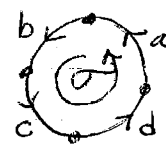
Notes on Khovanov-Homology

LHK, Spring 2007

I. Remarks on Standard Homology

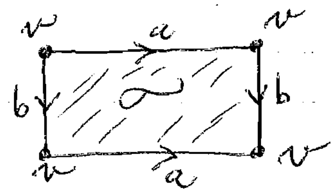
1° Cell Complexes



$\partial\sigma = a + b + c + d$

$\partial(y \to x) = x - y$

2° ex: Torus T



$\partial\sigma = a + b - a - b = 0$

$\partial a = \partial b = 0$

$C_0 = \langle\langle v \rangle\rangle,\ C_1 = \langle\langle a, b \rangle\rangle,\ C_2 = \langle\langle \sigma \rangle\rangle$

($\langle\langle x_1, \ldots, x_n \rangle\rangle$ = free $\mathbb{Z}$-module with basis $\{x_1, \ldots, x_n\}$)

$$C_2 \xrightarrow[\partial]{} C_1 \xrightarrow[\partial]{} C_0 \xrightarrow{\varepsilon = "\partial"} \mathbb{Z} = C_{-1}$$

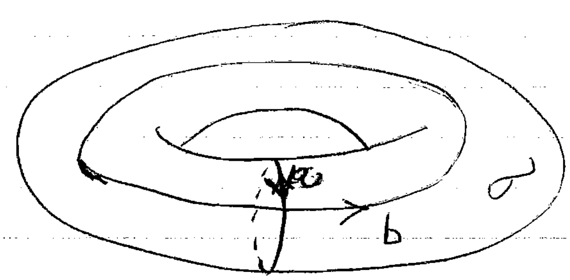
$(\varepsilon(v) = 0)$

$$\Rightarrow H_n(C) = H_n(T) = \frac{\mathrm{Ker}(\partial : C_n \to C_{n-1})}{\mathrm{Image}(\partial : C_{n+1} \to C_n)}.$$

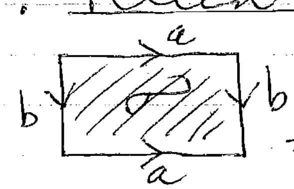
$H_2(T) \cong \mathbb{Z}$

$H_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$

$H_0(T) \cong \mathbb{Z}$



ex: Klein Bottle: Kl

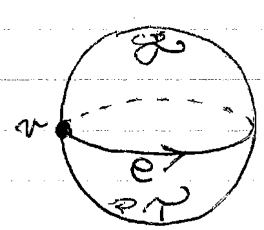


$\partial\sigma = 2b,\ \partial a = \partial b = 0$

$\Rightarrow H_2(Kl) \cong \{0\},\ H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$

$H_0(Kl) \cong \mathbb{Z}$

3° S^2 – *the two-sphere*



$$\partial\sigma = e \quad , \quad \partial e = 0$$
$$\partial\tau = -e$$

$$C_2 = \langle\langle \sigma, \tau \rangle\rangle \xrightarrow{\partial} C_1 = \langle\langle e \rangle\rangle \xrightarrow{\partial} C_0 = \langle\langle v \rangle\rangle,$$

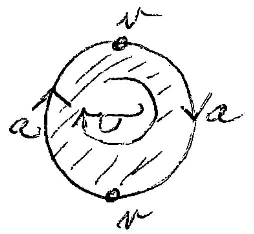
$$\partial(r\sigma + s\tau) = re - se = (r-s)e$$

$$\Rightarrow \partial(r\sigma + s\tau) = 0 \iff r - s = 0$$

so $Z_2(C) = \{\operatorname{Ker}\partial : C_2 \to C_1\} = \langle\langle n(\sigma+\tau)\rangle\rangle$

$$\Rightarrow \quad H_2(C) \cong \mathbb{Z}$$
$$H_1(C) \cong \{0\}$$
$$H_0(C) \cong \mathbb{Z}.$$

4° $\mathbb{P}^2$ – *the projective plane*



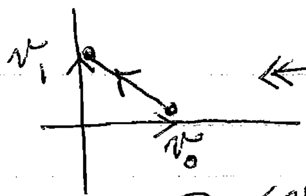
$$\partial\sigma = 2a$$
$$\partial a = 0$$
$$\Rightarrow \quad H_2(\mathbb{P}^2) \cong \{0\}$$
$$H_1(\mathbb{P}^2) \cong \mathbb{Z}/2\mathbb{Z}$$
$$H_0(\mathbb{P}^2) \cong \mathbb{Z}.$$

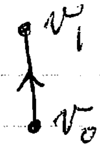
5° *Exercise.* Define cell decomposition for the n-spheres $S^n = \{\vec{v} \in \mathbb{R}^{n+1} \mid \|\vec{v}\| = 1\}$ and show $H_K(S^n) \cong \begin{cases} \mathbb{Z} & n = K \\ \mathbb{Z} & n = 0 \\ \{0\} & n \neq 0, K \end{cases}$.

6° $\mathbb{P}^n = S^n/\sim$ where $\sim$ denotes antipodal identifications ($\vec{v} \sim -\vec{v}$). Find the chain complex and homology groups for $\mathbb{P}^n$.

7° Let S_g be an orientable surface of genus g. Show $H_2(S_g) \cong \mathbb{Z}$ & $H_1(S_g) \cong \mathbb{Z}^{2g}$.

8° Simplices and Simplicial Complexes

(a) 

$\Longleftrightarrow$ 

$\Longleftrightarrow$ $\langle v_0, v_1 \rangle$

$$\partial \langle v_0, v_1 \rangle = \langle v_1 \rangle - \langle v_0 \rangle$$

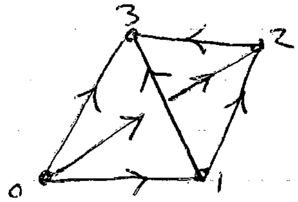
one-simplex and its boundary.

(b) 

$\langle 0, 1, 2 \rangle$

2-simplex

$$\partial \langle 0,1,2 \rangle = \langle 1,2 \rangle - \langle 0,2 \rangle + \langle 0,1 \rangle$$

(c) 

$\langle 0,1,2,3 \rangle$
3-simplex

$$\partial \langle 0,1,2,3 \rangle = \langle 1,2,3 \rangle - \langle 0,2,3 \rangle + \langle 0,1,3 \rangle - \langle 0,1,2 \rangle$$

(d) $\langle 0,1,2,\dots,n \rangle = \sigma$ n-simplex

$$\partial \sigma = \sum_{i=0}^{n} (-1)^i \langle 0,1,\dots,i-1,\hat{i},i+1,\dots,n \rangle$$

($\hat{a}$ = eliminate a from list).

__Lemma.__ If σ is an n-simplex, then $\partial\partial\sigma = 0$.

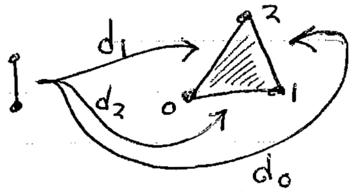
__Proof.__ __Exercise!__

9. Singular Homology

Use geometry of simplices.

Face inclusions : $d_i : \Delta^n \longrightarrow \Delta^{n+1}$

$i = 0, 1, \ldots, n, n+1$



etc.

X a topological space.

$f : \Delta^n \longrightarrow X$ a continuous map.

$\partial f \in C_{n-1}(X) = \{$linear combinations of maps from $\Delta^{n-1} \longrightarrow X \}$

$$\partial f = \sum_{i=0}^{n+1} f \circ d_i \, (-1)^i$$

Check that $\to C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X) \xrightarrow{\partial} \cdots$ is a chain complex. i.e. $\partial^2 = 0$.

$$H_n(X) \underset{def}{=\!=} H_n(C_*(X)) = \frac{Ker(\partial : C_n \to C_{n-1})}{Im(\partial : C_{n+1} \to C_n)}$$

This defn of homology is invariant under homeoms of X essentially by definition. One has to do some work to show how to compute it, and to reduce the big chain complex $C_*(X)$ to manageable proportions.

See any good book on algebraic topology for more details.

Exercise: Given $f : X \to Y$ continuous map. Define $f_* : C_*(X) \to C_*(Y)$ by: $\alpha : \Delta^n \to X$, $f_*(\alpha) = f \circ \alpha : \Delta^n \to Y$. Check that this induces a homomorphism of homology groups.

<u>Hint</u>: $f: X \longrightarrow Y$. Show that for $\alpha: \Delta^n \longrightarrow X$, $\partial f_*(\alpha) = f_*(\partial\alpha)$. //

Another useful concept in dealing with topological spaces is the notion of <u>homotopy of maps</u>, and <u>homotopy type of spaces</u>.

$f, g: X \longrightarrow Y$ (continuous maps) are homotopic $(f \sim g)$ means $\exists$ $F: X \times I \longrightarrow Y$ continuous such that $f = F|X\times\{0\}$, $g = F|X \times \{1\}$.

Two spaces X and Y are said to have the same homotopy type if $\exists$ continuous maps $f: X \to Y$, $g: Y \to X$ such that $f\circ g \sim \mathbb{1}_Y$ and $g \circ f \sim \mathbb{1}_X$.

<u>ex</u>: (i) $S^1 \times I$ and S^1 are homotopy equivalent.
(i) $D^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$ and $\{(0,0) \in \mathbb{R}^2\}$ are homotopy equivalent.

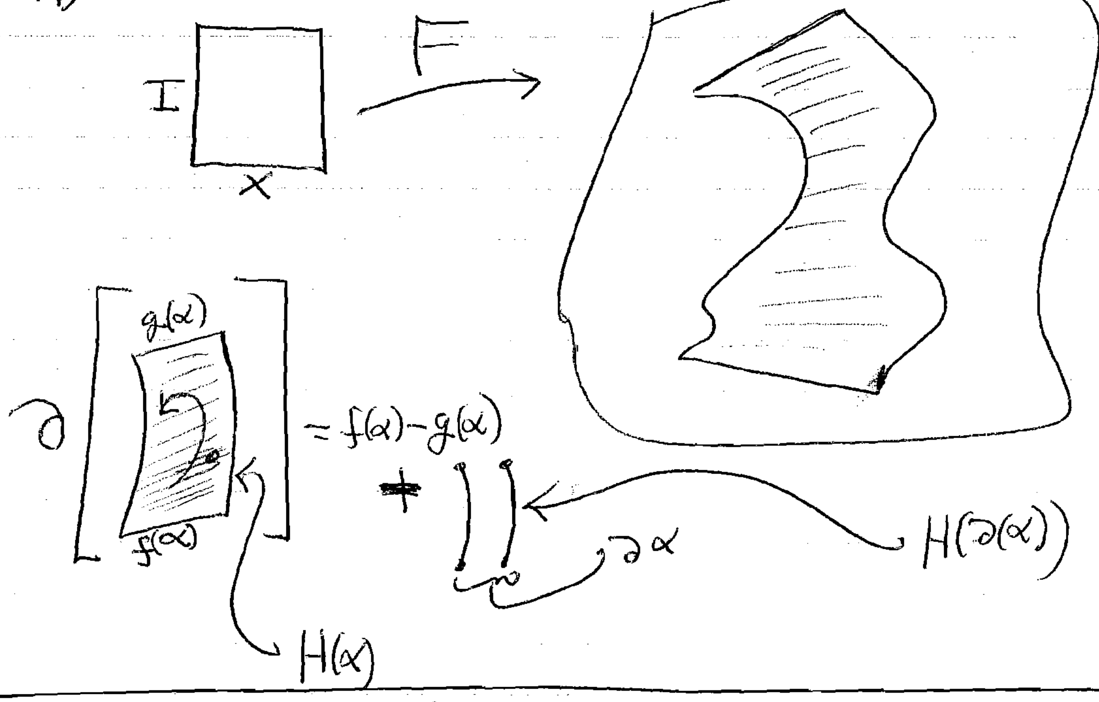
<u>ex</u>: Show that if $f: X \to Y$, $g: Y \longrightarrow X$ then $f_*: H_*(X) \to H_*(Y)$ and $g_*: H_*(Y) \to H_*(X)$ satisfy $f_* \circ g_* = \mathbb{1}_Y$, $g_* \circ f_* = \mathbb{1}_X$. Thus f_* and g_* induce isomorphisms on homology.

In order to prove the above you will need a little more about homolgy than we are providing here. The main idea is to create <u>a version of homotopy at the chain complex level</u>. We say (<u>by defn.</u>) that $f: C \longrightarrow C'$, $g: C \longrightarrow C'$ maps of chain complexes are <u>homotopic</u> if $\exists$ an map $H: C_n \longrightarrow C_{n+1}$ such that

$$\partial H(\alpha) = f(\alpha) - g(\alpha) + H(\partial\alpha)$$

for $\alpha \in C_n(X)$ (for all n).

This is an algebraic version of a homotopy:



<u>Note</u>: $f: C \to C'$ a map of chain complexes means $f: C_n \to C_n'$ homom & $f \circ \partial = \partial \circ f$ at all levels.

Show that if $f, g: C \to C'$ are homotopic then they induce the <u>same</u> map on homology.

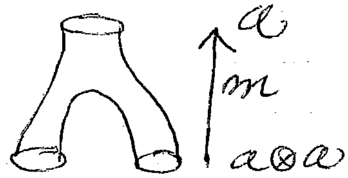
<u>Problem</u>: If $f: C \to C'$, $g: C \to C'$ maps of chain complexes, and $f \circ g \sim 1\!\!1_C$, $g \circ f \sim 1\!\!1_{C'}$ (these are homotopies of chain maps) then f and g induce isomorphisms of $H_*(C)$ and $H_*(C')$.

II. Elements of Frobenius Algebras

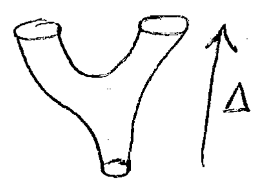
We will look at algebras A/k with
a <u>multiplication</u> $m : A \otimes A \longrightarrow A$
<u>comultiplication</u> $\Delta : A \longrightarrow A \otimes A$
<u>unit</u> $\eta : k \longrightarrow A$
<u>counit</u> $\varepsilon : A \longrightarrow k$
Here k is a ground-ring (e.g. $k = \mathbb{Z}$ or $k = \mathbb{Q}$ or $k = \mathbb{R}$ ...). A is a module over k.

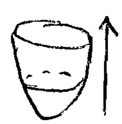
<u>Diagrams</u> : $A \longleftrightarrow \bigcirc$ (a circle).

$m : A \otimes A \longrightarrow A \longleftrightarrow$



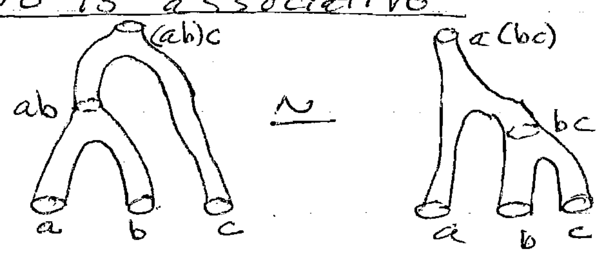
$\Delta : A \longrightarrow A \otimes A$



$\eta : k \longrightarrow A$  , $\varepsilon : A \longrightarrow k$ 

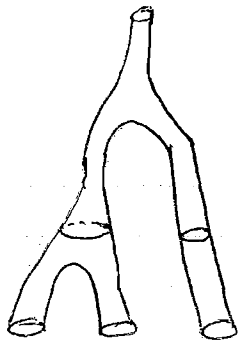
<u>Properties</u> : We want the algebraic operations in the algebra to correspond to topological equivalences of cobordisms between collections of circles. Thus we want

a) <u>m is associative</u>

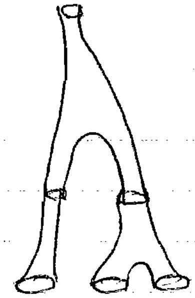


$[ab = m(a \otimes b)]$

$(ab)c = a(bc) \leftrightarrow$ (over)



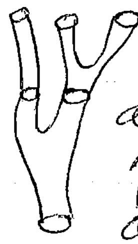
$$a \xleftarrow{m} a\otimes a \xleftarrow{m\otimes 1} a\otimes a\otimes a$$



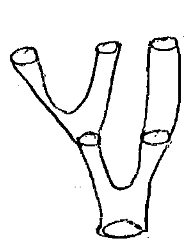
$$a \xleftarrow{m} a\otimes a \xleftarrow{1\otimes m} a\otimes a\otimes a$$

$$m\circ(m\otimes 1) = m\circ(1\otimes m)$$

b) <u>Δ is co-associative</u>



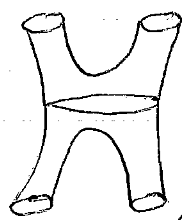
$$a\otimes a\otimes a \xleftarrow{1\otimes\Delta} a\otimes a \xleftarrow{\Delta} a$$



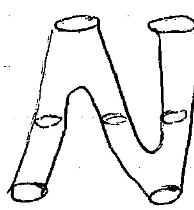
$$a\otimes a\otimes a \xleftarrow{\Delta\otimes 1} a\otimes a \xleftarrow{\Delta} a$$

$$(1\otimes\Delta)\circ\Delta = (\Delta\otimes 1)\circ\Delta$$

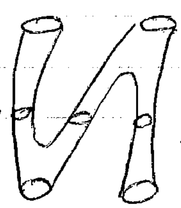
c) <u>Commutation of Saddle Points</u>



$$a\otimes a \xleftarrow{\Delta} a \xleftarrow{m} a\otimes a$$



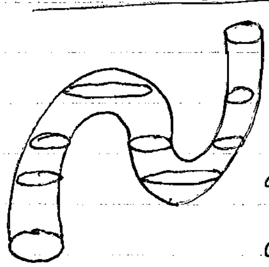
$$a\otimes a \xleftarrow{m\otimes 1} a\otimes a\otimes a \xleftarrow{1\otimes\Delta} a\otimes a$$



$$a\otimes a \xleftarrow{1\otimes m} a\otimes a\otimes a \xleftarrow{\Delta\otimes 1} a\otimes a$$

$$\Delta\circ m = (m\otimes 1)\circ(1\otimes\Delta) = (1\otimes m)\circ(\Delta\otimes 1)$$

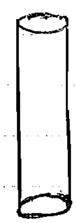
d) <u>Snake Identities</u>



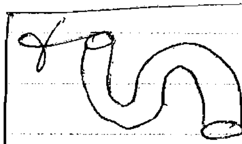
$$a\otimes a\otimes a \xleftarrow{1\otimes\Delta} a\otimes a \xleftarrow{1\otimes\eta} a\otimes k \xleftarrow{\cong} a$$

$$a \xleftarrow{\varepsilon\otimes 1} a\otimes a \quad m\otimes 1$$

$\cong$

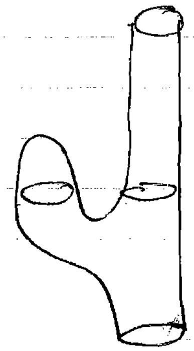


$$a \xleftarrow{\mathbb{1}} a$$

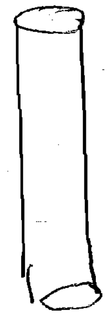


$\cong$

(e) Max and Saddle Cancellation

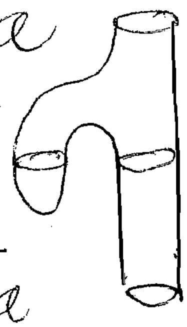


$$a \xrightarrow{\Delta} a\otimes a \xrightarrow{\varepsilon\otimes 1} a \quad \simeq$$

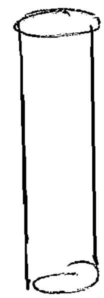


$$a \xrightarrow{\mathbb{1}} a$$

(f)



$$k\otimes a \simeq a \xrightarrow{\eta\otimes 1} a\otimes a \xrightarrow{m} a \quad \simeq$$



$$a \xrightarrow{\mathbb{1}} a$$

Min and Saddle Cancellation

Example 1. $\mathcal{A}$ generated by $1, X$ $/k$.

$1.1=1$, $1.X = X.1 = X$, $X.X = 0$

($m(a,b) = a.b$) & we write $1X = X1 = X$, $X^2 = 0$. $\Delta(X) = X\otimes X$

$\Delta(1) = 1\otimes X + X\otimes 1$.

$\eta(1) = 1$

$$\boxed{\begin{array}{l}\varepsilon(X) = 1\\ \varepsilon(1) = 0\end{array}}.$$

Check: a) ✓

b)
$$\begin{cases}(\Delta\otimes 1)\circ\Delta(X) = (\Delta\otimes 1)(X\otimes X) = X\otimes X\otimes X\\ (1\otimes\Delta)\circ\Delta(X) = (1\otimes\Delta)(X\otimes X) = X\otimes X\otimes X\end{cases} ✓$$

$$\begin{aligned}(\Delta\otimes 1)\circ\Delta(1) &= (\Delta\otimes 1)(1\otimes X + X\otimes 1)\\ &= (1\otimes X + X\otimes 1)\otimes X + (X\otimes X)\otimes 1\\ &= 1\otimes X\otimes X + X\otimes 1\otimes X + X\otimes X\otimes 1\\ (1\otimes\Delta)\circ\Delta(1) &= (1\otimes\Delta)(1\otimes X + X\otimes 1)\\ &= 1\otimes(X\otimes X) + X\otimes(1\otimes X + X\otimes 1)\\ &= 1\otimes X\otimes X + X\otimes 1\otimes X + X\otimes X\otimes 1\end{aligned}$$

✓

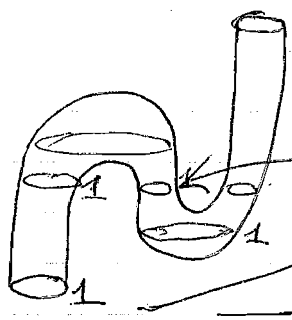
(c)
$$\begin{cases} \Delta\circ m(X\otimes X)=\Delta(X^2)=\Delta(0)=0 \\ (m\otimes 1)\circ(1\otimes\Delta)(X\otimes X)=(m\otimes 1)(X\otimes X\otimes X)=0 \\ (1\otimes m)\circ(\Delta\otimes 1)(X\otimes X)=(1\otimes m)(X\otimes X\otimes X)=0 \end{cases}$$ ✓

$$\begin{cases} \Delta\otimes m(1\otimes X)=\Delta(X)=X\otimes X \\ (m\otimes 1)\circ(1\otimes\Delta)(1\otimes X)=m\otimes 1\,(1\otimes X\otimes X)=X\otimes X \\ (1\otimes m)\circ(\Delta\otimes 1)(1\otimes X)=(1\otimes m)(1\otimes X\otimes X+X\otimes 1\otimes X)=X\otimes X \end{cases}$$ ✓

(Simil for $X\otimes 1$.)

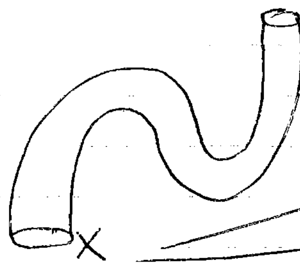
$$\begin{cases} \Delta\otimes m(1\otimes 1)=\Delta(1)=1\otimes X+X\otimes 1 \\ (m\otimes 1)\circ(1\otimes\Delta)(1\otimes 1)=m\otimes 1\,(1\otimes 1\otimes X+1\otimes X\otimes 1)=1\otimes X+X\otimes 1 \\ (1\otimes m)(\Delta\otimes 1)(1\otimes 1)=1\otimes m\,(1\otimes X\otimes 1+X\otimes 1\otimes 1)=1\otimes X+X\otimes 1 \end{cases}$$ ✓

(d)



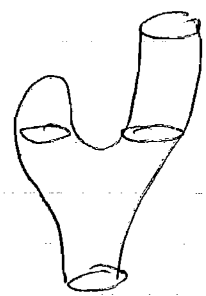
$1\otimes X + X\otimes 1$

$$1 \xleftarrow{\varepsilon\otimes 1} 1\otimes X+X\otimes 1 \xleftarrow{m\otimes 1} 1\otimes 1\otimes X+1\otimes X\otimes 1$$ ✓



$$X \xleftarrow{\varepsilon\otimes 1} X\otimes X \xleftarrow{m\otimes 1} X\otimes 1\otimes X+X\otimes X\otimes 1$$ ✓

(e)



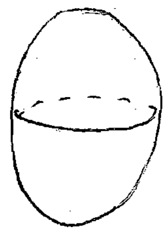
$$1 \xrightarrow{\Delta} 1\otimes X+X\otimes 1 \xrightarrow{\varepsilon\otimes 1} 1$$ ✓

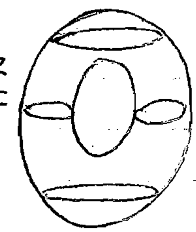
$$X \xrightarrow{\Delta} X\otimes X \xrightarrow{\varepsilon\otimes 1} X$$ ✓

(f) equiv to $m(1\otimes\alpha)=\alpha$ ✓

Thus we have shown that
$\mathcal{A}$ gen by $1, X$ with $X^2=0$,
$\Delta(1) = 1\otimes X + X\otimes 1$, $\Delta(X) = X\otimes X$, $\varepsilon(1)=0, \varepsilon(X)=1$
is a Frobenius algebra.

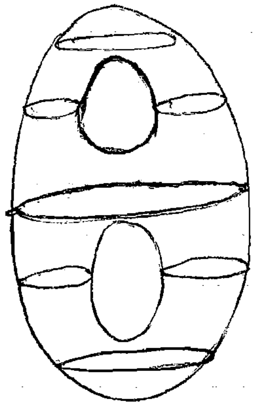
Note:

$S^2 =$  $\quad 1 \xrightarrow{\eta} 1 \xrightarrow{\varepsilon} \phi$

$T^2 =$  $\quad 1 \xrightarrow{\Delta} 1\otimes X + X\otimes 1 \xrightarrow{m} 2X \xrightarrow{\varepsilon} 2$

Thus the "value" of S^2 is ϕ
& the "value" of T^2 is 2.

Other surfaces?

 $\quad 1 \longrightarrow 2X \xrightarrow{\Delta} 4X\otimes X \xrightarrow{m} \phi$

So the value of all the other closed surfaces is ϕ.

(You might ask about non-orientable surfaces. We will come back to this theme.)

Example 2. Lee's Algebra $\mathcal{B}$

This is generated by $1, X$ with $X^2 = 1$
and $\Delta X = X\otimes X + 1\otimes 1$
$\Delta 1 = 1\otimes X + X\otimes 1$.

It is convenient to change basis so that $a = 1+X$, $b = 1-X$. Then

$$a^2 = (1+x)^2 = 1 + 2x + x^2 = 1 + 2x + 1 = 2a$$
$$b^2 = (1-x)^2 = 1 - 2x + x^2 = 1 - 2x + 1 = 2b$$
$$\begin{aligned}\Delta a = \Delta(1+x) &= \Delta(1) + \Delta(x) \\ &= 1\otimes x + x\otimes 1 + x\otimes x + 1\otimes 1 \\ &= (1+x)\otimes(1+x) = a\otimes a\end{aligned}$$

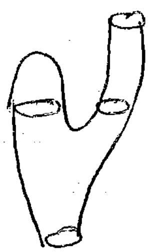
Similarly, $\Delta b = b\otimes b$.

So we have $1 = \dfrac{a+b}{2}$

$$\left\{\begin{array}{l} a^2 = 2a \\ b^2 = 2b \\ \Delta a = a\otimes a \\ \Delta b = b\otimes b \end{array}\right\} \qquad \boxed{ab = 1 - x^2 = 1 - 1 = 0}$$

$$\Delta(1) = \tfrac{1}{2}(\Delta a + \Delta b) = \tfrac{1}{2}(a\otimes a + b\otimes b).$$

What is ε?



want $(\varepsilon\otimes 1)\Delta(\alpha) = \alpha$.

$$(\varepsilon\otimes 1)\Delta(a) = (\varepsilon\otimes 1)(a\otimes a) = \varepsilon(a)\otimes a$$
$$\Rightarrow \varepsilon(a) = 1$$

Simil $\varepsilon(b) = 1$

So $\varepsilon(1) = \tfrac{1}{2}(\varepsilon(a) + \varepsilon(b)) = 1$ ✓

Lets check (c):

$$\left\{\begin{array}{l} \Delta\circ m\,(a\otimes a) = \Delta(2a) = 2\Delta(a) = 2a\otimes a \\ (m\otimes 1)\circ(1\otimes\Delta)\,(a\otimes a) = (m\otimes 1)(a\otimes a\otimes a) = 2a\otimes a \\ (1\otimes m)\circ(\Delta\otimes 1)\,(a\otimes a) = (1\otimes m)(a\otimes a\otimes a) = 2a\otimes a \end{array}\right.$$ ✓

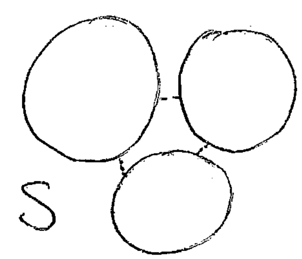
$$\left\{\begin{array}{l} (\Delta\otimes m)\,(a\otimes b) = \Delta(0) = 0 \\ (m\otimes 1)\circ(1\otimes\Delta)\,(a\otimes b) = (m\otimes 1)(a\otimes b\otimes b) = 0 \\ (1\otimes m)\circ(\Delta\otimes 1)\,(a\otimes b) = (1\otimes m)(a\otimes a\otimes b) = 0 \end{array}\right.$$ ✓

etc.

III. Frobenius Algebras Used for Constructing Chain Complexes Related to Bracket Polynomial States

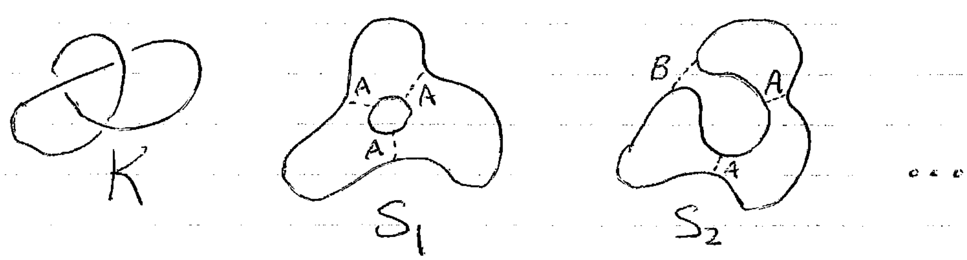
We shall associate to each bracket polynomial state S of a knot diagram K a module as follows: Let $\mathcal{A}$ be a given Frobenius algebra (e.g. use our example 1 alg with gens $1, x$: $x^2=0$, $\Delta(x)=x\otimes x$, $\Delta(1)=1\otimes x+x\otimes 1$, $\eta(1)=1$, $\varepsilon(x)=1$, $\varepsilon(1)=0$). Take one copy of $\mathcal{A}$ for each loop in the state and tensor theses $\mathcal{A}$'s to form

$$V(S)=\bigotimes_{\lambda\in \mathcal{L}oops(S)} \mathcal{A}_\lambda .$$

e.g. 

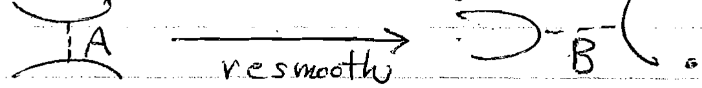
$\Longrightarrow V(S)=\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$

Recall that states S have some sites of type A and some sites of type B.

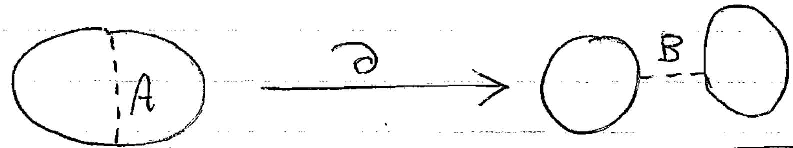
e.g. 

For any state S of K, we will define maps $V(S) \xrightarrow{\partial} V(S')$

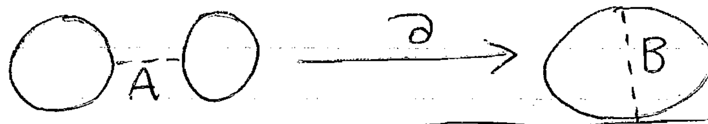
for any state S' obtained from S by a single re-smoothing of the form



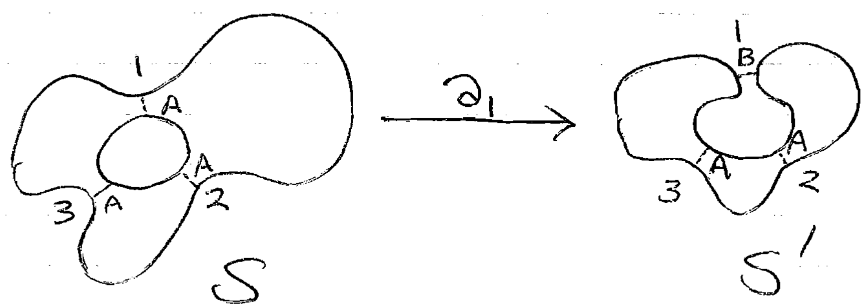
These maps are generated by the following two cases:

(i) 

$$\boxed{a \xrightarrow{\Delta} a \otimes a}$$

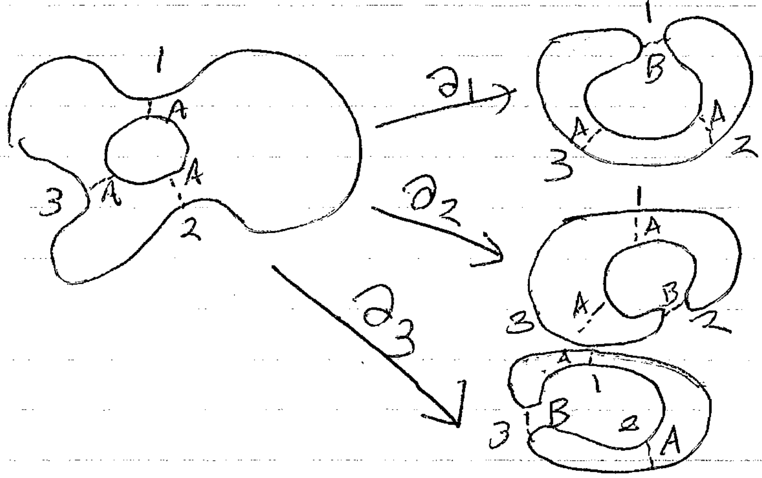
(ii) 

$$\boxed{a \otimes a \xrightarrow{m} a}$$

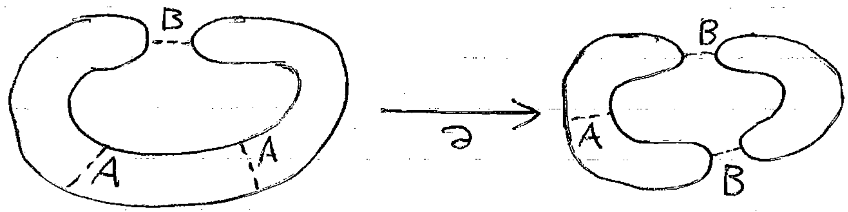
e.g. 

$$a \otimes a \xrightarrow{m} a$$

In the case of this state S there are <u>three</u> outgoing maps (all of the form $a \otimes a \xrightarrow[m]{} a$.

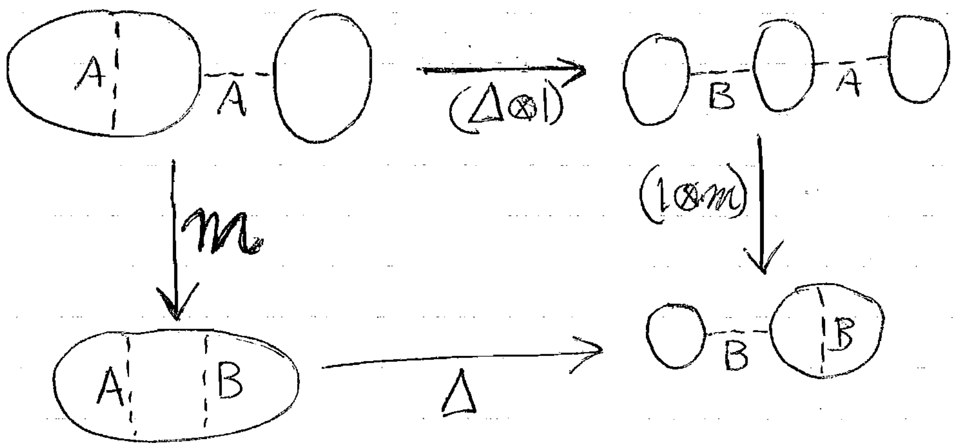


Each of these states have maps out in form of coproducts: e.g.



$$\boxed{a \xrightarrow{\Delta} a \otimes a}$$

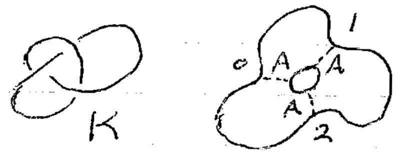
One can make composite maps in different orders, depending on which A-sites are chosen first or second. For example:



Since $\Delta \circ m = (1 \otimes m) \circ (\Delta \otimes 1)$ in a Frobenius algebra, we see that we'll get the same result in composing maps, independent of the order of choices of operations on the sites.

This situation lets us construct a well-defined chain complex associated with a knot or link diagram.

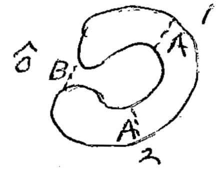
To create the chain complex $C(K)$ corresponding to a link diagram K, we start with the A-state S:



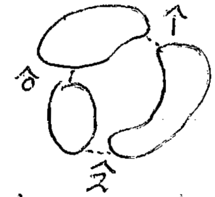
$S = \langle 0, 1, 2 \rangle$.

Label the sites $0, 1, \ldots, n$; and let $\langle 0, 1, 2 \rangle$ denote S $(n = 2)$.

Let $\hat{i}$ denote the result of re-smoothing the i^{th} site. Thus e.g.



$\langle \hat{0}, 1, 2 \rangle$,



$\langle \hat{0}, \hat{1}, \hat{2} \rangle$

In general let $\mathring{i}^{\varepsilon} = \hat{i}$ if $\varepsilon = 1$, $\mathring{i}^{\varepsilon} = \mathring{i}$ if $\varepsilon = 0$. Then a general state of K is
$S(\vec{\varepsilon}) = S = \langle 0^{\varepsilon_0}, 1^{\varepsilon_1}, \ldots, n^{\varepsilon_n} \rangle$.

Remember that each state $S(\vec{\varepsilon})$ is regarded as a tensor product of copies of the algebra $\mathcal{Q}$, one copy for each loop in the state. We then have

$$\partial_i : \langle 0^{\varepsilon_1}, \ldots, (i-1)^{\varepsilon_{i-1}}, \mathring{i}, (i+1)^{\varepsilon_{i+1}}, \ldots, n^{\varepsilon_n} \rangle$$
$$\downarrow$$
$$\langle 0^{\varepsilon_1}, \ldots, (i-1)^{\varepsilon_{i-1}}, \hat{i}, (i+1)^{\varepsilon_{i+1}}, \ldots, n^{\varepsilon_n} \rangle$$

where ∂_i is either multiplication or co-multiplication in the Frobenius algebra.

In the case of the A-state we have $\partial_0, \partial_1, \ldots, \partial_n$ all defined on $\langle 0, 1, \ldots, n \rangle$

$\partial_i : \langle 0, 1, \ldots, n \rangle \longrightarrow \langle 0, 1, \ldots, \hat{i}, \ldots, n \rangle$.

We the define $\partial : \langle 0, 1, \ldots, n \rangle \longrightarrow \bigoplus_{k=0}^{n} \langle 0, 1, \ldots, k-1, \hat{k}, k+1, \ldots n \rangle$

by $\partial = \partial_0 - \partial_1 + \partial_2 - \partial_3 \pm \cdots + (-1)^n \partial_n$.

Similarly, for a state with already-deleted sites (i.e. smoothed sites) we take the alternating sum of the maps that remain available.

Thus $\partial : \langle 0,1,2 \rangle \longrightarrow \langle \hat{0},1,2\rangle \oplus \langle 0,\hat{1},2\rangle \oplus \langle 0,1,\hat{2}\rangle$
$$\partial = \partial_0 - \partial_1 + \partial_2$$
while
$$\partial : \langle \hat{0},1,2\rangle \longrightarrow \langle \hat{0},\hat{1},2\rangle \oplus \langle \hat{0},1,\hat{2}\rangle$$
is given by $\partial_1 - \partial_2$ and

$$\partial^2 : \langle 0,1,2\rangle \longrightarrow \langle \hat{0},\hat{1},2\rangle \oplus \langle \hat{0},1,\hat{2}\rangle \oplus \langle 0,\hat{1},\hat{2}\rangle$$
$$\begin{aligned}\partial^2 &= (\partial_1 - \partial_2)\partial_0 - (\partial_0 - \partial_2)\partial_1 + (\partial_0 - \partial_1)\partial_2 \\ &= \partial_1\partial_0 - \partial_2\partial_0 - \partial_0\partial_1 + \partial_2\partial_1 + \partial_0\partial_2 - \partial_1\partial_2 \\ &= (\partial_1\partial_0 - \partial_0\partial_1) + (-\partial_2\partial_0 + \partial_0\partial_2) + (\partial_2\partial_1 - \partial_1\partial_2)\end{aligned}$$
$$\partial^2 = 0.$$
Thus $\partial^2 = 0$ because $\partial_i\partial_j = \partial_j\partial_i$ for $i \neq j$. This fact, in turn, is implied by the fact that orders of smoothing operations lead to commuting diagrams in the Frobenius algebra.

We therefore have a chain complex with $C_\ell(K) = \bigoplus_{\|\vec{\varepsilon}\| = \ell} S(\vec{\varepsilon})$ where $\|\vec{\varepsilon}\|$ denotes the number of 0's in the vector $\vec{\varepsilon}$.
$\partial : C_\ell(K) \longrightarrow C_{\ell-1}(K)$. And we define
$KH_\ell(K) = Ker(\partial : C_\ell \to C_{\ell-1}) / Im(\partial : C_{\ell+1} \to C_\ell)$.
This is the Khovanov homology of K. We shall have to revise and amplify certain matters about grading later.

Our task is to understand how these homology groups behave when we apply Reidemeister moves to the diagrams.

We will begin by doing some purely algebraic calculations, and then take a more abstract view by invoking the structure of cobordisms of one-manifolds as a category of mappings generating the differentials in (an abstract version of) these chain complexes.