geometry. This proves part (2). Part (3) follows from parts (1) and (2). Finally, to see part (3) use the same argument as in the proof of 19.3 to slide points onto the standard sphere. This completes the proof. ■

Note that it follows from our discussion that the dodecahedral space is obtained as the 2-fold branched covering M of S^3 with branch set a $(3,5)$ torus knot. It is a good exercise to show that $\pi_1(M) \cong \hat{G}$, and a more challenging exercise to show <u>directly</u> that $M_2(K_{3,5})$ and $S^3/\hat{G}$ are homeomorphic (even diffeomorphic) manifolds!

<u>Exercise</u>. To prove that $\hat{G}$ is perfect:

(i) Let G be the symmetry group of the icosahedron, $G \subset SO(3)$. Show that G is isomorphic to A_5, the group of even permutations on five letters. (HINT: Represent the five letters a,b,c,d,e as collections of four faces such that no two faces in any collection have edges or vertices in common.)



(ii) Show that A_5 is perfect. (Show that every element of A_5 is a product of 3-cycles, and that every 3-cycle is a commutator.)

(iii) Show that if S^3 = unit quaternions and if $u, v \in S^3$ such that $u^2 = v^2 = -1$ so that u and v are unit vectors in $S^3 \subset \mathbb{R}^3 = \{ai+bj+ck\}$ $(a,b,c \in \mathbb{R})$ with $u \perp v$ ($\perp$ denotes euclidean perpendicularity), then $uvu^{-1}v^{-1} = -1$. Thus -1 is a commutator in S^3. Show that -1 is a commutator in G. (*Hint:* This corresponds to finding two 180° rotations of the icosahedron having perpendicular axes.)

A few comments about the quaternions are germane to this last exercise. We regard

$$\mathbb{R}^4 = \{t+ai+bi+ck \mid t,a,b,c \in \mathbb{R}\}.$$

Quaternionic multiplication on $\mathbb{R}^4$ is generated by the identities $i^2 = j^2 = k^2 = ijk = -1$. (Plus associativity and distributivity.) The <u>pure quaternions</u> $\mathbb{R}^3 = \{ai+bj+ck\}$ constitute euclidean three-space, and the unit sphere $S^2 = \{ai+bj+ck = u \mid a^2+b^2+c^2 = 1\}$ has the property that $u \in S^3$ if and only if $u^2 = -1$. Thus any quaternion $g \in S^3$ can be written as $g = e^{u\theta} = \cos(\theta) + u\sin(\theta)$ where $0 \le \theta \le 2\pi$ and $u \in S^2$. We define $\pi : S^3 \longrightarrow SO(3)$ by the map $\pi(f)(v) = gv\overline{g}$ where $\overline{g} = e^{-u\theta}$. It is not hard to see that $\pi(g)$ <u>is a rotation about the axis</u> u <u>by the angle</u> 2θ. This is the quaternionic version of the double covering of $SO(3)$ by $SU(2)$.

Example 19.9. The Milnor Fibration: In [M3], Milnor proves the following theorem.

FIBRATION THEOREM. *Let* $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ *be a complex polynomial mapping with an isolated singularity at the origin. Let* $K = V(f) \cap S_{\epsilon}^{2n+1}$ *denote the link of the singularity. Then* $\phi : S_{\epsilon}^{2n+1} - K \longrightarrow S^1$ *is a smooth fibration, where the mapping* ϕ *is defined by* $\phi(z) = f(z)/|f(z)| = \arg(f(z))$.

Thus, links of isolated singularities have fibered complements. At this stage it is worth generalizing the term <u>knot</u> to denote <u>any codimension two smooth submanifold of a sphere</u>. Thus Milnor's theorem is that links of isolated singularities are fibered knots.

In particular, the fibration theorem states that the map

$$S^3 - \Sigma(a,b) \xrightarrow{\phi} S^1$$

$$\phi(z_0,z_1) = \arg(z_0^a+z_1^b)$$

gives the fiber structure for the (a,b) torus knot (or link if $\gcd(a,b) > 1$). Recall that we have explained the geometry of a fiber structure for $S^3-\Sigma(a,b)$ in Exercise 13.17.

In this example we see how the Fibration Theorem works in the case of the Brieskorn varieties. The reader is referred to Milnor's book for the full theorem.

Let $f(z) = z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n}$ and view this polynomial as a mapping $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$. Let $\mathbb{C}^*$ denote $\mathbb{C}-\{0\}$ and let $W = f^{-1}(\mathbb{C}^*) \subset \mathbb{C}^{n+1}$. Our first assertion is

LEMMA 19.10. $W \xrightarrow{f} \mathbb{C}^*$ *is a smooth fiber bundle.*

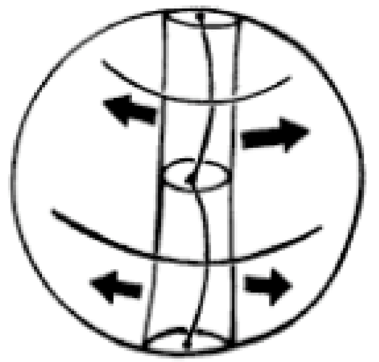
Proof: In order to prove this lemma we must examine how to locally trivialize the mapping. Recall that we have defined $\rho * z = (\rho^{1/a_0} z_0, \rho^{1/a_1} z_1, \cdots, \rho^{1/a_n} z_n)$ for positive real numbers ρ. Note that $f(\rho * z) = \rho f(z)$. For $0 \leq \theta \leq 2\pi$ we define $h_\theta(z)$ by the formula

$$h_\theta(z) = (\omega_0^\theta z_0, \omega_1^\theta z_1, \cdots, \omega_n^\theta z_n)$$

where $\omega_k = e^{i/a_k}$. We see that $\rho * h_\theta : f^{-1}(z) \to f^{-1}(\rho e^{i\theta} z)$. Thus these maps can be used to produce the local trivialization.

We now restrict the bundle of Lemma 19.10 to produce another bundle that is relevant to fibering the complement of $\Sigma(a_0, a_1, \cdots, a_n)$. Let $E_{\delta,\epsilon}$ denote the set defined below:

$$E_{\delta,\epsilon} = \{z \in \mathbb{C}^{n+1} \mid |f(z)| = \delta, \ |z| \leq \epsilon\}.$$



$E_{\delta,\epsilon} = f^{-1}(S^1_\delta) \cap D_\epsilon$

If we choose $0 < \delta << \epsilon$, then it is easy to see that
$\delta,\epsilon \longrightarrow S^1_\delta$, $z \longmapsto f(z)$ is a C^∞-fiber bundle. Since
sits inside the ball D^{2n+2}_ϵ of radius ϵ we see
its boundary is the boundary of a tubular neighborhood
$= \Sigma(a_0, a_1, \cdots, a_n) \subset S^{2n+1}_\epsilon$. In fact, $E_{\delta,\epsilon}$ deforms
ve a fiber structure on the complement of this tubular
borhood. The deformation involves expanding points of
via $z \longrightarrow \rho * z$ for ρ such that $|\rho * z| = \epsilon$. This
es a fiber structure $S^{2n+1}_\epsilon - N(\Sigma) \xrightarrow{\phi} S^1$ via
$= \arg(f(z))$. Here $\partial N(\Sigma) = E_{\delta,\epsilon} \cap S^{2n+1}_\epsilon$. This gives
of Milnor's theorem for this special case. He manage
ber the full complement by more careful analysis.

It is worth understanding the geometery of this fibra-
in more detail. We may take $\epsilon = 1$ and note that the
s of $\phi : S^{2n+1} - N(\Sigma) \longrightarrow S^1$ are deformation retracts

of $\{z \in \mathbb{C}^{n+1} \mid f(z) = 1\} = F$.

Now $F = \{z \in \mathbb{C}^{n+1} \mid z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n} = 1\}$, and F has as deformation retract a wedge of $(a_0-1)(a_1-1)\cdots(a_n-1)$ spheres of dimension n. This occurs as follows:

(1) $F \supset \{\bar{r} \in \mathbb{R}^{n+1} \mid r_0^{a_0} + r_1^{a_1} + \cdots + r_n^{a_n} = 1,\ r_i \geq 0\}$ $= \mathfrak{R}$.

(2) F is invariant under multiplication of the i^{th} coordinate by an $a_i{}^{th}$ root of unity.

(3) Thus $F \supset \{\bar{r} \in \mathfrak{R}\}\cdot(\Omega a_0 \times \Omega a_1 \times \cdots \times \Omega a_n)$ where Ωa_i = group of $a_i{}^{th}$ roots of unity. That is, $F \supset \{(r_0\omega_0, \cdots, r_n\omega_n) \mid \bar{r} \in \mathfrak{R},\ \omega_i \in \Omega a_i\} = \mathcal{Y}$.

It is good exercise to show

(a) this last set $\mathcal{Y}$ is a deformation retract of F.

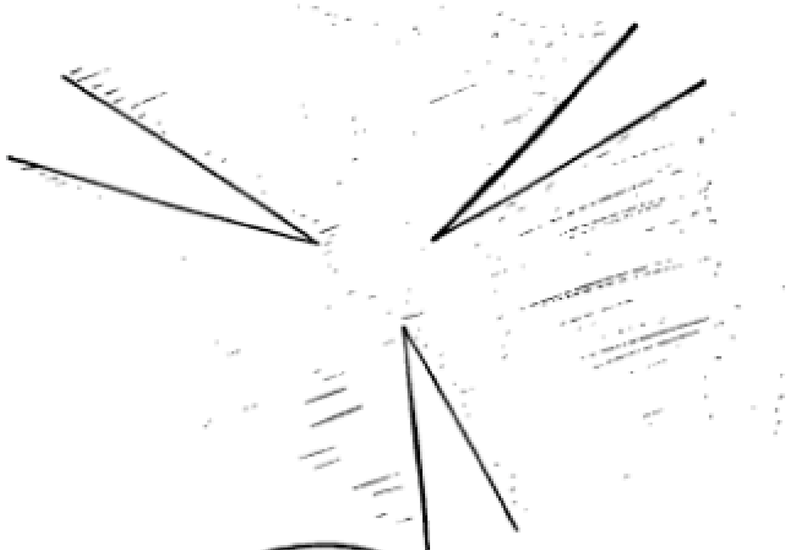
(b) $\mathcal{Y} \cong \Omega a_0 * \Omega a_1 * \cdots * \Omega a_n$ (where $*$ denotes join)

$$\cong \underbrace{(S^0 \vee \cdots \vee S^0)}_{(a_0-1)} \vee \cdots \vee \underbrace{(S^0 \vee \cdots \vee S^0)}_{(a_n-1)}$$

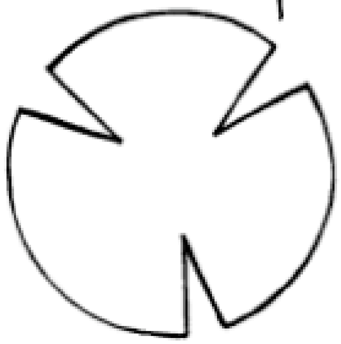
$$\cong \bigvee_{(a_0-1)(a_1-1)\cdots(a_n-1)} S^n .$$

See [BK] for more details.

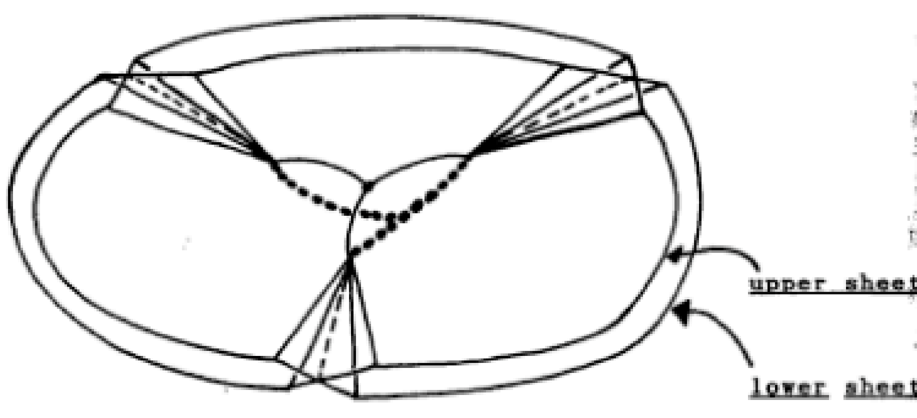
Here is a visualization for the two-variable case: The fiber is $F : z_0^{a_0} + z_1^{a_1} = 1$. Define $\pi : F \longrightarrow \mathbb{C}$ by $\pi(z_0, z_1) = z_1$. Then π is a branched covering of the complex plane branched along the $a_1{}^{st}$ roots of unity. We



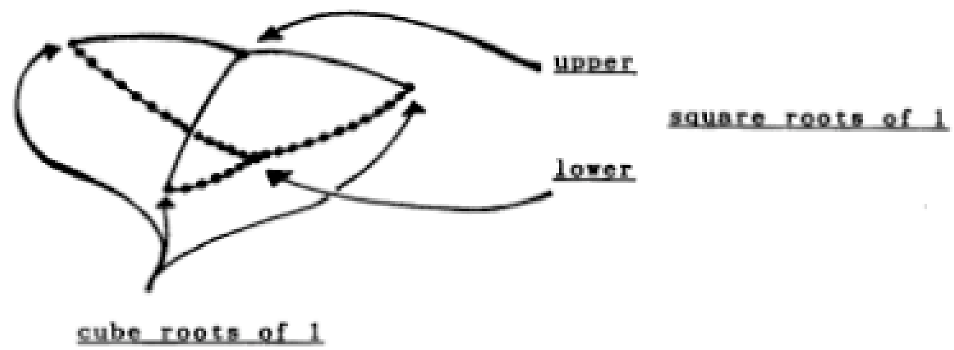
single slit plane



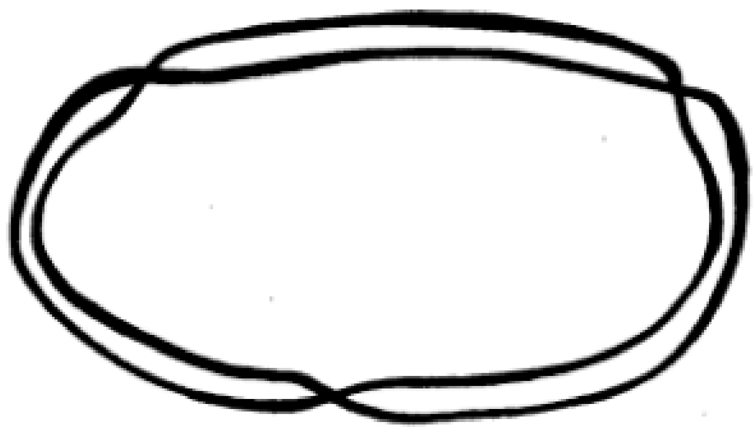
slit plane with a radial cut-off



can see F by creating a cut-and-paste picture of the branched covering. This is obtained by slitting the complex plane along rays emanating from each root of unity. For example, take $F : z_0^2 + z_1^3 = 1$. This surface construction is illustrated in the figure on p. 394. Note that $\Omega_2 * \Omega_3$ appears as the form



Note also how a projection of the (2,3) torus knot appears in the boundary of this representation:



These same patterns hold true for the more general case of

$z_0^{a_0} + z_1^{a_1} = 1$.

If we replace $z_0^{a_0} + z_1^{a_1} = 1$ by $z_0^{a_0} + z_1^{a_1} = \delta$ then, as δ approaches 0, the $\Omega a_0 * \Omega a_1$ part shrinks to a point, until at 0, F has degenerated to the cone on the (a_0, a_1) knot.

The structure of the fibration $\phi : S^{2n-1} - N(\Sigma) \longrightarrow S^1$ is given by the monodromy $h : F \longrightarrow F$ where F is the fiber. It is easy to see from our discussion that this monodromy consists in multiplying each coordinate by the corresponding root of unity. Thus

$h(z_0, z_1, \cdots, z_n) = (\omega_0 z_0, \cdots, \omega_n z_n)$ where $\omega_i = e^{2\pi i/a_i}$.

This means that $S^{2n+1} - N(\Sigma)$ is diffeomorphic to $F \times I/(h(x),0) \sim (x,1)$. From this description it is possible to compute many things—including the Alexander polynomial of $\Sigma \subset S^{2n+1}$.

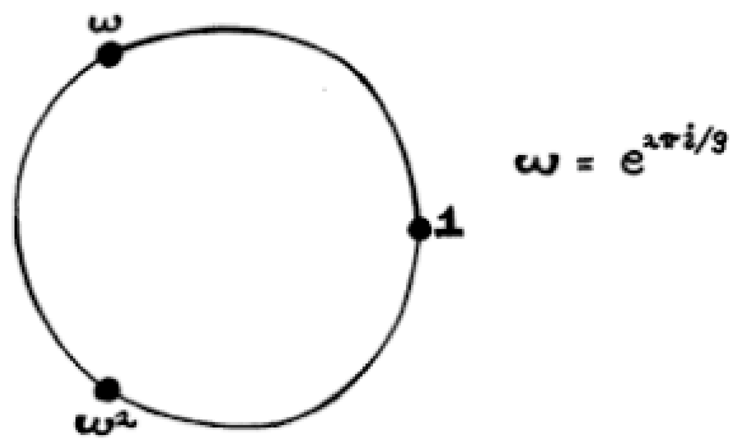
Exercise. Show that if $K \subset S^3$ is a fibered knot with fiber F and monodromy $h : F \longrightarrow F$, then $\Delta_K(t) \doteq \operatorname{Det}(H - tI)$ where H = the matrix of $h_* : H_1(F) \longrightarrow H_1(F)$ for some basis of this homology group. Use this description and our discussion of Brieskorn manifolds to recompute the Alexander polynomials of torus knots and links.

Example 19.11: The Empty Knot. The simplest Brieskorn polynomial is $f(z_0) = z_0^{a_0}$. Here $f : S^1 \longrightarrow S^1$, $z_0 \longmapsto z_0^{a_0}$ is the Milnor fibration. The "knot" $\Sigma(a_0)$ is the empty set! Nevertheless, this knot has a fiber, and it has a Seifert pairing with respect to this fiber. We calculate the pairing.

Let $F(a_0)$ be the fiber of the Milnor fibration for this empty knot. Then

$$F(a_0) = f^{-1}(1)$$
$$= \{\omega \in S^1 \mid \omega^{a_0} = 1\}$$

$\therefore$ $F(a_0) = \Omega a_0$, the set of $a_0{}^{th}$ roots of unity.



$F(3) = \Omega_3 = \{1, \omega, \omega^2\}$

The Milnor Fiber for the Empty Knot of Degree Three

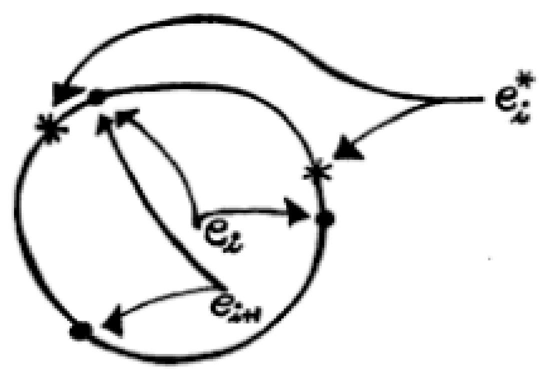
By letting $\omega = e^{2\pi i/a_0}$ denote an $a_0{}^{th}$ root of unity, we have that $\Omega a_0 = \{1, \omega, \omega^2, \cdots, \omega^{a_0 - 1}\}$ represents the "spanning surface" for the empty knot of degree a_0. The Seifert pairing is defined in reduced homology:

$$\theta a_0 : \tilde{H}_0(\Omega a_0) \times \tilde{H}_0(\Omega a_0) \longrightarrow Z.$$

We may take the push-off in the normal direction to Ωa_0 to be generated by a small counter-clockwise rotation of S^1. Let x^* denote the result of so pushing a chain x.

Note that the generators for $\tilde{H}_0(\Omega a_0)$ are $(1-\omega)$, $(\omega-\omega^2)$, $(\omega^2-\omega^3),\cdots,(\omega^{a_0-2}-\omega^{a_0-1})$. These form a basis. If we let $e_0 = (1-\omega)$, $e_1 = (\omega-\omega^2)$ and generally $e_k = (\omega^k-\omega^{k+1})$ where k is taken modulo a_0, then $\tilde{H}_1(\Omega a_0)$ has basis $\{e_0, e_1, \cdots, e_{a_0-2}\}$. Also $e_k = \omega^k e_0$ in the sense of the multiplicative action of the roots of unity on $\tilde{H}_0$.

The Seifert pairing is defined by the formula $\theta(a,b) = \ell k(a^*,b)$. Here we see that



$$\begin{cases} \theta(e_i, e_i) = \ell k(e_i^*, e_i) = +1 \\ \theta(e_i, e_{i+1}) = \ell k(e_i^*, e_{i+1}) = -1 \end{cases}$$

and otherwise $\theta(e_i, e_j) = 0.$

This means that <u>for the empty knot of degree a the Seifert pairing has matrix</u>

$$A_a = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & \ddots & 1 \\ & & & & 1 \end{bmatrix}$$

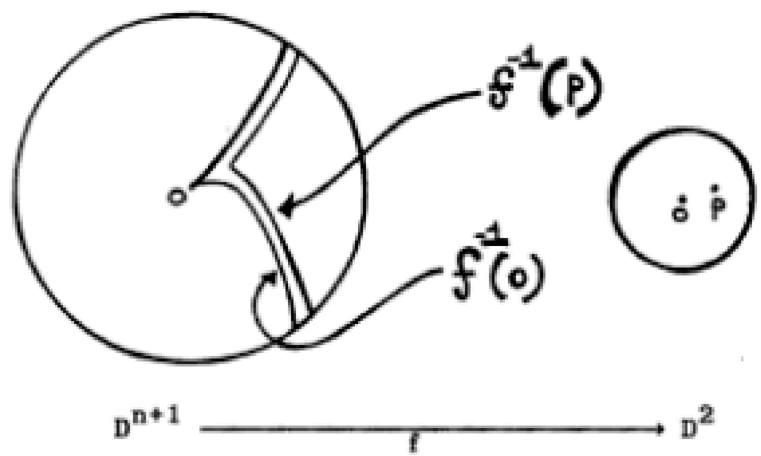
<u>with respect to the basis</u> $\{e_0, e_1, \cdots, e_{a-2}\} = \mathcal{B}$.

We have already encountered this matrix in Chapter 12 where the intersection on the a-fold cyclic branched covering of D^4 along a pushed-in Seifert surface for a knot K has the form $\theta \otimes A_a + \theta^T \otimes A_a^T$ (T denotes transpose), and θ is the given Seifert pairing for $K \subset S^3$. This connection with the formalism of the empty knot is not spurious. It is in fact, the first instance of a unified arena of constructions which happen both in studying singularities and in studying knot theory. Most of the rest of this chapter will be devoted to an outline of these constructions, which we have elsewhere called the <u>cyclic suspension</u> ([N]) and the <u>knot product</u> ([KN],[K6]).

Example 19.12: The Cyclic Suspension. In this example we explain how the empty knot of degree a is related to the a-fold cyclic branched covering. Note that any knot (codimension two embedding in a sphere) can be regarded as the link of a singularity that is not necessarily algebraic. First view the figure on p. 371. Here we have indicated a singularity associated with a knot $K \subset S^3$ obtained by forming the cone $CK \subset D^4$. The cone is illustrated as a projection into three-dimensional space.

If we really want to think of $CK \subset D^4$ as analogous to an algebraic singularity, then there should be a function $f : D^4 \longrightarrow D^2$ such that $f^{-1}(0) = CK$. This is analogous to a mapping from $\mathbb{C}^n \longrightarrow \mathbb{C}$ in the complex case. Such a mapping can always be obtained. The construction is as follows: Let $K \subset S^n$ be a smooth codimension two submanifold with trivial normal bundle $N(K) \cong K \times D^2 \longrightarrow S^n$. Then an obstruction theory argument shows that there is a mapping $\alpha : S^n\text{-}\mathrm{Int}N(K) \longrightarrow S^1$ that is smooth and that restricts to $pr : K \times S^1 \longrightarrow S^1$, the projection on the boundary of the tubular neighborhood. The mapping α represents a generator of $H^1(S^n-K)$ when K is connected and an oriented sum of generators in the case of a link. In the case of a knot in the three-sphere, this mapping may be visualized by first constructing a spanning surface for K, then splitting S^3 along the spanning surface, then writing a Morse function to $[0,1]$ from the split manifold. In any case, α may be chosen smooth, so that $\alpha^{-1}(p)$ is a smooth spanning surface for K, for a dense set of $p \in S^1$ (via the Morse lemma [M]). When K is a fibered knot, α is a smooth fibration. By its construction, α may be extended to $\bar{\alpha} : S^n \longrightarrow D^2$ by taking its union with the projection $K \times D^2 \longrightarrow D^2$. Finally, let $f : D^{n+1} \longrightarrow D^2$ be the <u>cone on</u> $\bar{\alpha}$. That is, $f(ru) = r\bar{\alpha}(u)$ where $u \in S^n$ and $0 \le r \le 1$. We shall call $f : D^{n+1} \longrightarrow D^2$ a <u>generator</u> of the knot $K \subset S^n$. See [KN] for discussion of the details of construction and uniqueness of generators.

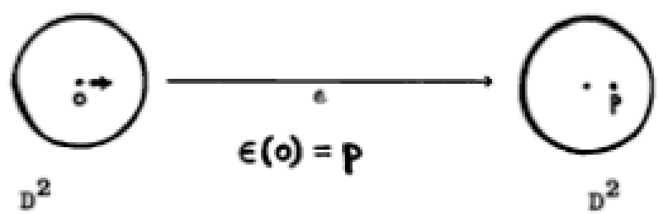
Note that it follows from the discussion that a generator $f : D^{n+1} \longrightarrow D^2$ for a fibered knot, <u>itself</u> gives rise to a fibration $f : D^{n+1} - CK \longrightarrow D^2 - \{0\}$. This is exactly analogous to the fibration $\mathbb{C}^n - V(f) \longrightarrow \mathbb{C}^*$ discussed in Lemma 19.10 for the Brieskorn polynomials. In general, if $f : D^{n+1} \longrightarrow D^2$ is a generator then $f^{-1}(p)$ will generically be a codimension two submanifold of D^{n+1} with boundary ambient isotopic to K.



We now observe how to use a generator $f : D^2 \longrightarrow D^2$ to construct cyclic branched covers:

(i). Let $\underline{a} : D^2 \longrightarrow D^2$ denote the mapping $\underline{a}(z) = z^a$ for a fixed positive integer a. Note that $\underline{a}$ is a generator of the empty knot of degree a. [In this context, the cone over the empty set is a single point.]

(ii) Given any generator $f : D^{n+1} \longrightarrow D^2$ let f^* denote a slight displacement of f that is obtained by composing f with a diffeomorphism $\epsilon : D^2 \longrightarrow D^2$ that moves 0 to a point p with $f^{-1}(p)$ a smooth submanifold. We require that $\epsilon|\partial D^2$ is the identity.



Thus $f^{*-1}(0) \subset D^{n+1}$ is a smooth spanning manifold for $K \subset \partial D^{n+1}$.

(iii) Form the <u>pull-back</u> <u>diagram</u>

$$\begin{array}{ccc} W & \longrightarrow & D^2 \\ \big\downarrow & & \big\downarrow \underline{a}. \\ D^{2n+1} & \xrightarrow{f^*} & D^2 \end{array}$$

Then $W = \{(x,z) \in D^{n+1} \times D^2 \mid f^*(x) = z^a\}$ is (manifestly) the *a-fold cyclic branched covering of* D^{n+1} *branched along* $f^{*-1}(0) - F$.

This topological construction for the branched covering makes W the precise analog of a variety $\epsilon = z^a - f(x)$ (where $f(x)+\epsilon$ corresponds to $f^*(x)$). Again see [KN] for the precise comparison theorems. By creating branched coverings in this fashion, we get much more than just a relation with the case of algebraic varieties. We also get the embedding $W \subset D^{n+1} \times D^2$ and hence an embedding $\partial W \subset \partial(D^{n+1} \times D^2) \cong S^{n+2}$. Now ∂W *is the a-fold cyclic branched cover of* S^n *with branch set* K. Let $M_a(K) \longrightarrow S^n$ denote this branched cover. Then we have proved the

THEOREM [KN]. *Let* $K \subset S^n$ *be a knot (codimension two embedding) and let* $M_a(K) \longrightarrow S^n$ *denote the a-fold cyclic covering of* S^n *with branch set* K. *Then there is a natural embedding of* $M_a(K)$ *in* S^{n+2}. *Thus we have a tower of embeddings and branched coverings:*

$$\begin{array}{ccccc}
 & & & & \cdots \\
 & & & & \downarrow \\
 & & M_{a'}(M_a(K)) & \longrightarrow & S^{n+4} \\
 & & \downarrow & & \\
M_a(K) & \longrightarrow & S^{n+2} & & \\
\downarrow & & & & \\
K \subset S^n & & & &
\end{array}$$

These embeddings are the topological analogs of the embeddings already discussed for algebraic varieties. That is, the embedding $\mathrm{Link}(f(x)+z^{a}) \subset S^{n+2}$ is obtained as ambient isotopic to $M_{\underline{a}}(\mathrm{Link}(f(x)) \subset S^{n+2}$ where $\mathrm{Link}(f(x)) \subset S^{n}$.

In the particular case of the Brieskorn manifolds, everything actually begins with the empty knots! For consider the diagram

$$\begin{array}{ccc} W & \longrightarrow & D^2 \\ \downarrow & & \downarrow \underline{a} \\ D^2 & \xrightarrow[\underline{b}^*]{} & D^2 \end{array}.$$

This describes the construction of a surface $W \subset D^2 \times D^2$ whose boundary $\partial W \subset S^3$ is the (a,b) torus knot (link).

If we let $K \otimes [a] \subset S^{n+2}$ denote the knot obtained from $K \subset S^n$ by embedding the branched covering along K into S^{n+2}, then we have

$$[a_0] \otimes [a_1] \subset S^3 \quad \text{torus knot (link)}$$

and, generally, $[a_0] \otimes [a_1] \otimes \cdots \otimes [a_n] \subset S^{2n+1}$ is ambient isotopic to the Brieskorn manifold

$$\Sigma(a_0, a_1, \cdots, a_n) \subset S^{2n+1}.$$

This result is not just formal. For by analyzing the construction of the cyclic suspension $K \otimes [a] \subset S^{n+2}$ more closely one can conclude information about the Seifert pairing for a surface $F_a \subset S^{n+2}$ with $\partial F_a = K \otimes [a]$. The result is

THEOREM [KN]. *Let* $F \subset S^n$ *be a spanning manifold for* $K \subset S^n$ *with Seifert pairing* $\theta : H_*(F) \times H_*(F) \longrightarrow Z$. *Then* $K \otimes [a] \subset S^{n+2}$ *has a spanning manifold* F_a *with Seifert pairing* $\theta \otimes \Lambda_a$ *where* Λ_a *is the Seifert pairing for the empty knot of degree* a.

We indicate briefly in the next example how this result is proved. Please note that this explains how Λ_a appears in the formula for the intersection form on the branched covering $N_a(F)$ of Chapter 12. In these terms

$$\begin{array}{ccc} N_a(F) & \longrightarrow & D^2 \\ \downarrow & & \downarrow \underline{a} \\ D^4 & \xrightarrow{\ f\ } & D^2 \end{array}$$

where f is a generator for $K \subset S^3$. We have $M_a(K) = \partial N_a(F) \subset \partial(D^4 \times D^2) = S^5$ with Seifert pairing $\theta \otimes \Lambda_a$ for θ a Seifert pairing for $K \subset S^3$. We did not yet indicate that $N_a(F)$ itself embeds in S^5 with this Seifert pairing $\theta \otimes \Lambda_a$. Nevertheless, this is the case and the proof is a generalization of our argument that pushed Brieskorn fibers $\Sigma z_i^{a_i} = \epsilon$ to Milnor fibers in the sphere. The upshot is an embedding $N_a(F) \subset S^5$ with Seifert form $\theta \otimes \Lambda_a$. Hence it has intersection form $\theta \otimes \Lambda_a + (\theta \otimes \Lambda_a)^T$ (since the intersection form is the sum of the Seifert form and its transpose in this dimension). This intersection form was the result of direct calculation in Chapter 12.

As a specific example, consider the dodecahedral space $\Sigma(2,3,5)$. According to the above results, $\Sigma(2,3,5)$ bounds a manifold $N(2,3,5) \subset S^5$ and $N(2,3,5)$ has intersection form $\pm(\theta+\theta^T)$ where θ is the Seifert form for a $(3,5)$ torus knot. Reference to the table after Exercise 12.7 then shows that $\text{Sign}\, N(2,3,5) = \pm 8$. Thus these results about Seifert forms and embeddings lead to various signature calculations. An exactly analogous calculation shows that $\Sigma(3,5,2,2,2) = \Sigma$ also bounds a manifold of signature ± 8. This leads [M3], [BK], to the identification of $\Sigma(3,5,2,2,2)$ as an exotic sphere. Thus, the Milnor sphere is three cyclic suspensions of a $(3,5)$ torus knot. It is obtained by classical branched covering constructions.

Example 19.13: Products of Knots. The cyclic suspension generalizes to a product construction that corresponds to the link of the sum of two singularities. This is obtained by replacing (in the cyclic suspension) the empty knot generator $\underline{a} : D^2 \longrightarrow D^2$ by any generator $\lambda : D^{m+1} \longrightarrow D^2$ for a fibered knot $L \subset S^m$. The pull-back diagram then becomes:

$$\begin{array}{ccc} W & \longrightarrow & D^{m+1} \\ \downarrow & & \downarrow \lambda \\ D^{n+1} & \xrightarrow[f^*]{} & D^2 \end{array}$$

We define $K \otimes L = \partial W \subset \partial(D^{n+1} \times D^{m+1}) \cong S^{n+m+1}$. Thus, given a knot $K \subset S^n$ and a fibered knot $L \subset S^m$ (fibered is needed to make the construction well-defined) there is a new composite or product knot $K \otimes L \subset S^{n+m+1}$.

The construction is built to be a straightened version of the link of the sum of two singularities. In fact it is true that if $f(x)$ and $g(y)$ are polynomial singularities with separate sets of variables x and y, then

$$\mathrm{Link}(f+g) \cong \mathrm{Link}(f) \otimes \mathrm{Link}(g)$$

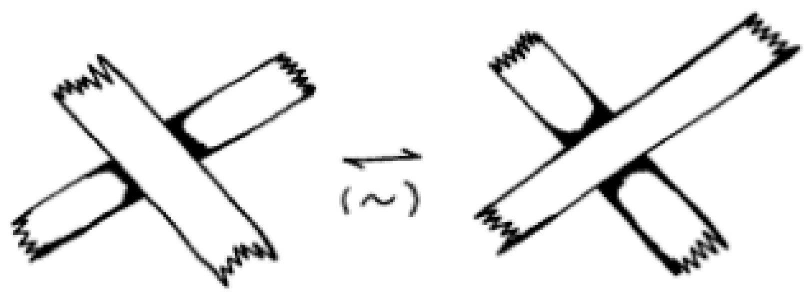
where $\cong$ denotes ambient isotopy of the corresponding knots.

In terms of the construction, the generators $f : D^{n+1} \longrightarrow D^2$ and $\lambda : D^{m+1} \longrightarrow D^2$ give rise to a new generator $\phi : D^{n+1} \times D^{m+1} \longrightarrow D^2$ that is essentially the difference map $f(x)-\lambda(y)$. One can show that a nonsingular fiber of ϕ has the homotopy type of the join of non-singular fibers of f and λ individually. Furthermore, this join structure is preserved via a deformation of the large nonsingular fiber into the join sphere $S^{n+m+1} = \partial(D^{n+1} \times D^{m+1})$. From this one shows that $K \otimes L$ bounds a manifold $\mathcal{F} \subset S^{n+m+1}$ with Seifert form the <u>tensor product</u> $\theta_K \otimes \theta_L$ of respective Seifert forms for K and L individually.

The product construction has useful corollaries. We conclude by mentioning just one. Let A :  $\subset S^3$ denote the Hopf Link. Then $K \subset S^n \longrightarrow K \otimes A \subset S^{n+4}$ takes spherical knots to spherical knots, and it generates the isomorphism of Levine knot concordance groups $C_n \xrightarrow{\cong} C_{n+4}$. See [KN], [L1].

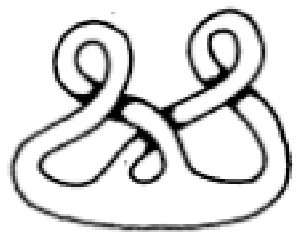
A good deal more can be said about the knot product. Other geometric interpretations are available, and there are connections with spinning and twist-spinning as well.

Example 19.14: The 8-fold Periodicity of $\Sigma(k,2,2,2,\cdots,2)$ *(an odd number of 2's).* Let Σ_k^{4n+1} denote the Brieskorn manifold $\Sigma(k,2,2,2,\cdots,2)$ with $(2n+1)$ 2's (other than k). Σ_k^{4n+1} bounds a handle-body whose structure is analogous (see [E], [K7] for details) to the spanning surface for a $(2,k)$ torus link. Furthermore, the operation of band exchange



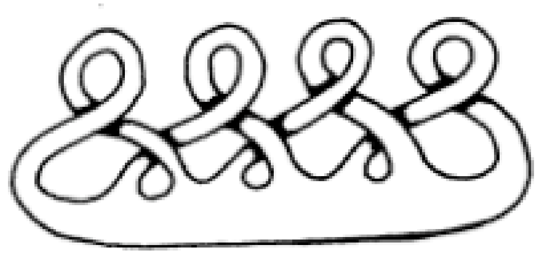
results in a diffeomorphism (via handle-sliding) of this handle-body and hence a diffeomorphism of its boundary. As a result, we obtain an 8-fold periodicity in the list of manifolds Σ_k^{4n+1}, $k = 2,3,4,\cdots$. The periodicity follows from a corresponding periodicity in the band-exchange classes of the corresponding spanning surfaces. This is a good example of how low dimensional knot theory can influence the properties of high dimensional manifolds. The (2,k) torus links have spanning surfaces in the pattern:

$(2,2) = K_2$



$(2,3) = K_3$

$(2,4) = K_4$

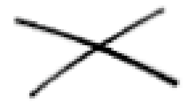


$(2,5) = K_5$

Topological script will be used to notate the periodicity (See Chapter VI, Sections 6.3 and 6.13 of these notes.)

K_2 K_3 K_4

are the corresponding script representations.

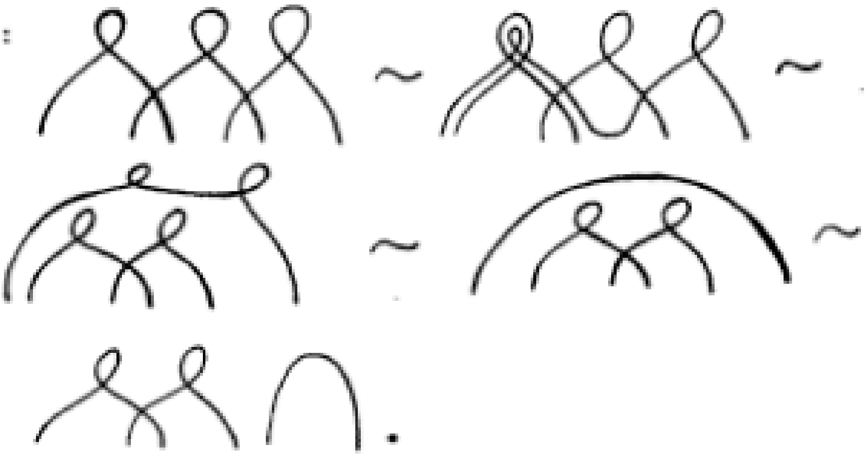
Now we note that we use  for either

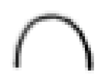
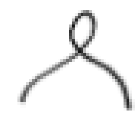
 or  since these are exchange equivalent. Also we have equivalences

and

since these can be accomplished by ambient isotopy and exchange on the corresponding bands. Thus

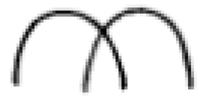
K_4:



Let  be denoted by A_0 and  by $A_1 = K_2$.

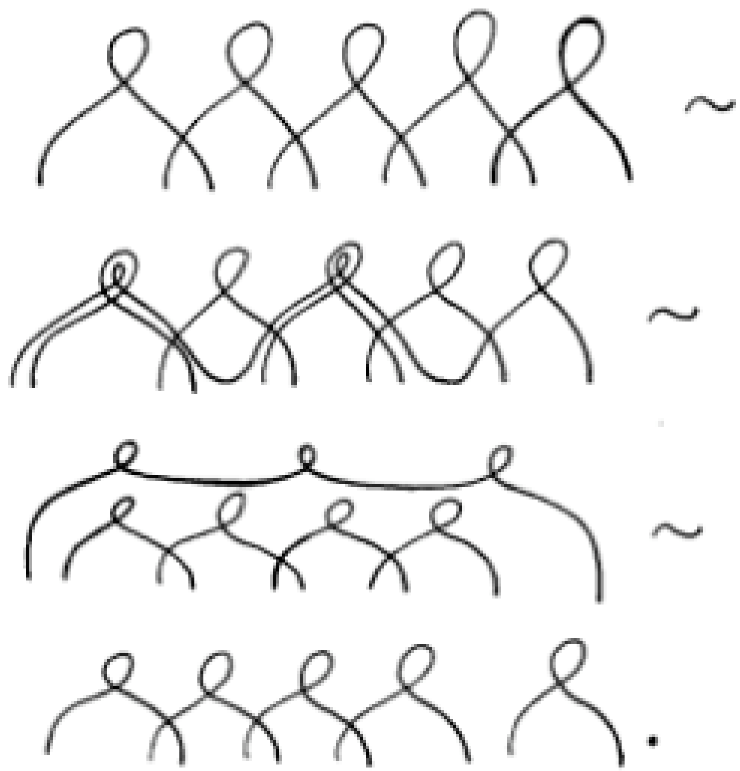
Thus $\underline{K_4 \sim K_3 \# A_0}$.

K_5:

Let H_0 denote  . Since H_0 represents the trivial knot: $X \# H_0 \sim X$.

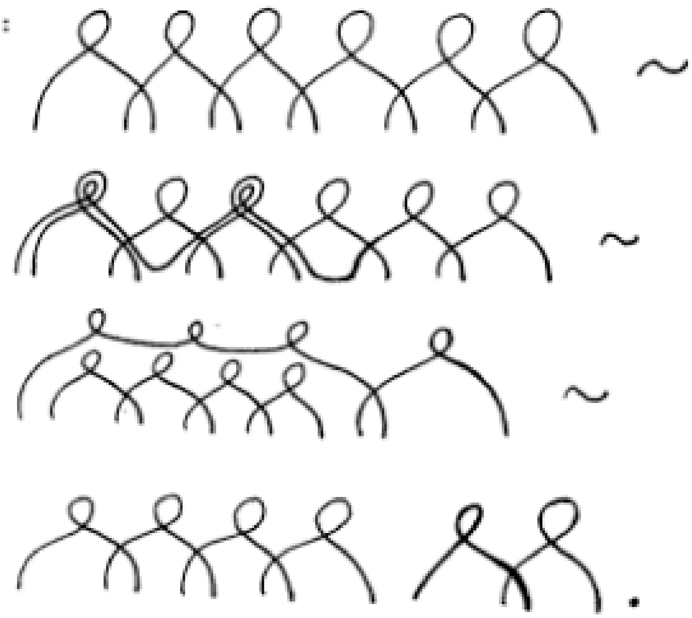
Thus $\underline{K_5 \sim K_3 \# H_0 \sim K_3}$.

K_6:



$\therefore$ $\underline{K_6 \sim K_5 \# A_1}$.

K_7:



Thus $\underline{K_7 \sim K_5 \# K_3}$.

Continuing in this pattern, we find that:

$K_2 \sim A_1$	$K_3 \sim K_3$
$K_4 \sim K_3 \# A_0$	$K_5 \sim K_3 \# H_0 \sim K_3$
$K_6 \sim K_5 \# A_1$	$K_7 \sim K_5 \# K_3$
$K_8 \sim K_7 \# A_0$	$K_9 \sim K_7 \# H_0 \sim K_7$
$K_{10} \sim K_9 \# A_1$	

Thus $K_{10} \sim K_7 \# A_1 \sim K_5 \# K_3 \# A_1 \sim K_3 \# K_3 \# A_1$. But we know (Section 6.3 of Chapter 6) that $K_3 \# K_3 \sim H_0 \# H_0$ $\sim$ (blank). Hence $K_{10} \sim A_1 \sim K_2$.

This begins the <u>8-fold</u> <u>periodicity</u>: $K_{\ell+8} \sim K_\ell$. The basic list is

$$K_2 \sim K_2$$
$$K_3 \sim K_3$$
$$K_4 \sim K_3 \# A_0$$
$$K_5 \sim K_5$$
$$K_6 \sim K_3 \# A_1 \sim H_0 \# A_1 \sim A_1$$
$$K_7 \sim K_3 \# K_3 \sim H_0$$
$$K_8 \sim A_0$$
$$K_9 \sim H_0$$

with $K_{\ell+8} \sim K_\ell$.

To go into the precise details of the relationship between the corresponding manifolds and these links would take us too far afield. However, the list of manifolds is as follows:

K_2 — T^{4n+1} = tangent sphere bundle to S^{2n+1}
K_3 — Σ^{4n+1} = Kervaire sphere
K_4 — $\Sigma^{4n+1} \# S^{2n+1} \times S^{2n}$
K_5 — Σ^{4n+1}
K_6 — T^{4n+1}
K_7 — S^{4n+1}
K_8 — $S^{2n+1} \times S^{2n}$
K_9 — S^{4n+1} (see [K7], [DK]).

The Kervaire sphere is exotic in many dimensions (for example, Σ^9 is exotic). Under these circumstances the exoticity is detected by the Arf invariant, which in this context corresponds to the Arf invariant of the corresponding $(2,k)$ torus knot. The connected sum of two Kervaire spheres is diffeomorphic to the standard sphere S^{4n+1}. The handle-sliding geometry of this diffeomorphism is depicted via topological script in the equivalence $K_3 \# K_3 \sim H_0 \# H_0$.

$$\Sigma^9 \# \Sigma^9 \cong S^9$$

Epilogue: This final chapter has been a sketch of relationships between knot theory and manifolds in geometric topology. We have hardly touched on the beginnings of many topics such as the work of Thurston, or the Kirby Calculus and its application to 4-manifolds. The subject of knots and algebraic varieties could expand to another book. Therefore it is time for this writing to stop. I hope these pages have given the reader a taste for the surprising variety, fascination and mathematical pleasure that is the theory of knots.

> "Existence, by nothing bred,
> Breeds everything.
> Parent of the universe,
> It smooths rough edges,
> Unties hard knots,
> Tempers the sharp sun,
> Lays blowing dust,
> Its image in the well spring never fails.
> But how was it conceived? - this
> Image
> Of no other sire."

[From *The Way of Life* by Lao Tsu, translated by Witter Bynner; Capricorn Books, 1944.]