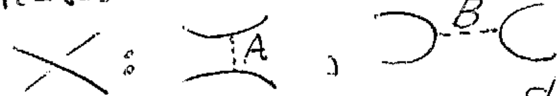
A Short Introduction to Khovanov Homology

by Louis H. Kauffman
<kauffman@uic.edu>

1° Bracket Polynomial

$$\langle K \rangle = \sum_S A^{n_+(S) - n_-(S)} d^{\|S\| - 1}, \quad d = -A^2 - A^{-2}$$

$n_+(S)$ = # A-smoothings in S
$n_-(S)$ = # B-smoothings in S



$\|S\|$ = # of loops in S

S is a state of the diagram K, obtained by taking a choice of smoothing at each crossing.

e.g.

$n_+(S) = 3$
$n_-(S) = 0$
$\|S\| = 2$

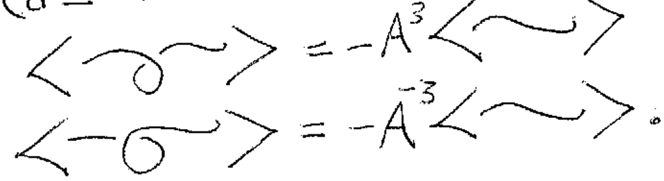
$\Rightarrow$ (a) $\langle K \rangle$ is an invariant of regular isotopy (equiv rel generated by Reidemeister II + Reidemeister III moves).

(b)

$\langle O K \rangle = d \langle K \rangle$

$(d = -A^2 - A^{-2})$

(c)



(d) $wr(K) = \sum_{c \in \text{Crossings}(K)} \varepsilon(c)$, <u>K oriented link</u>

$\varepsilon(\;) = +1$, $\varepsilon(\;) = -1$.

$wr(K)$ called the <u>writhe</u> of K.

$wr(K)$ is also an invariant of regular isotopy.

(e) Define $f_K(A) = (-A^3)^{-wr(K)} \langle K \rangle$.

Then $f_K(A)$ is invariant under R I, II, III & so is an invariant of ambient isotopy of K.

(f) $K^* =$ mirror image of K
(switch all crossings)

$\Rightarrow f_{K^*}(A) = f_K(A^{-1})$

$\langle K^* \rangle (A) = \langle K \rangle (A^{-1})$.

Hence $K \sim K^*$ ($\sim \equiv$ ambient isotopy)

$\Rightarrow f_K(A) = f_K(A^{-1})$.

(g) <u>Jones Polynomial</u> $V_K(t)$

$$\boxed{\begin{array}{l} t^{-1} V_{\;} - t V_{\;} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{\;} \\ V_{\bigcirc} = 1 \\ K \sim K' \Rightarrow V_K(t) = V_{K'}(t) \end{array}}$$

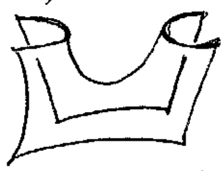
$$\boxed{V_K(t) = f_K\left(t^{-\frac{1}{4}}\right)}$$

2. Toward Khovanov Homology

Khovanov's discovery of Khovanov Homology (a new invariant related to the Jones polynomial) was motivated by

(a) the idea that $\langle K \rangle (A)$ or $V_K(t)$ <u>ought</u> to be a shadow of some larger invariant, perhaps in analogy to the way homology groups generalize Betti numbers.

& (b) the smoothings $\asymp$ and $)($ naturally relate to one another in the form of a saddle:



indicating that the bracket polynomial state sum <u>should</u> have something to do with surfaces, cobordisms and embeddings of surfaces in four-dimensional space.

3. <u>Toward an Euler Characteristic</u>

(a) First we rewrite the bracket state sum via

$$[K] \underset{\text{def}}{=} A^{-c(K)} \langle K \rangle d .$$

$(c(K) = \#\text{ crossings in } K)$

Verify that

$$[\times] = [\asymp] + A^{-2}[)\,(]$$
$$[O K] = d[K]$$
$$[O] = d, \quad d = -A^{2} - A^{-2}.$$

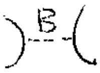
Let $q = -A^{-2}$. Then

$$\boxed{\begin{array}{l} [\times] = [\asymp] - q[)\,(] \\ [O K] = d[K] \\ [O] = d = q + q^{-1} \end{array}}$$

We will use this version of the bracket and expand on <u>augmented states s</u> where each loop in s is labeled + or − :

$$\boxed{O = O^{+} + O^{-} \quad \Longleftrightarrow \quad d = q + q^{-1}}$$

Using augmented states s, we have the formula $[K] = \sum_s (-q)^{n_-(s)} q^{\lambda(s)}$

where $n_+ = \#$ , $n_- = \#$ 

$\lambda = \#(+) - \#(-)$ where + and − denote the labels on the state loops.

Let $\boxed{j(s) = n_-(s) + \lambda(s)}$.

Then
$$[K] = \sum_s q^{j(s)} (-1)^{n_-(s)}$$
$$= \sum_{n,j} (-1)^n q^j \Big(\sum_{\substack{n_-(s)=n \\ j(s)=j}} 1 \Big).$$

Let C_{nj} denote the module over $\mathbb{Z}_2$ with <u>basis</u> the states s of K with $n_-(s) = n$ and $j(s) = j$.

Then $\dim(C_{nj}) = \sum_{\substack{n_-(s)=n \\ j(s)=j}} 1$.

and we can write:
$$[K] = \sum_{n,j} (-1)^n q^j \dim(C_{nj}).$$

[We work modulo 2 for convenience here.]

Now we could hope/dream that $\{C_{nj}\}$ for a fixed j would assemble to form a chain complex with $\partial : C_{nj} \longrightarrow C_{n+1,j}$, $\partial^2 = 0$ (∂ would have to preserve j).

We would have

$$C_{*j} : C_{0j} \xrightarrow{\partial} C_{1j} \xrightarrow{\partial} C_{2j} \longrightarrow \cdots$$

and could consider both Euler characteristics and homology:

$$\chi(C_{*j}) = \sum_n (-1)^n \dim(C_{nj})$$

$$H_n(C_{*j}) = \frac{\mathrm{Ker}(\partial : C_{nj} \to C_{n+1,j})}{\mathrm{Image}(\partial : C_{n-1,j} \to C_{n,j})}$$

and as usual

$$\chi(C_{*j}) = \chi(H_{*j}) = \sum_n (-1)^n \dim(H_{nj}).$$

Then we would have

$$[K] = \sum_j q^j \Big(\sum_n (-1)^n \dim(C_{nj})\Big)$$

$$= \sum_j q^j \chi(C_{*j}) = \sum_j q^j \chi(H_{*j})$$

We define the *q-graded Euler characteristic* by the formula

$$\chi_q(C) = \sum_{j,n} q^j \dim(C_{nj})(-1)^n$$
$$= \sum_j q^j \chi(C_{*j}).$$

Then we would have

$$C(K) : \ C(K)_{nj} = C_{nj} \text{ as before.}$$
$$H(K) : \ H(K)_{nj} = H_n(C_{*j})$$

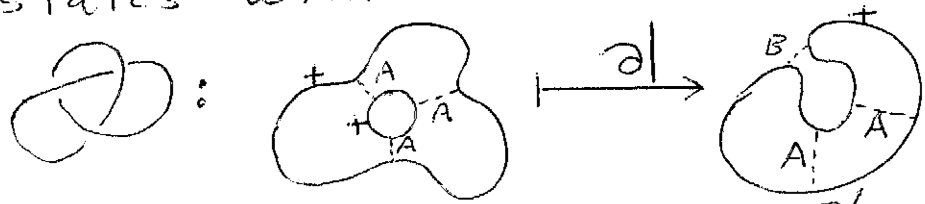
and
$$[K] = \chi_q(C(K)) = \chi_q(H(K)).$$

Thus we would realize the idea of section 2(a) – to express the bracket summation as an Euler characteristic of a larger theory. Of course we would want $H(K)$ *itself* to be an invariant of K.

Khovanov discovered that this program really does work. Here's how:

4. Making the chain complex.

It is natural to go from $C_{n,j}$ to $C_{n+1,j}$ by smoothing A-sites of states in $C_{n,j}$ to obtain states in $C_{n+1,j}$. (To simplify matters, lets work over $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ so that elements of C_{nj} are just combinations of states with coefficients 0 or 1.)

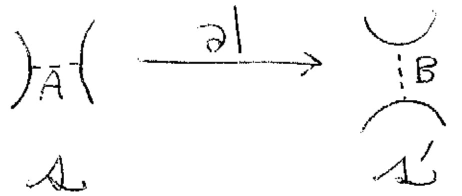


S $\qquad\qquad$ S'

$n_-(S) = 0$ $\qquad$ $n_-(S') = 1$

$j(S) = 0 + 2 = 2$ $\qquad$ $j(S') = 1$

Over $\mathbb{Z}_2$, we shall define the boundary of a state s to be the linear combination of a set of states obtained via re-smoothing one A-site at a time from s. <u>We need to explain how to do single re-smoothings.</u> For this we use the invariance of $j(s)$.



resmooth a single A-site

$n_-(s) = n \qquad n_-(s') = n+1$

$j(s) = n + \lambda(s)\,, \quad j(s') = n+1+\lambda(s')$

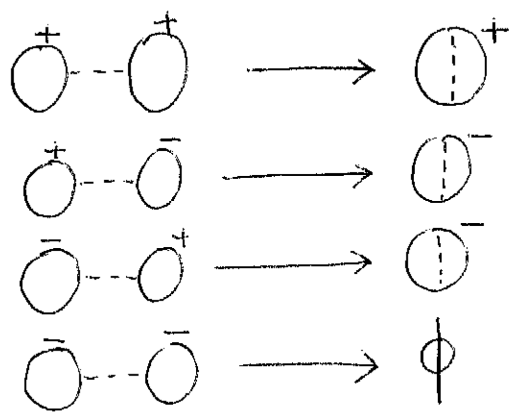
$$j(s) = j(s') \iff \lambda(s) = 1 + \lambda(s')$$
$$\iff \boxed{\lambda(s') = \lambda(s) - 1}.$$

There are two cases to consider in re-smoothing:

(A) two loops $\longrightarrow$ one loop

(B) one loop $\longrightarrow$ two loops

$\boxed{A}$



a) : $\lambda = 2 \to \lambda' = 1$

b) : $\lambda = 0 \to \lambda' = -1$

c) : $\lambda = 0 \to \lambda' = -1$

d) : $\lambda = (-2)$ can not get smaller.

We see that for <u>two loops $\longrightarrow$ one loop</u> the boundary rule is <u>determined by $\lambda' = \lambda - 1$</u>.

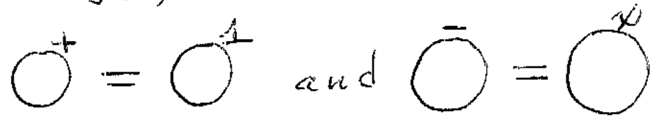
Note also that if you think of the labels as a multiplication rule for combining

$$\overset{\alpha}{\bigcirc}\!-\!\overset{\beta}{\bigcirc} \xrightarrow{m} \bigcirc^{\alpha\beta} = m(\alpha,\beta)$$

then: <u>$+$ acts as an identity element,</u>

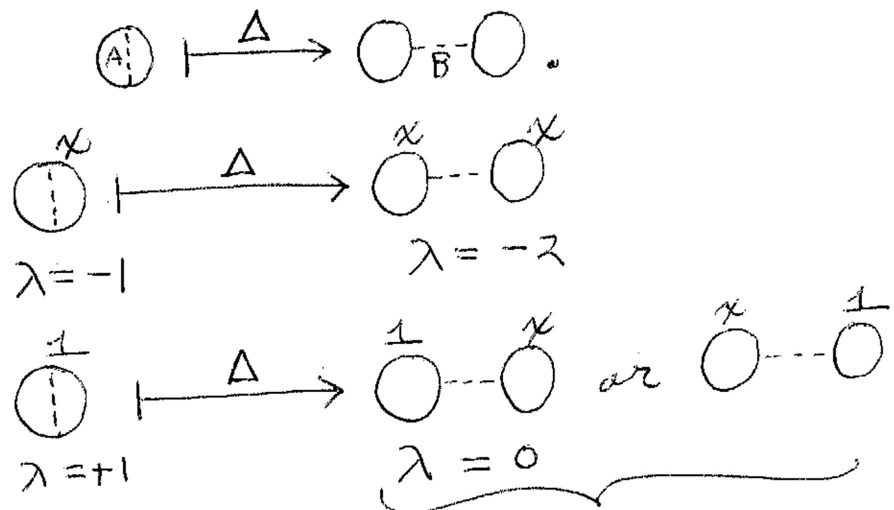
<u>$-$ has square zero.</u>

Accordingly, we write

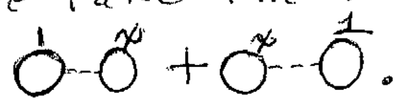


with $x^2 = 0$. Letting $k = \mathbb{Z}_2$, we have the algebra $k[x]/(x^2) = V$. $m: V \otimes V \longrightarrow V$ denotes the multiplication in V.

[B] Now we consider



Since the 1-label has two choices, we take the linear combination:

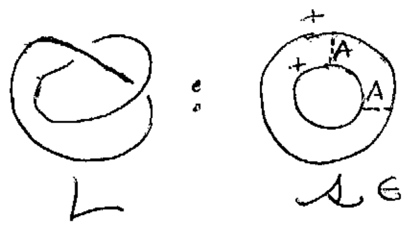


Turning all this into algebra, we have

$$\Delta : V \longrightarrow V \otimes V$$
$$\Delta(x) = x \otimes x$$
$$\Delta(1) = 1 \otimes x + x \otimes 1.$$

The algebra $k[x]/(x^2) = V$ with coproduct $\Delta : V \longrightarrow V \otimes V$ as above, encodes the information we need to define $\partial : C_{n,j} \longrightarrow C_{n+1,j}$.

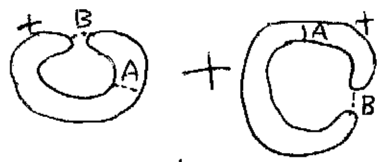
e.g.



$\lambda(s) = 2$
$j(s) = 2$

$s \in C_{0,2}$

$\partial(s) =$



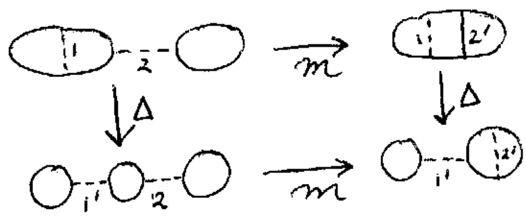
$\in C_{1,2}$

$\lambda = 1$
$j = 1 + 1 = 2.$

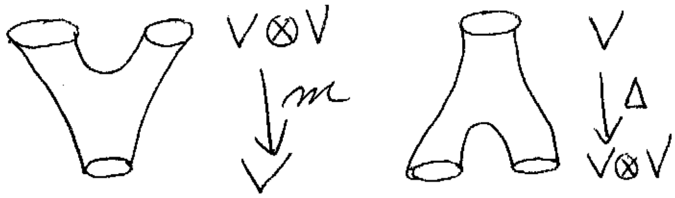
We now want to verify that $\partial^2 = 0$.

Since we are over $\mathbb{Z}_2 = k$, it will suffice to see that $\partial|$ (acting on single smoothings) is <u>independent of the order of its action</u>. Then applying boundary twice will result in two copies of everything and so gives zero.

<u>example</u>:

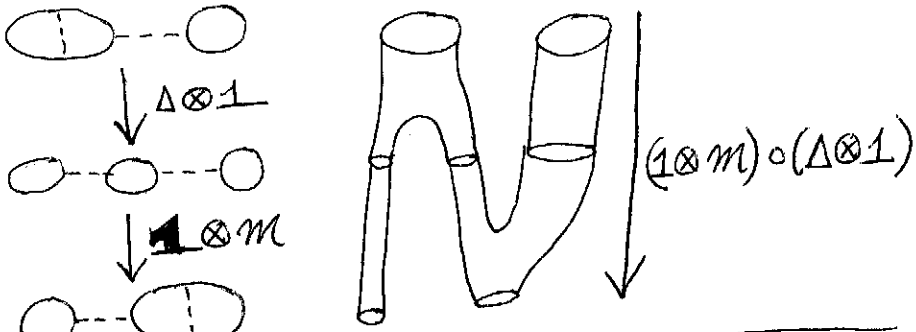


Use surface cobordisms:

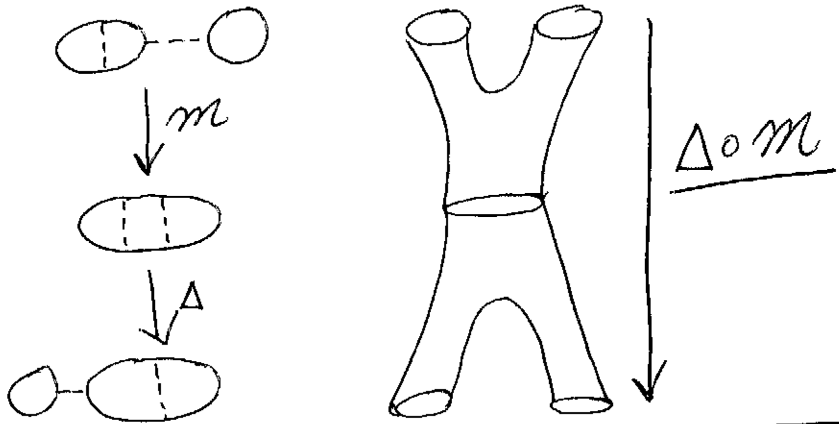


$V \otimes V \xrightarrow{m} V \qquad V \xrightarrow{\Delta} V \otimes V$

Then



$\xrightarrow{\Delta \otimes 1}$, $\xrightarrow{1 \otimes m}$, $(1 \otimes m) \circ (\Delta \otimes 1)$



$\xrightarrow{m}$, $\xrightarrow{\Delta}$, $\underline{\Delta \circ m}$

Algebraically, we need

$$(1 \otimes m) \circ (\Delta \otimes 1) = \Delta \circ m$$

(and $(m \otimes 1) \circ (1 \otimes \Delta) = \Delta \circ m$).

Topologically, the corresponding surface cobordisms are homeomorphic.

All these things work!

e.g. $(1\otimes m)\circ(\Delta\otimes 1)(1\otimes 1) = (1\otimes m)(1\otimes x\otimes 1 + x\otimes 1\otimes 1)$

$= 1\otimes x + x\otimes 1$

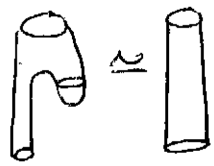
$= \Delta(1)$

$= \Delta\circ m(1\otimes 1)$ ✓

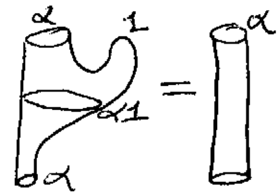
The surface cobordisms suggest that we should have

 $V \xrightarrow{\varepsilon} k$ and  $k \xrightarrow{\eta} V$.

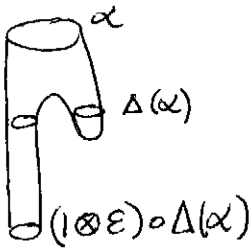
η is easy: $\eta(1_{(k)}) = 1_{(V)}$. This is the unit for the algebra V. ε is a counit and we want



just as



So:



want

$(1\otimes\varepsilon)\circ\Delta(\alpha) = \alpha \quad \forall \alpha\in V$

$(\varepsilon\otimes 1)\circ\Delta(\alpha) = \alpha \quad \forall \alpha\in V.$

e.g. $(\varepsilon\otimes 1)\circ\Delta(x) = (\varepsilon\otimes 1)(x\otimes x)$

$= \varepsilon(x)\otimes x$

$= \varepsilon(x)x \qquad (k\otimes V = V)$

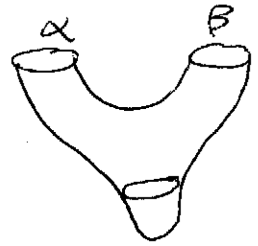
$\therefore$ want $\boxed{\varepsilon(x)x = x}$.

So we define $\varepsilon(x) = 1$.

$$
\begin{aligned}
(\varepsilon\otimes 1)\circ\Delta(1) &= (\varepsilon\otimes 1)(1\otimes x + x\otimes 1)\\
&= \varepsilon(1)x + \varepsilon(x)1\\
&= \varepsilon(1)x + 1
\end{aligned}
$$

So we take $\boxed{\varepsilon(1) = 0}$ $\boxed{\varepsilon(x) = 1}$.

We then have a non-degenerate pairing



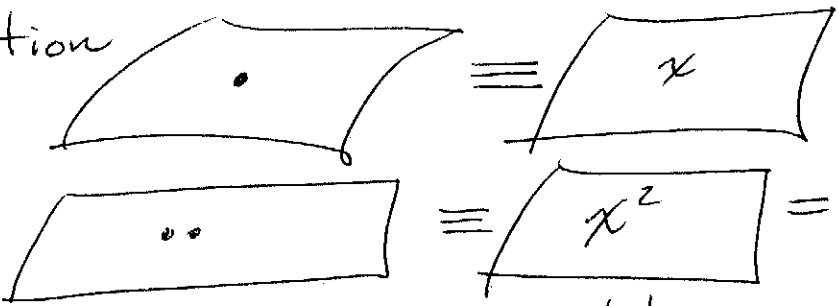
$$V\otimes V \xrightarrow{\langle\alpha|\beta\rangle = \varepsilon(\alpha\beta)} k$$

and V is a <u>Frobenius algebra</u>.

<u>Claim:</u> $\boxed{\alpha = \varepsilon(\alpha x)1 + \varepsilon(\alpha)x \quad \forall \alpha\in V}$

(Just check for $\alpha = 1, x$.)

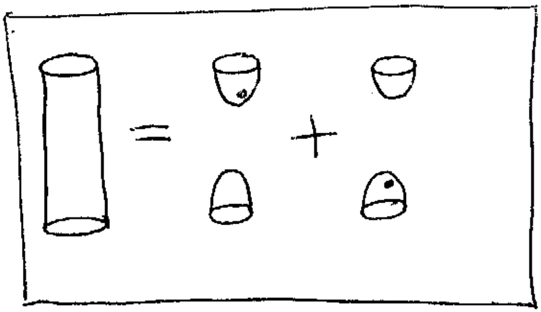
We draw a cobordism picture of this relation via the dotting convention



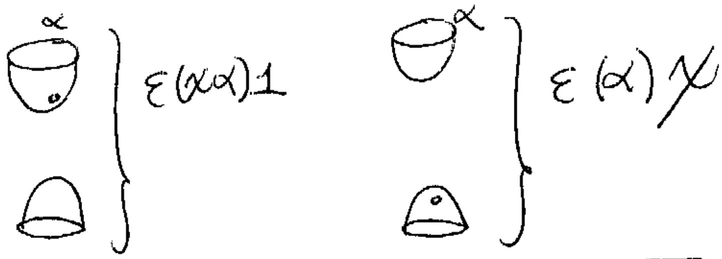
$= \phi$.

(a bit of surface with a dot on it corresponds to multiplication by x in the algebra).

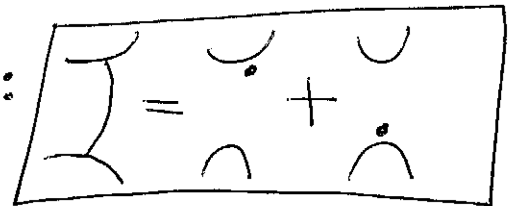
Then



Tube-Cutting Relation

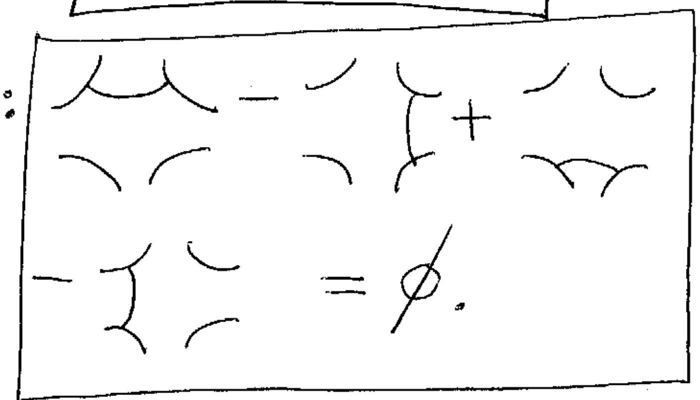


Shorthand:



Exercise:

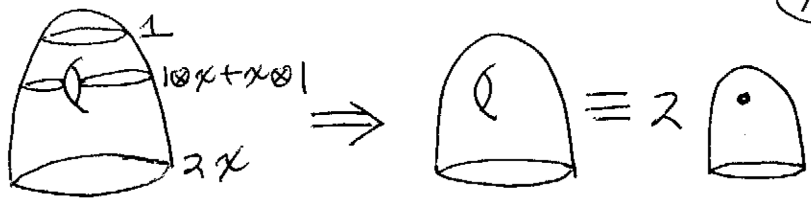
[4 bits of surface with tubes between as shown. Note there are no dots in this relation.]



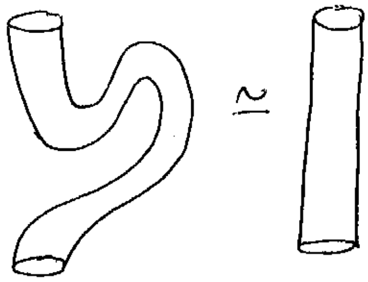
Dror Bar-Natan's: 4Tu Relation

The tube-cutting relation and the 4Tu Relation are useful in analyzing Khovanov homology by interpreting chain maps as linear combinations of surface cobordisms.

Note:



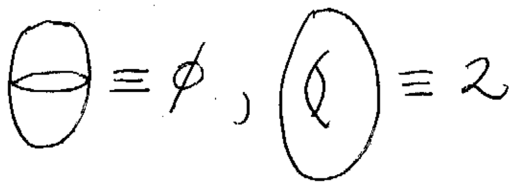
Exercise 1. a)



works in the algebra.

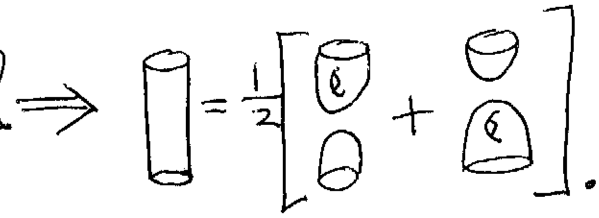
b)

c)



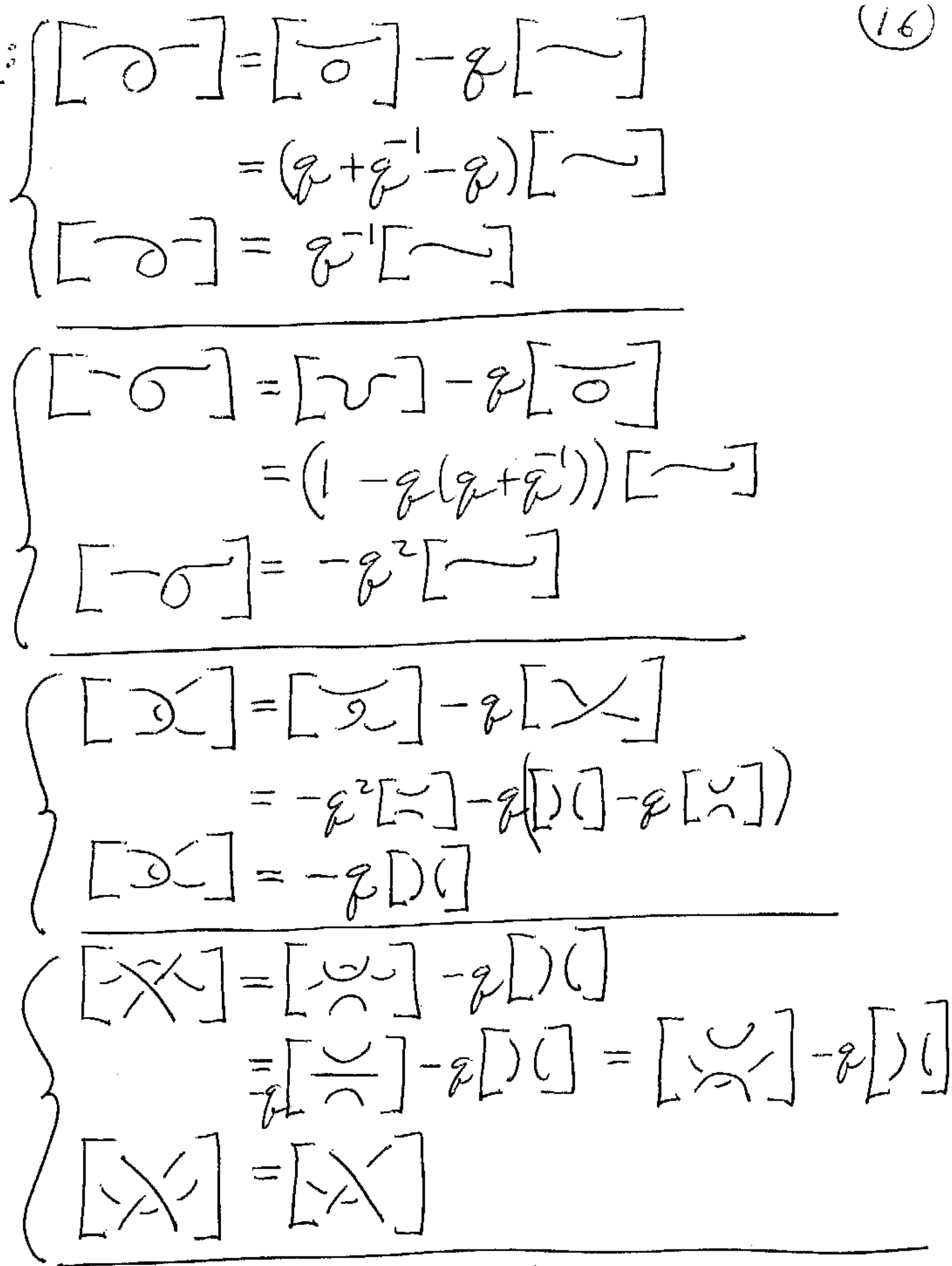
Exercise 2. Show that

{4Tu Relation on surface cobordisms} ⇒



Thus the tube-cutting relation and the 4Tu relation are equivalent in the context of the Khovanov Frobenius algebra.

Note:



The Khovanov Homology is invariant under Reidemeister moves with the <u>grading shifts</u> that correspond to the behaviour of [K].

5. Other Frobenius Algebras

Lee's Algebra

$$\left\{\begin{array}{l} k[x]/(x^2-1) = \mathcal{A} \\ x^2 = 1 \\ \Delta(1) = 1\otimes x + x\otimes 1 \\ \Delta(x) = x\otimes x + 1\otimes 1 \\ \varepsilon(x) = 1,\ \varepsilon(1) = 0 \end{array}\right\}$$

This also gives a link homology theory. Now the second grading j is not preserved. But

$$j(\partial\alpha) \geq j(\alpha)$$

for each chain α. This means that one can use j to filter the chain complex for Lee homology.

The result is a spectral sequence that starts from Khovanov homology and converges to Lee homology.

Lee homology is simple:

$$\dim \mathcal{L}ee^*(K) = 2^{\#comp(L)}$$

and behaves well under link concordance.

Rasmussen uses this relation to define invariants of links that give lowerbounds for the 4-ball genus & determine it for torus links.

More about Lee's algebra:

$$\boxed{\begin{array}{l} x^2=1, \quad \varepsilon(x)=1, \quad \varepsilon(1)=0. \\ \Delta(1)=1\otimes x+x\otimes 1 \\ \Delta(x)=x\otimes x+1\otimes 1 \end{array}}$$

Let $r=\frac{1+x}{2}$, $g=\frac{1-x}{2}$ (we are now over $\mathbb{Q}$.)

$$\Rightarrow \quad r^2=\tfrac{1}{4}(1+2x+x^2)=\frac{1+x}{2}=r$$

$$g^2=\tfrac{1}{4}(1-2x+x^2)=g$$

& $r+g=1$, $rg=0$

$$xr=r, \quad xg=-g.$$

So we can write

$$\Big| = \Big|_r \oplus \Big|_g$$

[See Dror Bar-Natan, The Karoubi Envelope and Lee's Degeneration of Khovanov Homology, math.GT/0606542]

$$\Rightarrow \ \big)\big(\ \simeq \ \big)_r\big(_r \oplus \big)_r\big(_g \oplus \big)_g\big(_r \oplus \big)_g\big(_g$$

$\Rightarrow$ (after a little work) Lee's homology has an up-to-homotopy vanishing differential.

"In a beautiful article, Eun Soo Lee introduced a second differential on the Khovanov complex of a knot (or link) and showed that the resulting (double) complex has non-interesting homology. ... this is a very interesting result — ..."

Proposition. $\dim \mathcal{L}_*(L) = 2^k$, $k = \#\,\mathrm{comp}(L)$.

Description: Given an orientation θ of L, take the "Seifert State" obtained by taking all oriented smoothings:

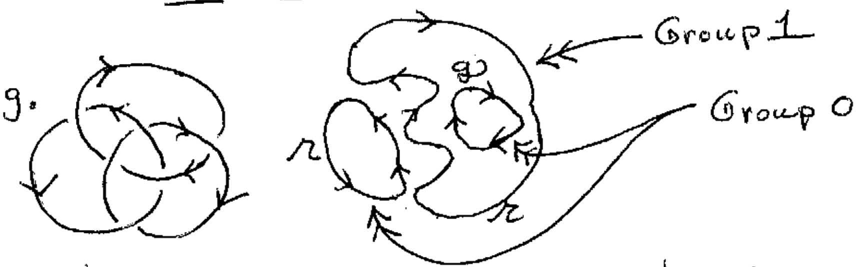
Divide loops into Group 0 and Group 1:

Group 0: counterclockwise and separated from unbounded region by an even # of circles.

or clockwise and separated from unbounded region by an odd # of circles.

Group 1: counterclockwise, odd.
or clockwise, even.

e.g.



Label each loop in Group 0 by r, and each loop in Group 1 by g.
This defines a cycle s_θ + a generator of homology.

$$\mathcal{L}_*(L) = \{ [s_\theta] \mid \theta \text{ is an orientation for } L \}$$

A more general Frobenius algebra yielding invariant link homology:

$$\mathcal{A}_{h,t}: \quad x^2 = hx + t1$$
$$\Delta(1) = 1\otimes x + x\otimes 1 - h(1\otimes 1)$$
$$\Delta(x) = x\otimes x + t(1\otimes 1)$$
$$\eta(1) = 1, \ \varepsilon(1) = 0, \ \varepsilon(x) = 1.$$

6. Remark on Spectral Sequences

It is useful, in learning about spectral sequences, to do the basic exercise about exact couples. An exact couple is a triangle:

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

(abelian groups, modules/R, R comm ring)

that is exact: $\operatorname{Ker} i = \operatorname{Image} k$
$\operatorname{Ker} j = \operatorname{Image} i$
$\operatorname{Ker} k = \operatorname{Image} j$.

Define: $\partial : E \longrightarrow E$, $\partial = j\circ k$ $\Big\}$ So $\partial^2 = \phi$.

$$E' = H(E) = \operatorname{Ker}\partial / \operatorname{Im}\partial.$$

$$D' = i(D), \quad i' : D' \to D', \ i' = i|_{D'}.$$

$$j'(i(x)) = [jx] = \text{homology class of } jx.$$

$$k'[z] = kz.$$

Show that
$$\begin{tikzcd} D' \arrow[rr, "i'"] & & D' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$
is also an exact Δ. Thus one can get an (infinite!) sequence of exact Δ's from a given exact Δ.

Given a filtration $C_*^{\ell} \subset C_*^{\ell+1}$ on a chain complex C_* one has long exact sequences:

$$\cdots \to H_{p+q+1}(C^p, C^{p-1}) \to H_{p+q}(C^{p-1}) \to$$
$$\to \underset{\substack{\| \\ D'_{pq}}}{H_{p+q}(C^p)} \to \underset{\substack{\| \\ E'_{pq}}}{H_{p+q}(C^p, C^{p-1})} \to \underset{\substack{\| \\ D'_{p-1q}}}{H_{p+q-1}(C^{p-1})} \to \cdots$$

This gives a basic exact couple corresponding to a filtration on C_*.

<u>Note</u>: $E'_{pq} \xrightarrow{k} D'_{p,q} \xrightarrow{j} E'_{p-1,q}$

$$\begin{matrix} E'_{pq} & & E'_{p-1,q} \\ \| & & \| \\ H_{p+q}(C^p, C^{p-1}) & \xrightarrow[\partial]{} & H_{p+q-1}(C^{p-1}, C^{p-2}) \end{matrix}$$

This shows how the homology at the beginning of the spectral sequence corresponds to the largest truncation of the filtration. Higher homology of the iteration sees longer and longer stretches of the filtration...

7. Rasmussen Invariant (uses spectral sequence from Khovanov to Lee.)

We have the j-grading on $C_*(K)$ for a diagram K and the fact that for Lee's algebra $j(\partial s) \geq j(s)$. Rasmussen uses a normalized version of this grading denoted by $q(s)$ (adjusted for invariance of the normalized Jones polynomial.)

Then one makes a filtration

$$F^k C_*(K) = \{ v \in C_*(K) \mid q(v) \geq k \}$$

and given $\alpha \in Lee_*(K) = \mathcal{L}_*(K)$ define

$$S(\alpha) \underset{\text{def}}{=} \max \{ q(v) \mid [v] = \alpha \}$$

$$s_{min}(K) \underset{\text{def}}{=} \min \{ S(\alpha) \mid \alpha \in \mathcal{L}_0(K), \alpha \neq 0 \}$$

$$s_{max}(K) \underset{\text{def}}{=} \max \{ S(\alpha) \mid \alpha \in \mathcal{L}_0(K), \alpha \neq 0 \}$$

$$\boxed{s(K) = \frac{s_{min}(K) + s_{max}(K)}{2}}$$

Facts: 0) $s_{max}(K) = s_{min}(K) + 2$ so $s(K) \in \mathbb{Z}$.

1) $s(K)$ is a concordance invar of K.
2) $s(K)$ is additive under connected sum.
3) $s(K^*) = -s(K)$
4) If K is a positive knot diagram (all $\nearrow$ crossings) then $s(K) = -r + n + 1$ where $r = \#$ of loops in canonical smoothing, $n = \#$ crossings.
5) $s(K_{pr}) = (p-1)(r-1)$ for K_{pr} a (p,r) torus knot.
6) $|s(K)| \leq 2g^*(K)$ where $g^*(K)$ is the least genus spanning surface for K in the four ball.
7) $g^*(K_{pr}) = \frac{(p-1)(r-1)}{2}$ (Milnor's Conjecture).

We shall stop here, with this introduction, and recommend the reader to the following papers.

1. L. Kauffman, New Invariants in Knot Theory, Amer. Math. Monthly (1987).
2. L. Kauffman, "Knots and Physics", World Sci. Pub. Co. (1991, 1994, 2001).
3. M. Khovanov, A categorification of the Jones polynomial. Duke Math. J. 101 (2000), n. 3, 359-426, math QA/9908171.
4. D. Bar-Natan, On Khovanov's categorification of the Jones polynomial, Algebr. Geom. Top. 2 (2002) 337-370, math QA/0201043.
5. D. Bar-Natan, Khovanov's homology for tangles and cobordisms, Geom. Topol. 9 (2005) 1443-1499. math.GT/0410495.
6. E.S. Lee, An endomorphism of the Khovanov invariant, Adv. Math. 197 (2), 554-586 (2005). math.GT/0210213.
7. J. Rasmussen, Khovanov homology and the slice genus, math.GT/0402131.
8. P. Turner, Five Lectures on Khovanov Homology, math.GT/0606464.