Math 313 — HW#3 — Selected Solutions

<43,44; 49,50, >

p43

2.2.1 (a) Given an $\varepsilon > 0$, show $\exists N$ s.t. $n > N \Rightarrow \frac{1}{6n^2+1} < \varepsilon$.

<u>Solution.</u>: Need $\frac{1}{\varepsilon} < 6n^2+1$

So $6n^2 > \frac{1}{\varepsilon} - 1$

$n^2 > \frac{1}{6\varepsilon} - \frac{1}{6}$

$n > \sqrt{\frac{1}{6}\left(\frac{1}{\varepsilon} - 1\right)}$.

Let N be any natural number $> \sqrt{\frac{1}{6}\left(\frac{1}{\varepsilon}-1\right)}$. //

b) omit. c) omit.

2.2.2

<u>Def.</u> (x_n) <u>verconges</u> to x if $\exists \varepsilon > 0$ $\forall N \in \mathbb{N}$ $n \geq N \Rightarrow |x_n - x| < \varepsilon$.

The statement $\underline{\forall N \in \mathbb{N}, n \geq N \Rightarrow |x_n - x| < \varepsilon}$ is equivalent to $\forall n \in \mathbb{N}$ $|x_n - x| < \varepsilon$.

i.e. $(|x_1 - x| < \varepsilon) \wedge (|x_2 - x| < \varepsilon) \wedge (|x_3 - x| < \varepsilon) \wedge \cdots$

So e.g. if $x_n = 117$ for all n and $x = 11117$, then $|x_n - x| = 11{,}000$ and so if $\varepsilon = 11{,}001$ then $|x_n - x| < \varepsilon$ for all n. Thus (x_n) is vergent to x.

2.2.3 (a) Need to exhibit a college where all students are < 7 ft tall.

(b) Need to exhibit a college where there is no professor who gives only A'S & B'S.

(c) Need to exhibit a college where there are ~~students~~ students of height less than 6 feet.

2.2.4

n:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
x_n:	1	0	1	0	0	1	0	0	0	1	0	0	0	0	1	...

Let $\varepsilon = \frac{1}{2}$. Then there does not exist an N s.t. $|x_n| < \varepsilon$ $\forall n > N$, (since there will be $n > N$ s.t. $|x_n| = 1$).

$\therefore$ (x_n) does not converge to 0.

[In answering the book's question:
$\varepsilon > 1 \Rightarrow$ there is a response N, but
$\varepsilon \leq 1 \Rightarrow$ there is no response N.]

2.2.5 $[[x]] = \lfloor x \rfloor$ = greatest integer less than or equal to x.

(a) $a_n = \lfloor 1/n \rfloor \Rightarrow$ $\lfloor a_1 \rfloor = \lfloor 1/1 \rfloor = 1$

$\lfloor a_2 \rfloor = \lfloor 1/2 \rfloor = 0$

$\lfloor a_3 \rfloor = \lfloor 1/3 \rfloor = 0$

'' $\lfloor a_n \rfloor = 0$ $\forall n > 1$.

$\Rightarrow \lim_{n\to\infty} a_n = 0$.

(b) The definition in (a) does <u>not</u> say that the sequence $\{1,0,2,0,3,0,4,0,5,0,\ldots\}$ converges to ∞. In fact it is oscillatory ~~but~~ has a subsequence $1, 2, 3, 4, \ldots$ that does converge to ∞.

2.2.8 (a) $(-1)^n$ is frequently in $\{1\}$.

(b) eventually $\Rightarrow$ frequently

(c) <u>Rephrasing of 2.2.3B</u>:

$\boxed{\text{A sequence } (a_n) \text{ converges to } a \text{ if, } a_n \text{ is eventually in any } \varepsilon\text{-nbhd } V_\varepsilon(a) \text{ of } a.}$

(d) If an infinite number of terms of (x_n) are equal to 2, then (x_n) is <u>frequently</u> in the interval $(1.9, 2.1)$. It is not neco eventually in that interval. e.g. $x_n = \begin{cases} 2 & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$

P49 | 2.3.1 | $a_n = a \ \forall n$

$|a_n - a| = 0 \ \forall n.$

Given $\varepsilon > 0$, Thus $\forall n \geq 1$, $|a_n - a| = 0 < \varepsilon$.

$\Longrightarrow a_n \to 0$ as $n \to \infty$.

2.3.2 | (a) $x_n \geq 0$ for all $n \in \mathbb{N}$.

Suppose $(x_n) \to 0$.

Then given $\varepsilon > 0$ $\exists N \in \mathbb{N}$, $n > N \Rightarrow |x_n| < \varepsilon$.

$\Rightarrow 0 \leq x_n < \varepsilon \Rightarrow 0 \leq \sqrt{x_n} < \sqrt{\varepsilon}$.

Thus we prove that $(\sqrt{x_n}) \longrightarrow 0$ as follows: Given $\varepsilon > 0$, let $N \in \mathbb{N}$ s.t. $\forall n > N$, $|x_n| < \varepsilon^2$ (this can be done since $(x_n) \to 0$). Then $0 \leq x_n < \varepsilon^2 \ \forall$ ∴ $0 \leq \sqrt{x_n} < \varepsilon$. So $|\sqrt{x_n}| < \varepsilon \ \forall n > N$. //

(b) Lets first generalize (a). Suppose (x_n) is any sequence of real numbers (not nec ≥ 0) and suppose $(x_n) \longrightarrow a \in \mathbb{R}$. We will prove that $(x_n - a) \longrightarrow 0$. (Proof is obvious). Hence by (a)

$(\sqrt{|x_n - a|}) \longrightarrow 0.$

(b) (continued) (We can assume $x > 0$.)

We know $(x_n) \longrightarrow (x)$. $\qquad V_{\sqrt{x}\varepsilon-\varepsilon^2}(x)$ //

∴ Given $\varepsilon > 0$, $\exists N$ s.t. $n > N$

$$\Longrightarrow \quad x_n \in \left(x - (2\sqrt{x}\varepsilon\ [\text{illegible}]\ \varepsilon^2),\ x + (2\sqrt{x}\varepsilon\ [\text{illegible}]\ \varepsilon^2)\right)$$

(we can assume ε chosen small enough so this is a positive open interval)

But $V_{\sqrt{x}\varepsilon-\varepsilon^2}(x) = \left(x - 2\sqrt{x}\varepsilon + \cancel{\varepsilon}^2,\ x + 2\sqrt{x}\varepsilon - \cancel{\varepsilon}^2\right)$

$$\cap \|$$

$$x_n \in \left(x - 2\sqrt{x}\varepsilon + \varepsilon^2,\ x + 2\sqrt{x}\varepsilon + \varepsilon^2\right)$$

$$\Downarrow$$

$$\sqrt{x_n} \in \left(\sqrt{x - 2\sqrt{x}\varepsilon + \varepsilon^2},\ \sqrt{x + 2\sqrt{x}\varepsilon + \varepsilon^2}\right)$$

$$\|$$

$$\left((\sqrt{x} - \varepsilon),\ (\sqrt{x} + \varepsilon)\right)$$

Thus we have shown ~~that~~
for $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $n > N$

$$\Longrightarrow \sqrt{x_n} \in (\sqrt{x} - \varepsilon, \sqrt{x} + \varepsilon).$$

This proves $\sqrt{x_n} \longrightarrow x$. //

2.3.3 omitted.

2.3.4 omitted.

2.3.5 omitted.

2.3.6 (a) omitted.

(b) If $|b_n| \to |b|$ it does not follow that $b_n \to b$.
e.g. $b_n = (-1)^n$, $b = 1$.

2.3.7 The algebraic limit theorem asks to know that both a_n & b_n are convergent. But here we can show that if (a_n) bounded & $\lim_{n\to\infty} b_n = 0 \Rightarrow \lim_{n\to\infty}(a_n b_n) = 0$.

Proof omitted.

2.3.8 (a) ~~[illegible]~~ $x_n = n$, $y_n = -n$
$x_n + y_n = 0$.
(b) not possible
(c) $b_n = 1$ for all n.
(d) not possible
(e) $\{1/n\} = (a_n)$, $(b_n) = ((-1)^n)$.
$(a_n b_n) = \left(\frac{(-1)^n}{n}\right)$. $a_n \to 0$, $a_n b_n \to 0$, b_n does not converge.

2.3.9 A convergent seq (x_n) satis $x_n > 0$ for all $n \in \mathbb{N}$. Suppose $x_n \to x$. Then you can conclude that $x \geq 0$.

Suppose $a_n \leq b_n$ $\forall n$ + $a_n \to a$
$b_n \to b$.
The best you can conclude is $a \leq b$.

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etc.

2.3.10 omit

2.3.11 omit proof

Give example when $y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$ converges but x_n does not.

Let $x_n = (-1)^n$.

$$y_1 = x_1/1 = -1$$
$$y_2 = \frac{-1+1}{2} = 0$$
$$y_3 = \frac{-1+1-1}{3} = -1/3$$
$$y_4 = \frac{-1+1-1+1}{4} = 0$$
$$y_5 = \frac{-1+1-1+1-1}{5} = -1/5$$
$$\cdots$$

$$y_n = \begin{cases} -1/n, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$y_n \to 0$ + x_n does not converge

2.3.12 omit