Regular Isotopy and Generalized Polynomials for Knots and Links

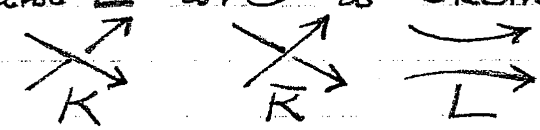
by Louis H. Kauffman

During the summer of 1984 a new invariant appeared in knot theory. The invariant emerged in two stages. In stage 1 Vaughan Jones reported his research a Laurent polynomial invariant of oriented knots and links K, $V_K(t)$. The Jones polynomial $V_K(t)$ is defined as a normalized trace function on certain representations of braid groups.

It is very powerful, and quite good at distinguishing many knots from their mirror images. On top of these properties, the Jones polynomial satisfies a Conway type identity of the form:

$$t V_K - t^{-1} V_{\overline{K}} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_L$$

where K, $\overline{K}$ and L are a skein triple: $K = \overline{K} \oplus L$:



This identity, along with topological invariance and the normalization $V_{\bigcirc} = 1$, allows recursive calculation of the Jones polynomial just as can be done for the Conway polynomial — without recourse to the trace definition.

Very soon after Jones spoke on his work, a number of mathematicians saw how to create a two-variable generalization of both the Jones and Conway polynomials. In our notation, the generalized polynomial will be denoted P_K. It is a Laurent polynomial in two variables: α, z. The generalized polynomial satisfies an exchange identity in the form: $\alpha P_K - \bar{\alpha} P_{\bar{K}} = z P_L$ (see axioms below). It was identified independently by Freyd-Yetter [ ], Hoste [ ], Lickorish-Millet [ ], and Ocneanu [ ]. Of these authors, Freyd and Yetter use an induction via braids; Hoste, Lickorish and Millet use induction in the category of knots and links; Oceanu generalizes Jones' representation theory.

To date a number of other papers have appeared on these invariants including [ ] and [ ]. **This paper will be based on an approach of our own.** We separate (partially) the roles of the variables α and z by writing

$$P_K = \Gamma_K / \alpha^{w(K)}$$

where $w(K)$ is the <u>writhe</u> of a diagram for K and $\Gamma_K(\alpha, z)$ is a <u>Laurent</u> polynomial invariant of <u>regular</u> <u>isotopy</u> of diagrams. (See section 1.) In this formulation, we have the simpler exchange identity $\Gamma_{\times} - \Gamma_{\times} = z\Gamma_{\rightrightarrows}$. The α's come in at the bottom of the skein decomposition.

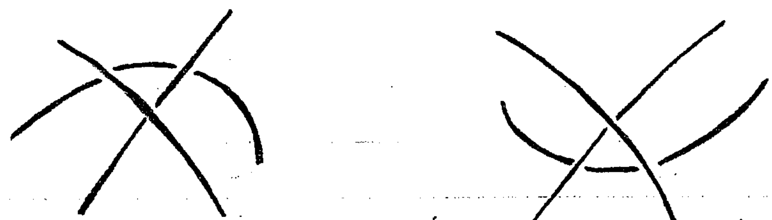
The **paper** is organized as follows: In section 1 we give axioms for Γ_K and P_K and do sample calculations. In section 2 we prove by induction that Γ_K (hence P_K) is well-defined. In section 3 we give a sketch of the relationship to representations of braid groups.

1. Regular Isotopy and Generalized Polynomial

Definition . Two (oriented) link diagrams are **regularly isotopic** ($\approx$) if one can be obtained from the other by a finite sequence of Reidemeister moves of type II and type III.

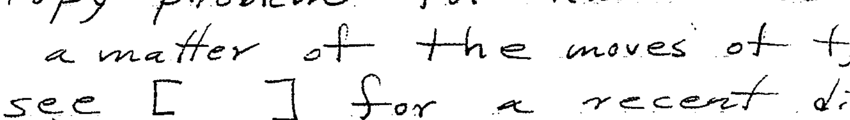
Recall that moves of types II and III are represented as in the figure below:

II. 

III. 

We have chosen the term regular isotopy because a regular isotopy of link diagrams projects to a regular **homotopy** of the corresponding plane curve **immersions** (see [ ])

Recall also that two links are ambient isotopic ($\sim$) if one may be obtained from the other via moves of type I, II, and III where type I is exemplified by

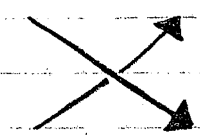
I.  .

Since moves of type I can be "saved-up" until the end of a sequence, the ambient isotopy problem for knots and links is primarily a matter of the moves of type II and III (see [ ] for a recent discussion of this point). Thus it makes sense to distinguish regular isotopy from ambient isotopy.

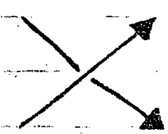
Since regular isotopies project to regular homotopies we see that the <u>Whitney degree</u>, $d(K)$, of the plane curve immersion corresponding to the diagram K must be an invariant of regular isotopy. The Whitney degree is the total turn of a tangent vector to the immersion. For our purposes it may be defined as the sum $d(K) = \sum |C|$ where C runs over the Seifert circles of the diagram, and $|C| = +1$ or -1 according as the Seifert circle is oriented counter-clockwise or clockwise, respectively. (See [ ].)

A more specifically knot-theoretic invariant of regular isotopy is the <u>writhe</u>, $w(K)$. This is defined as follows:

<u>Definition</u> . Define <u>crossing</u> <u>signs</u> $\varepsilon = \pm 1$ as shown below:



$\varepsilon = +1$



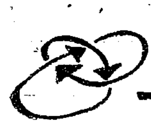
$\varepsilon = -1$.

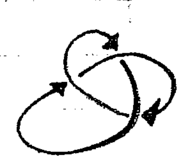
The writhe, $w(K)$, of a diagram K is given by the formula

$$w(K) = \sum_p \varepsilon(p)$$

where p runs over all crossings of K.

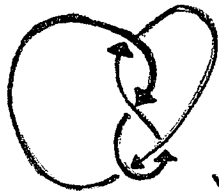
<u>Example</u> . Let T denote the trefoil, and $T^!$ its mirror image (obtained by reversing all the crossings). Then $w(T) = +3$ while $w(T^!) = -3$.

 T

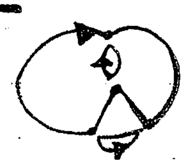
 T!

Thus we see that the trefoil is **not regularly** isotopic to its **mirror image**.

Example ___. Let E denote the figure-eight knot as shown below:



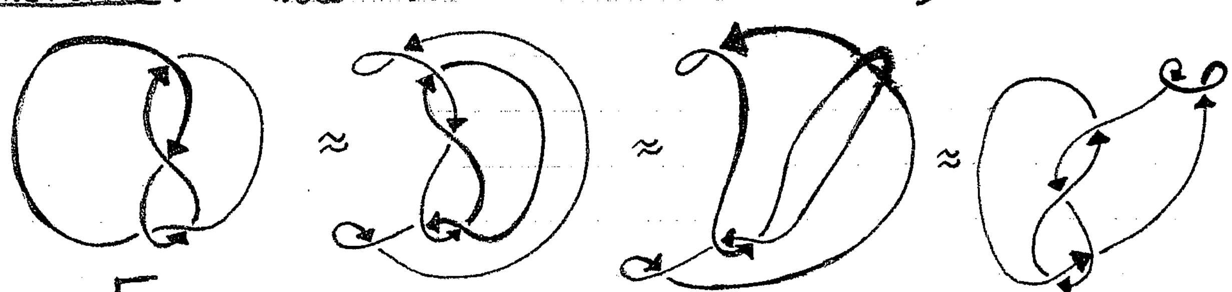
E . $W(E)=0.$

$\Big[$  $\Rightarrow d(E) = -1-1+1 = -1 \Big]$

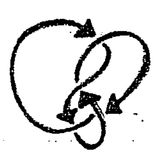
As is well-known, the figure-eight is ambient isotopic to its mirror image. If we try to turn this ambient isotopy into **a regular** homotopy, then a typical attempt will **appear as in Figure 20.5.** The **regular isotopy takes E to F** where F is the **mirror image of E**, with reversed orientation and an added "**curl**" of form $\leftarrow\!\!\circ\!\!-$.
This curl compensates the difference in Whitney degree between E and $\hat{E}$ where $\hat{E}$ denotes $E^!$ with reversed orientation.
Since $d(E) = -1$ and $d(\hat{E}) = +1$, we know that E **and** $\hat{E}$ **are** not regularly isotopic.

<u>Question</u>. **Is E regularly** isotopic to $E^!$?

<u>Exercise</u>: Show that the answer is yes!



E F

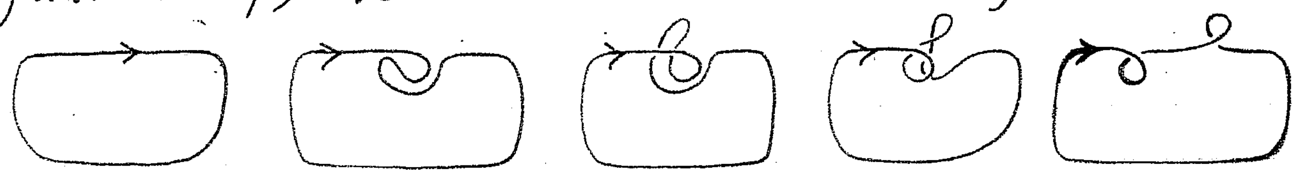


$E^!$

<u>Is $E \approx E^!$??</u>

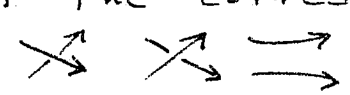
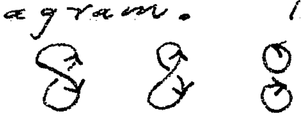
<u>Figure</u>

Example ___. Two unknotted diagrams $\mathcal{U}$ and $\mathcal{U}'$ (i.e. $\mathcal{U}$ and $\mathcal{U}'$ are ambient isotopic to $\bigcirc$) are regularly isotopic if and only if $d(\mathcal{U}) = d(\mathcal{U}')$ and $w(\mathcal{U}) = w(\mathcal{U}')$. We extend this as an exercise for the reader. Note the regular isotopy version of the Whitney trick:



We now give axioms for a Laurent polynomial in two variables, $\Gamma_K(\alpha, z)$ that is an invariant of regular isotopy. The gamma polynomial is, as we shall see, a direct relative of the generalized polynomial P_K of Freyd-Yetter, Hoste, Jones, Lickorish-Millet and Ocneanu. The Γ-polynomial has some conceptual advantages in that it partially separates the roles of α and z, becomes an ambient isotopy invariant for an associated knotted-twisted-band, and it gives a geometric interpretation of the variable α in terms of writhing.

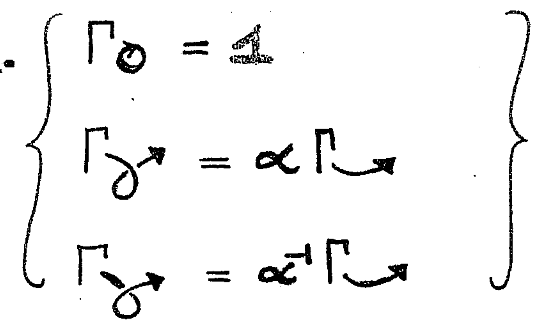
Definition/Axioms ___. In the following axioms a partial diagram denotes a full diagram containing the indicated partial. Change of partial diagram induces change of the corresponding full diagram. Thus

 can mean .

This convention will be followed throughout.

Axiom 1. To each oriented knot or link diagram K there is associated a two-variable Laurent polynomial $\Gamma_K(\alpha, z) \in \mathbb{Z}[\alpha, \alpha^{-1}, z, z^{-1}]$. This Γ-polynomial is an invariant of regular isotopy: $K \approx K' \Rightarrow \Gamma_K = \Gamma_{K'}$.

Axiom 2.



Axiom 3.

As the axioms indicate, the variable α picks up information about writhing, while the z is in structural analogy to the Conway polynomial $\nabla_K(z)$. In fact, $\nabla_K(z) = \Gamma_K(1, z)$. Just as in the case of the Conway polynomial, these axioms are sufficient for recursive calculation. While we will prove inductively that they are consistent, there is, as yet, no known model for Γ that generalizes the Seifert pairing [ ] or states [ ] models for the Conway polynomial.

Definition . The generalized polynomial $P_K(\alpha, z)$ is defined by the formula

$$P_K = \Gamma_K / \alpha^{w(K)} .$$

Thus, by axiom 3 and the definition of the writhe $W(K)$, P_K is an invariant of ambient isotopy satisfying the exchange identity:

$$\alpha P_{\times} - \alpha^{-1} P_{\times} = z P_{\rightrightarrows} .$$

Other authors have used different labellings and signs for these variables. The translations are easy. For example, the original Jones polynomial $V_K(t)$ satisfies

$$t V_{\times} - t^{-1} V_{\times} = (\sqrt{t} - 1/\sqrt{t}) V_{\rightrightarrows} .$$

Thus $V_K(t) = P_K(t, \sqrt{t} - 1/\sqrt{t})$.

We now give some sample calculations:

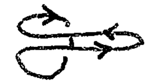
<u>Example</u> . $\mathcal{U}_1 = \bigcirc$, $\mathcal{U}_2 = \bigcirc\bigcirc$, $\mathcal{U}_3 = \bigcirc\bigcirc\bigcirc$, ..

We assert that

$$\Gamma_{\mathcal{U}_n} = \delta^{n-1}$$

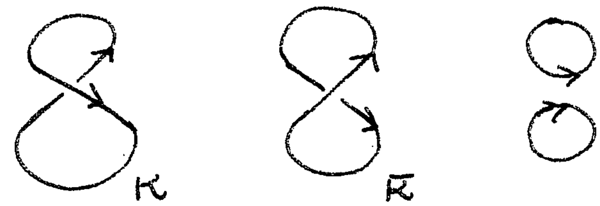
where $\delta = (\alpha - \alpha^{-1})/z$. Here $\mathcal{U}_n$ is an unlink diagram of n components with zero writhe. The value δ^{n-1} does not depend on the orientations of these components.

Since by axiom 2, $\Gamma_{\circlearrowright} = 1$, our first task is to compute $\Gamma_{\circlearrowright\circlearrowright}$:

$\circlearrowright \;\approx$ 

Thus, by axiom 2, $1 = \Gamma_{\circlearrowright} = \Gamma_{\infty} = \alpha\alpha^{-1}\Gamma_{\circlearrowright} = \Gamma_{\circlearrowright}$.

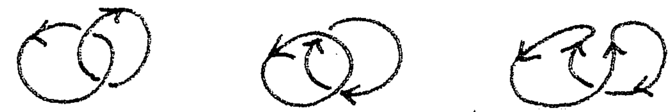
Next:



$K \qquad \bar{K} \qquad L$

Hence $\Gamma_K - \Gamma_{\bar{K}} = z\Gamma_L$ and $\Gamma_K = \alpha$, $\Gamma_{\bar{K}} = \alpha^{-1}$. Thus $\alpha - \alpha^{-1} = z\Gamma_L$. Therefore $\Gamma_L = (\alpha - \alpha^{-1})/z = \delta$. Similar calculations check that $\Gamma_{\mathcal{U}_n} = \delta^{n-1}$.

Example 9.

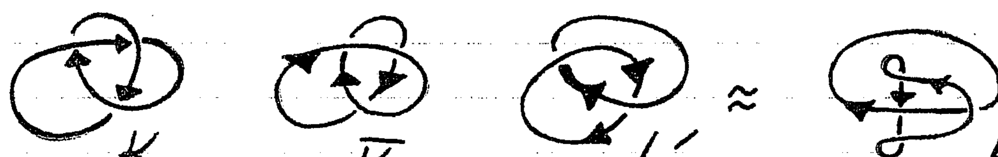


$L \qquad \bar{L} \qquad \mathcal{U}$

$$\Gamma_L = \Gamma_{\bar{L}} + z\Gamma_{\mathcal{U}} = \delta + z\alpha = (\alpha - \alpha^{-1})z^{-1} + z\alpha .$$

Note that $P_L = \alpha^{-w(L)}\Gamma_L = \alpha^{-2}\Gamma_L$.

Example .



$K \qquad \bar{K} \qquad L' \approx L''$

$$\Gamma_{L'} = \Gamma_{L''} = \Gamma_L \text{ (above)}$$

$$\therefore \Gamma_K = \Gamma_{\bar{K}} + z\Gamma_L$$

$$= \alpha + z(\alpha - \alpha^{-1})z^{-1} + z^2\alpha$$

$$\therefore \Gamma_K = (2\alpha - \alpha^{-1}) + z^2\alpha$$

$$\therefore P_K = \alpha^{-3}\Gamma_K = (2\alpha^{-1} - \alpha^{-4}) + z^2\alpha^{-2} .$$

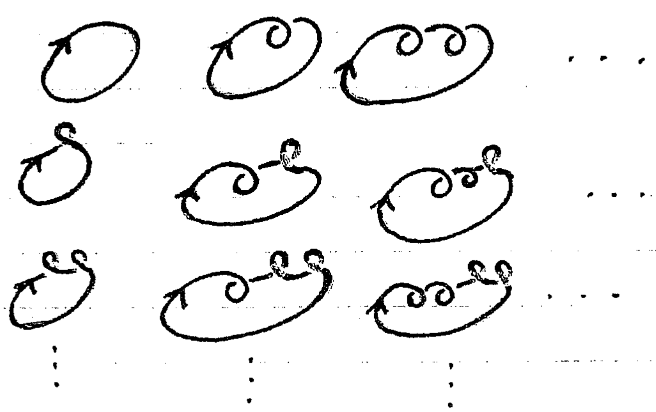
Note that if we had started with the mirror image $K^!$, then all the writhing calculations would be replaced by their negatives, and the exchange identities would be reversed. Hence

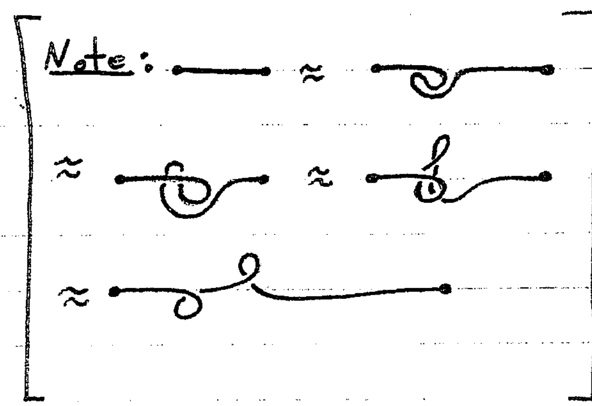
$$\Gamma_{K^!} = \Gamma_K(-\alpha^{-1}, z), \text{ and}$$

$$P_{K^!} = P_K(-\alpha^{-1}, z) \text{ as well.}$$

Since, in this case, P_K is not symmetric under the substitution $\alpha \mapsto -\bar{\alpha}^{1}$ we conclude that <u>the trefoil is not ambient isotopic to its mirror image</u>. This is an example of the power of the generalized polynomial.

Since by these examples, we know how to compute Γ on unknot and unlink diagrams. We see that the "skein" for regular isotopy, and the Γ-**polynomial** is generated by unlinks formed from the unknots with curls:





The curls just contribute various powers of α in the computations. It is then easy to generalize the known skein theory for the Conway polynomial. (See [ ] for a statement of specific results for the generalized polynomial.)

Just as in the trefoil example, we have: (The proof is omitted.)

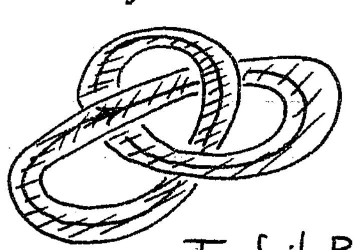
<u>Proposition</u> . Let $K!$ denote the oriented mirror image of the link K. Then
$\Gamma_{K!}(\alpha, z) = \Gamma_K(-\bar{\alpha}^{1}, z)$ and $P_{K!}(\alpha,z) = P_K(-\bar{\alpha}^{1}, z)$.

Knotted, Twisted Bands

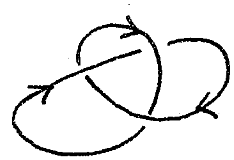
The Γ-polynomial can be construed as an ambient isotopy invariant of a <u>band</u> embedded in $\mathbb{R}^3$.

Definition ___. A <u>band</u> $B \subset \mathbb{R}^3$ is an embedding of an annulus $A = S^1 \times [0,1]$. This embedding a knot that is the image of $S^1 \times [\frac{1}{2}]$ and will be referred to as the <u>core</u> of the band. It also determines a link of two components that is the boundary of the band. This link consists of two (parallel with respect to the band) copies of the core.

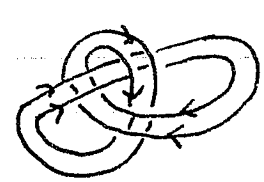
We orient a band by choosing an orientation for the core. The two components of the associated link are given orientations parallel to the core. (Thus they donot take orientation induced on the boundary of band as oriented surface).



<u>Trefoil Band B</u>



<u>K=Core B</u>

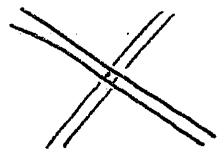


<u>L = Link B</u>

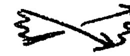
We let $C(B)$ and $L(B)$ denote the core and associated link of the band B.

It is well-known that a band $B \subset \mathbb{R}^3$ is determined up to ambient isotopy by the ambient isotopy class of its core and the <u>linking</u> <u>number</u> $lk(L(B))$ of its associated link.

A planar diagram for a band B is <u>flat</u> if its singularities are all of the type



.

Thus the trefoil band above is flat. In general, any band B has a planar diagram such that the singularities are crossings as above and <u>twists</u>  ,  . It is easy to see that the linking number of $L(B)$ is equal to the <u>sum of the twist signs</u>, $T(B)$, plus the writhe of the core of B:

$$\ell k(L(B)) = T(B) + w(C(B)).$$

This formula has differential gemetric versions for non-planar diagrams. (See [ ]).

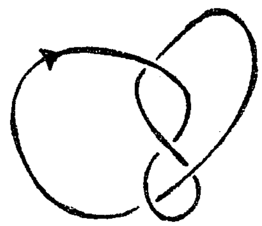
Now we know that $\alpha^{-w(CB)}\Gamma_{CB} = P_{CB}$ is an ambient isotopy invariant of CB and hence of B. Therefore

$$\alpha^{T(B)-\ell k(L(B))}\Gamma_{CB}$$

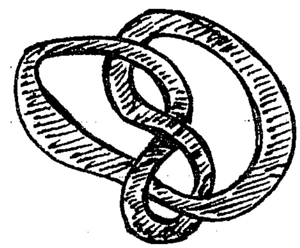
is an ambient isotopy inveriant of B. If B is flat, then $T(B) = 0$. The linking number $\ell k(L(B))$ is an ambient isotopy invariant of B. We have proved:

<u>Proposition</u> __. Let B be a band (diagram) then $\alpha^{T(B)}\Gamma_{CB}$ is an ambient isotopy invariant of B. In particular, if B is a flat diagram, then the Γ-polynomial itself, Γ_{CB}, is an ambient isotopy invariant of the band B.

Turning this around, let $B(K)$ denote the band (flat band) obtained from K by taking a parallel copy of K. Then Γ_K is an ambient isotopy invariant of $B(K)$.



K.

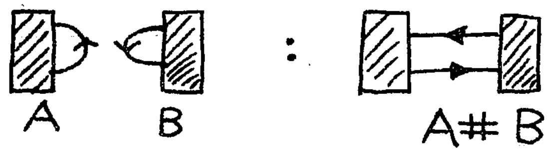


$B(K)$.

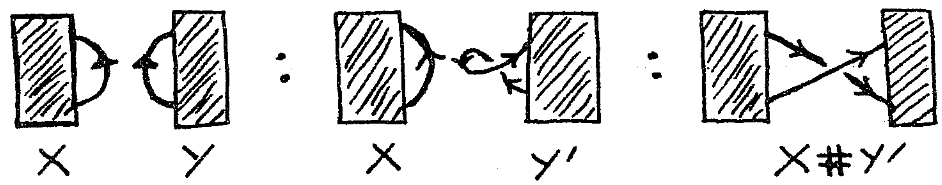
This point of view provides a geometric interpretation for the extra variable α, as a measure of twisting in the band.

2. Inductive Analysis of the Γ-Polynomial

A little extra care is needed for dealing with connected sums in the regular isotopy category:



Example



Here a connected sum is formed between X and Y'. The result is not regularly isotopic to a connected sum formed without the extra twist.

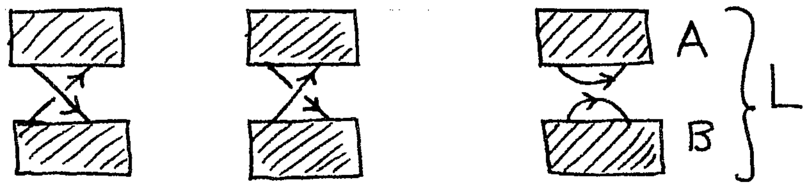
The following proposition follows via the inductive calculations implicit in the axioms (and our indication of skein generators in section 1):

Proposition ___. Let A and B be link diagrams and $A\#B$ a connected sum obtained directly from these diagrams, then
$$\Gamma_{A\#B} = \Gamma_A \Gamma_B .$$

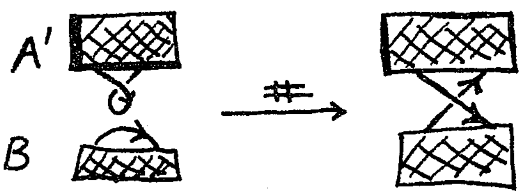
The proof is omitted. As an immediate corollary we have:

Corollary ___. Let $L = A \cup B$ be a split link whose diagram falls into the pieces A and B. Then
$$\Gamma_L = \delta\, \Gamma_A \Gamma_B$$
where $\delta = (\alpha - \alpha^{-1})/z$.

Proof. We assume the pieces can be configured as shown below. The other case is handled with an extra twist (by the reader.).

(i) 

K $\bar{K}$ L

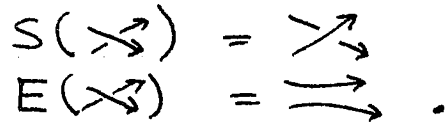
(ii) 

Thus $\Gamma_K = \Gamma_{A'\#B} = \Gamma_{A'}\Gamma_B = \alpha\Gamma_A\Gamma_B$. Similarly, $\Gamma_{\bar{K}} = \alpha^{-1}\Gamma_A\Gamma_B$. Then by (i) $\Gamma_K - \Gamma_{\bar{K}} = z\Gamma_L$. Hence $(\alpha - \alpha^{-1})\Gamma_A\Gamma_B = z\Gamma_L$. This completes the proof. //

The rest of this discussion will proceed as follows. We first develop some systematic notation for indicating recursive calculations. Then we turn the tables and make a recursive prescription for calculating Γ_K. We then show that this prescription is independent of the choices involved and that it gives an ivariant of regular isotopy that satisfies the axioms. This will complete the logical development of the Γ-polynomial.

Let $\varepsilon_i(K)$ denote the sign of the ith crossing of K. Thus, for example, the writhe may be written as $w(K) = \sum_{i=1}^{n} \varepsilon_i(K)$ where $i = 1, \ldots, n$ indexes all crossings of K.

Let $S_i(K)$ denote the result of switching the i^{th} crossing of K, and $E_i(K)$ the result of splicing this crossing:



Using this notation, the exchange identity becomes $\Gamma_K - \Gamma_{S_iK} = \varepsilon_i(K) z \, \Gamma_{E_iK}$.

In order to calculate the Γ-polynomial, we choose a sequence of crossings $\{1, \ldots, n\}$ so that $S_n S_{n-1} \cdots S_1 K = K'$ is either unknotted or a split link. This can always be done. Repeated application of the exchange identity yields the formula:

$$(*) \quad \Gamma_K = \Gamma_{K'} + \sum_{i=1}^{n} \varepsilon_i(K) \Gamma_{\mathcal{D}_i K}$$

where $\mathcal{D}_i K = E_i S_{i-1} \cdots S_1 K$. Using $\Gamma_u = \alpha^{w(u)}$ when u is unknotted, and $\Gamma_{A \cup B} = \delta \Gamma_A \Gamma_B$, this gives an inductive procedure to calculate Γ.

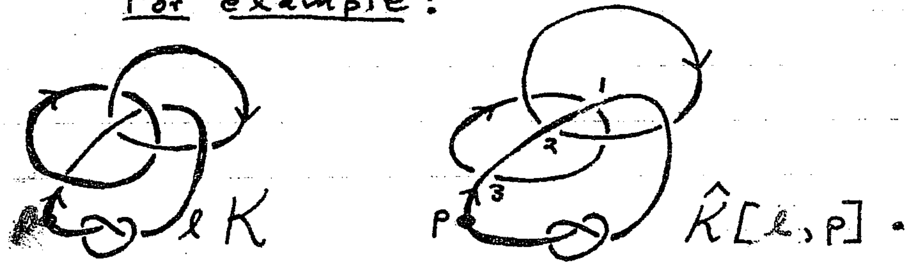
We systematize the unknotting or unlinking sequence by

(i) When K has a single component diagram, choose a basepoint p on this diagram (p is not a crossing). Form the <u>reference knot-diagram</u> $\widehat{K}(p)$ obtained by overcrossing in the diagram for K at every first encounter with a crossing in the (oriented) trip along the diagram from p. (See Figure 20.6 for an example.) The reference diagram is unknotted, and we obtain a switching sequence from K to $\widehat{K}$

by traversing either diagram from the basepoint, and noting those crossings that differ (are switched) between K and $\hat{K}$. Label these crossings $n, n-1, \ldots, 1$ in order from the basepoint. Then
$\hat{K} = S_n S_{n-1} \cdots S_1 K$.

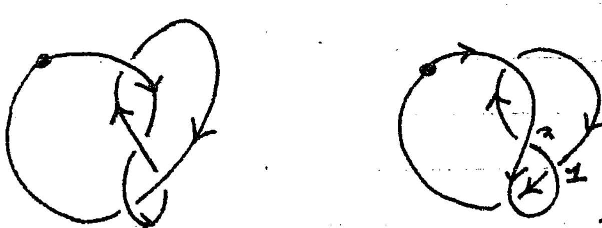
(ii) If K is a link, then the same procedure may be applied to any (ℓ^{th}) component of K, producing $\hat{K}[\ell]$, with this component lifted above all the other components in the diagram. In this case we switch only those crossings that constitute undercrossings between the ℓ^{th} component of K and the other components.

For example:



We may use $\hat{K}[\ell, p]$ to denote the result of lifting the ℓ^{th} component, and choosing an order of switching relative to a base-point p. In the example above,
$\hat{K} = S_3 S_2 S_1 K$.

Calculations based upon the above procedures appear to depend upon the choices of basepoint and (in the case of links) lifting component. We can define away the choice of lifting component by adding up the results from each choice, and dividing by the number of components. A special case of this arises in the classical definition of linking number: $lk(A,B) = \frac{1}{2} \sum_{p \in A \sqcap B} \varepsilon(p)$ where $A \sqcap B$ denotes all crossings between A and B: Half of the members of $A \sqcap B$ are the crossings switched when lifting A above B; the other half are the crossings switched when lifting B above A. By defining the linking number as above, it is easy to see that it is a well-defined invariant, and to prove subsequently that it can be computed from one lift sequence or the other. Thus <u>linking number is a special case of the</u> general argument. (In fact if $lk(K)$ denotes the linking number of K when K has two components, and 0 otherwise, then $lk(K)$ is the coefficient of z in the Conway polynomial $\nabla_K(z) = \Gamma_K(1,z)$.)



K $\qquad$ $\hat{K}$

$\hat{K} = S_2 S_1 K$

<u>Figure ___ . Knot K, Reference Knot $\hat{K}$</u>

We are now in a position to state an induction hypothesis through which we can verify that Γ_K is well-defined. First we define $\Gamma_K = \delta \Gamma_A \Gamma_B$ if K (as a diagram) is a disjoint union of diagrams A and B ($\delta = z^{-1}(\alpha - \alpha^{-1})$). Here it is assumed that Γ_A and Γ_B have been previously defined.

If one diagram X lies completely over (all over-crossings) another diagram Y, then X will be said to overlay Y.

We also carry the following part of the axioms into the inductive assumption:

1. $\Gamma_{\bigcirc} = 1$.
2. $\Gamma_{\times} - \Gamma_{\times} = z\,\Gamma_{\rightrightarrows}$.

It is to be proved that whenever Γ is defined, it satisfies these properties.

We are now ready to make the formula (*) into a definition. Note that nowhere in the inductive assumptions do we take any mention of topological invariance. We will prove that Γ is an invariant of regular isotopy <u>after</u> it is defined for all links.

<u>**Definition 20.18**</u>. Suppose that Γ has been defined for all oriented link diagrams with $\leq n$ crossings. Let K be a diagram with n crossings.

1. If K has one component, let p be a basepoint on K and $\hat{K}$ the reference unknot constructed from p. Let $S_n S_{n-1} \cdots S_1 K = \hat{K}$ be the corresponding switching sequence.

Define Γ_K by the formula

$$\Gamma_K = \alpha^{w(\hat{K})} + z\sum_{i=1}^{n} \varepsilon_i(K)\,\Gamma_{\mathcal{D}_i K}$$

where $\mathcal{D}_i = E_i S_{i-1} \cdots S_1$.

(2) If $K = K_1 \cup \cdots \cup K_t$ has t components, let L_s be the overlay obtained by switching crossings on K_s so that the switched K_s overlies all other components. If P_s is a base-point on K_s giving rise to a switching sequence $L_s = S_n S_{n-1} \cdots S_1 K$, define

$$\mathcal{G}_s(K) = \delta\,\Gamma_{K_s}\,\Gamma_{\widetilde{K}_s} + z\sum_{i=1}^{n} \varepsilon_i(K)\,\Gamma_{\mathcal{D}_i K}$$

where $\widetilde{K}_s$ is the diagram obtained from K by deleting the component K_s.

Then define Γ_K by the formula

$$\Gamma_K = \frac{1}{t}\sum_{s=1}^{t} \mathcal{G}_s(K) \quad .$$

It remains to prove that <u>the resulting values of Γ_K are independent of the choice of basepoint(s), and they continue to satisfy the induction hypothesis.</u> Call this statement the <u>Inductive Step</u>.

<u>Proof of the **Inductive** Step</u>. We first verify independence of the basepoint in the case where K is a link. Here the inductive definition of Γ_K is given in (2) above. We must show that $\mathcal{G}_s(K)$ (see above) does not depend on the order of the switching sequence

$S_n S_{n-1} \cdots S_1$. In particular, it suffices to show that $G_s(K)$ is invariant under a cyclic permutation of this ordering, since this is the effect of sliding the basepoint past a crossing in the sequence. Since K_s and $\tilde{K}_s$ depend on the choice of component only, it suffices to show that

$$\Delta = \sum_{i=1}^{n} \varepsilon_i(K)\, \Gamma_{D_i K}$$

is independent of cyclic permution of $n, n-1, \ldots, 1$. Accordingly, let Δ' be the corresponding expression for the order $1, n, n-1, \ldots, 2$. Then

$$\Delta = \varepsilon_1 \Gamma_{E_1 K} + \varepsilon_2 \Gamma_{E_2 S_1 K} + \varepsilon_3 \Gamma_{E_3 S_2 S_1 K} + \cdots + \varepsilon_{n-1} \Gamma_{E_{n-1} S_{n-2} \cdots S_1 K} + \varepsilon_n \Gamma_{E_n S_{n-1} \cdots S_1 K}.$$

$$\Delta' = \varepsilon_2 \Gamma_{E_2 K} + \varepsilon_3 \Gamma_{E_3 S_2 K} + \cdots + \varepsilon_n \Gamma_{E_n S_{n-1} \cdots S_2 K} + \varepsilon_1 \Gamma_{E_1 S_n S_{n-1} \cdots S_2 K}$$

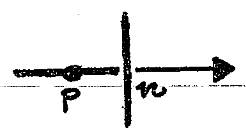
Thus

$$\begin{aligned}\Delta' - \Delta = {} & \varepsilon_1 \left[\Gamma_{S_n S_{n-1} \cdots S_2 E_1 K} - \Gamma_{E_1 K} \right] \\ & + \varepsilon_2 \left(\Gamma_{E_2 K} - \Gamma_{S_1 E_2 K} \right) \\ & + \varepsilon_3 \left(\Gamma_{E_3 S_2 K} - \Gamma_{S_1 E_3 S_2 K} \right) \\ & + \cdots \\ & + \varepsilon_n \left(\Gamma_{E_n S_{n-1} \cdots S_2 K} - \Gamma_{S_1 E_n S_{n-1} \cdots S_2 K} \right) \\ = {} & \varepsilon_1 \left[\Gamma_{S_n S_{n-1} \cdots S_2 E_1 K} - \Gamma_{E_1 K} \right] \\ & + \varepsilon_2 \varepsilon_1 \Gamma_{E_2 E_1 K} \\ & + \varepsilon_3 \varepsilon_1 \Gamma_{E_3 S_2 E_1 K} \\ & + \cdots \\ & + \varepsilon_n \varepsilon_1 \Gamma_{E_n S_{n-1} \cdots S_2 E_1 K}\end{aligned}$$

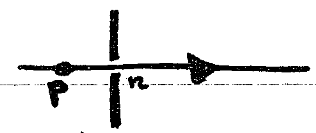
$\therefore \Delta' - \Delta = 0$ by induction. Here we have used the inductive definition of Γ, and the inductive order-independence of operations. This completes the inductive step showing that ΓK is well-defined when K is a link.

We now assume that K is a knot. Now the switching sequence is more crucial since it serves to undo the knot, and it can change when we slide the basepoint. Two cases will be distinguished:

Case 1.



K



$\tilde{K} = S_n S_{n-1} \cdots S_1 K$

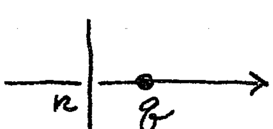
In this case, we encounter an under-crossing on traveling along the diagram from the base-point. This first encounter is labelled n. By the convention for constructing the reference knot $\tilde{K}$, this crossing must be switched to form $\tilde{K}$. Thus $\tilde{K} = S_n S_{n-1} \cdots S_1 K$. Note also that since n is the first crossing encountered after the basepoint, $\underline{E_n S_{n-1} S_{n-2} \cdots S_1 K \text{ is a split unknot diagram}}$. (That is the part of it from n back to ω lies entirely over the rest and hence is regularly isotopic to two disjoint unknots.) Note also that $E_n \tilde{K} = E_n S_{n-1} \cdots S_1 K$.

For this choice of basepoint we have

$$\Gamma_{K,p} = \Gamma \tilde{K} + z \sum_{i=1}^{n} \varepsilon_i(K)\, \Gamma E_i S_{i-1} \cdots S_1 K,$$

$$\Gamma \tilde{K} = \alpha^{w(\tilde{K})}.$$

Now consider the effect of sliding the base-point forward, past this crossing:



K

$\tilde{K}' = S_{n-1} S_{n-2} \cdots S_1 K.$

Then the crossing n remains unswitched and the reference unknot is $\tilde{K}' = S_{n-1}\cdots S_1 K$.

$$\therefore\ \Gamma_{K,q} = \Gamma_{\tilde{K}'} + z\sum_{i=1}^{n-1} \varepsilon_i(K)\,\Gamma_{E_i S_{i-1}\cdots S_1 K} .$$

Hence $\Gamma_{K,p} - \Gamma_{K,q} = \Gamma_{\tilde{K}} - \Gamma_{\tilde{K}'} \overset{\varepsilon_i(K)}{+z}\Gamma_{E_n\tilde{K}}$.

If $\Gamma_{\tilde{K}} = \alpha^{d-\varepsilon}$ then $\Gamma_{\tilde{K}'} = \alpha^{d+\varepsilon}$ where $\varepsilon = \varepsilon_n(K)$ (Since n is not switched in $\tilde{K}'$.)
By our induction hypothesis,

$\Gamma_{E_n\tilde{K}} = \delta\,\Gamma_A\Gamma_B$ where A and B are the components of $E_n\tilde{K}$. Since these are unknots, we see that $\Gamma_A\Gamma_B = \alpha^d$.
Thus

$$\Gamma_{K,p} - \Gamma_{K,q} = \alpha^{d-\varepsilon} - \alpha^{d+\varepsilon} + \varepsilon z\left(z^{-1}(\alpha - \alpha^{-1})\right)\alpha^d$$

$$= \alpha^{d-\varepsilon} - \alpha^{d+\varepsilon} + \varepsilon\left(\alpha^{d+1} - \alpha^{d-1}\right)$$

$$\therefore\ \Gamma_{K,p} - \Gamma_{K,q} = 0 .$$

This proves that Γ_K is independent of the choice of base-point in Case 1.

Case 2.

K $\qquad$ $\tilde{K} = S_n \cdots S_{i+1} S_{i-1} \cdots S_1 K$

$\tilde{K}' = S_n \cdots S_i \cdots S_1 K$

K

In this case we have labelled the first crossing encountered after p by i. It is not switched to form $\tilde{K}$, but it is switched to form $\tilde{K}'$ — the reference unknot for K with basepoint q. We retain that $E_i S_{i-1}\cdots S_1 K$ is split, and that

$S_n S_{n-1} \cdots S_{i+1} E_i S_{i-1} \cdots S_1 K$ is a split (overlaid) unknot. As indicated, i may occur anywhere within the sequence $n, n-1, \ldots, 1$. The same argument given for the link case then shows that $\Gamma_{K,p}$ and $\Gamma_{K,q}$ will remain unchanged under cyclic permutation of the order of sequence. Hence we may re-order the sequences for K,p and Kq to:

K,p : $i-1, i-2, \ldots, 1, 2, \ldots, n$

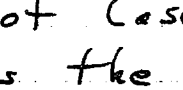
K,q : $i, i-1, i-2, \ldots, 1, 2, \ldots, n$

(meaning that the switching and eliminating be done in this order in each case.).

After this transformation, the calculation is identical to that of Case 1.

This completes the proof of the inductive step. //

We must now show that the inductive assumptions are inherited at the level of the newly defined Γ's. Here the exchange identity requires comment:

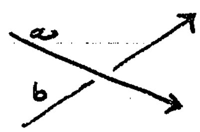
1) Suppose K has one component, and the crossing ⤱. Let $\overline{K} = SK$ be the result of switching this crossing, and $L = EK$, the result of eliminating it. By choosing the basepoint correctly on K () we can make

this crossing number 1 in the switching sequence for K. Then $\widehat{K} = S_n \cdots S_2 S_1 K$ and $S_1 K = \overline{K}$. The identity $\Gamma_K - \Gamma_{\overline{K}} = z\Gamma_L$ is then an immediate consequence of our definition.

We have used the independence of placement of the basepoint.

(2) Suppose K has more than one component and that is a crossing of a single component with itself. Then the identity follows inductively through the definition of $\mathcal{G}_s(K)$ and $\Gamma_K = \frac{1}{z}\sum_s \mathcal{G}_s(K)$.

(3) Suppose K has more than one component, and that the two strands in are from different components. Let these strands be denoted $\underline{a}$ and $\underline{b}$:



Then $\mathcal{G}_s(\;) - \mathcal{G}_s(\;) = z\,\mathcal{G}_s(\;)$ when $s \neq a, b$ by induction. (Recall that $\mathcal{G}_s$ is obtained by using a switching sequence obtained from lifting the component labelled s.). Thus we need only check the identity for $\mathcal{G}_a$ and $\mathcal{G}_b$. This follows by appropriate choice of basepoint as in (2).

This completes the whole inductive definition of Γ_K. We now prove that <u>Γ_K is an invariant of regular isotopy</u>:

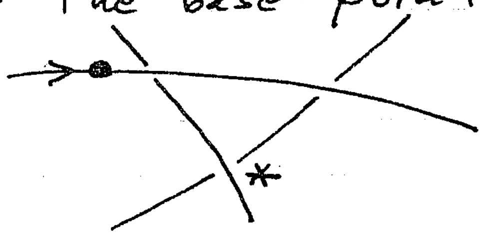
<u>Proof of invariance.</u> Once again, we distinguish two cases: K connected and K a multi-component link. If K is connected, then invariance

under moves of type II and III (i.e. regular isotopy invariance) can be seen by using an appropriate choice of basepoint:

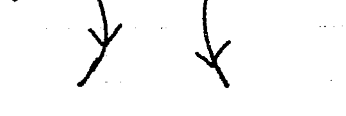
Thus in the case of a type II move, choose the basepoint as shown below:

Then the two crossings involved in the move are not switched by the switching sequence relating K to its reference knot $\tilde{K}$. Since we may inductively assume that each term in the expansion $\Gamma_K = \Gamma_{\tilde{K}} + \sum_{i=1}^{n} \varepsilon_i(K) z \Gamma_{D_i K}$ (**) is already verified to be independent of the type two move, we see that $\Gamma_K = \Gamma_{K'}$ where K' is the diagram obtained from K by performing the type II move.

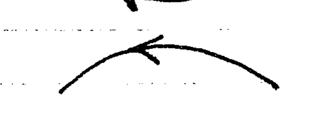
In a type III move we can arrange that two out of the three crossings are uninvolved in the switching sequence by choosing the base-point as shown below:



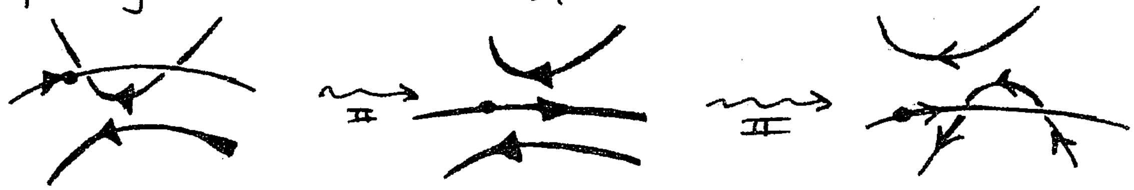
The remaining starred crossing may be switched or spliced in any given term of the expansion (** above). If switched, we can still do the type III move. If it is spliced, then the configuration will look like one of the two cases below:



or



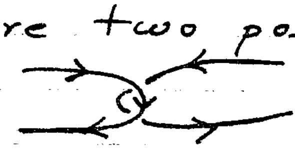
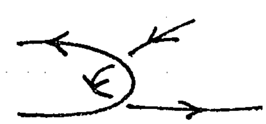
Thus, in the first case there is no change (since topologically equivalent underlying graphs give the same result), and in the second case two moves of type II transform the picture to that picture obtained by splicing after the type III move:



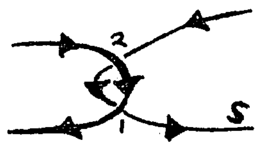
This completes the inductive proof of regular isotopy invariance for connected diagrams.

Of course, the above argument depends upon having also verified inductively the invariance for multi-component diagrams. Here the argument is a bit more subtle. Recall that for a link K we have defined $\Gamma_K = \frac{1}{\ell}\sum_{s=1}^{\ell} G_s(K)$ where $G_s(K)$ is a sum of type (**) that corresponds to switching all crossings on the s-th component that under-cross any other component. Thus the s-th component is lifted above all other components by this switching sequence.

If the components involved in a given move are not switched, then invariance proceeds inductively as before. Thus we must consider those cases of type II and of type III moves where one of the strands is involved in the lift sequence. For type II there are two possibilities:

(A)  and (B)  .

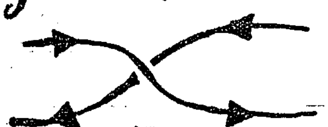
We consider these in turn.

(A) 

K

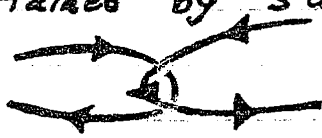
In $G_s(K)$ the arc labelled s will be lifted. As in our previous discussion, we may assume that the crossings labelled 1 and 2 are lifted first, and in that order. Thus we will have the terms $\varepsilon_1 \Gamma_{E_1K} + \varepsilon_2 \Gamma_{E_2S_1K}$ and in further terms in the expansion of $G_s(K)$ both crossings 1 and 2 will be switched.

Now here $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and in the general case of type A $\varepsilon_2 = -\varepsilon_1$.


E_1K

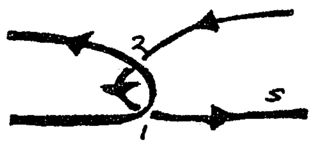

E_2S_1K

Since the diagrams E_1K and E_2S_1K are the same, we see that $G_s(K)$ is equal to $G_s(K')$ where K' is obtained by switching both 1 and 2.

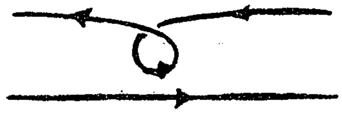

K'


K''

Then by induction, we may assume that every term in the expansion of $G_s(K'')$ is invariant under the type II move so that $G_s(K) = G_s(K') = G_s(K'')$ (as above). This completes the proof for case (A).

(B) 

K



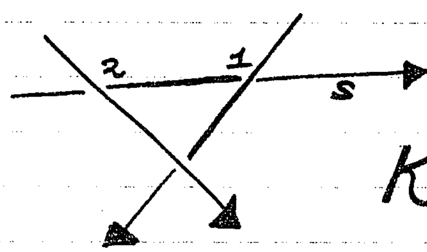
E_1K

E_2S_1K

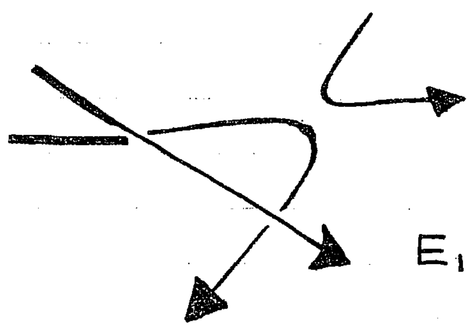
In the second case we see (diagram above) that $\Gamma_{E_1K} = \Gamma_{E_2S_1K}$. Hence the proof proceeds as in case (A).

This completes the verification for the type II move.

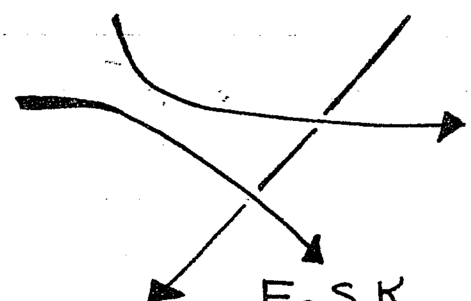
For the type III move we again divide into two cases labelled (A) and (B).

(A) 

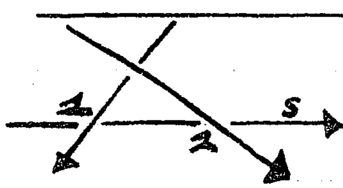
K



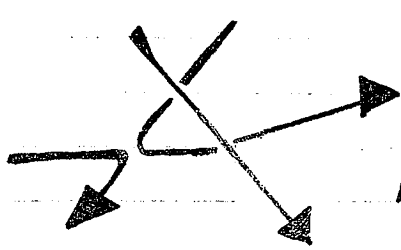
E_1K



E_2S_1K



K'

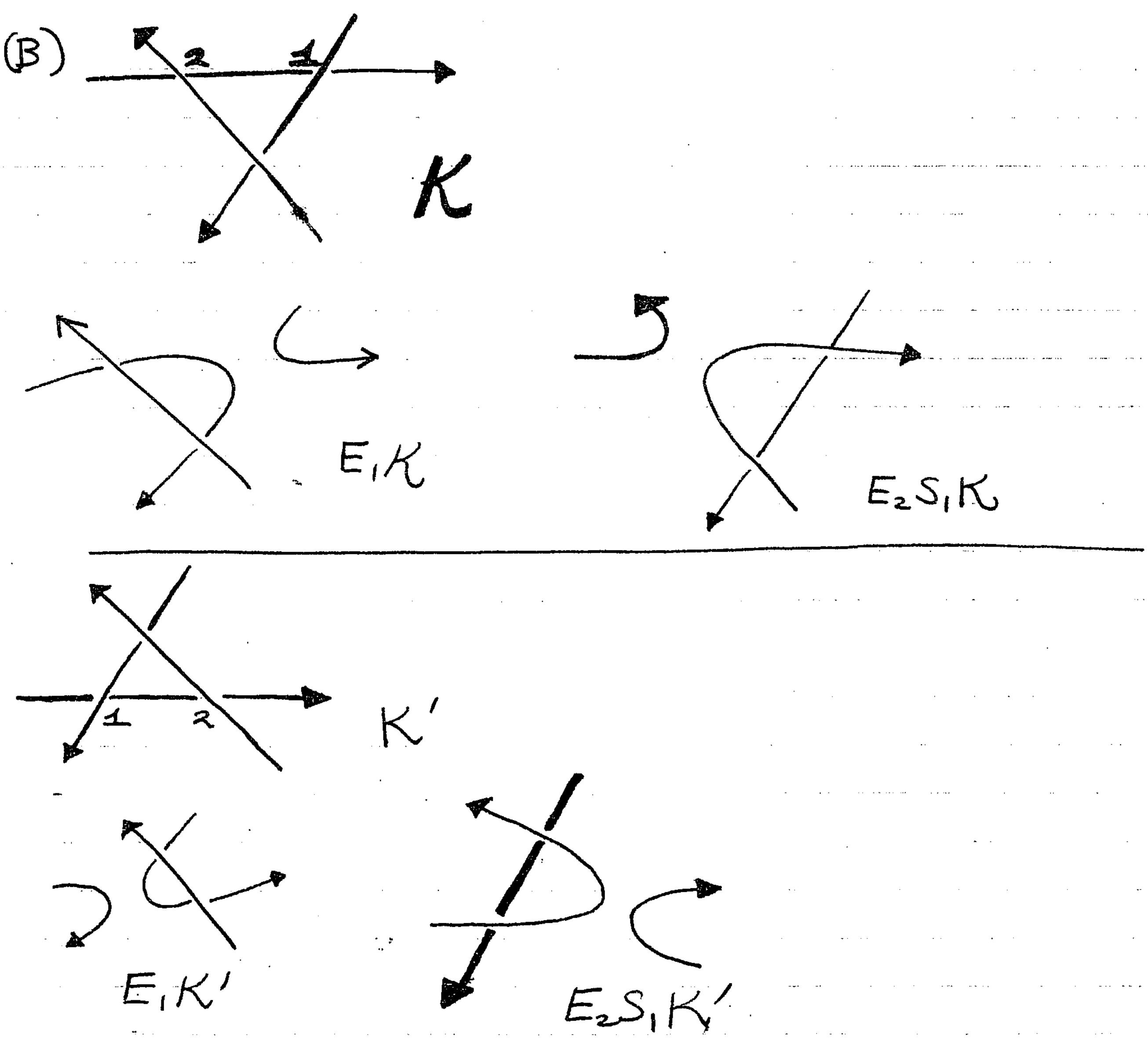


E_1K'

E_2S_1K'

The diagram above indicates how the situation looks for case (A) of the type III move. It shows that we will obtain the same result from $G_s(K)$ and $G_s(K')$ if we let K' be the result of performing the type III move. However, an examination of the figure reveals that it is neccessary to use the order-independence of the switching sequence. This can be assumed inductively since it follows from the exchange identity.

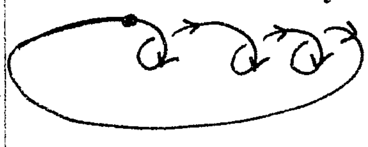
(B)



The same argument applies to case (B) as shown above.

This completes the proof of the theorem, and completes verification of the regular isotopy invariance of the Γ-polynomial. //

__Remark.__ Note that familiar specific evaluations are a consequences of well-definedness. Thus

 $K \Rightarrow \Gamma_K = \alpha^3$ directly from

the definition and choice of base-point.