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Form dynamics

Louis H. Kauffman
Department of Mathematics, University of Illinois at Chicago Circle, Chicago, Illinois 60680, USA

and

Francisco J. Varela
Universidad de Chile, Facultad Ciencias, Casilla 653, Santiago, Chile

This paper is an exposition and extension of ideas begun in the work of G. Spencer-Brown (Laws of Form). We discuss the relations between form and process, distinction and indication by the use of simple mathematical models. These models distill the essence of the ideas. They embody and articulate many concepts that could not otherwise be brought into view. The key to the approach is the use of *imaginary Boolean values*. These are the formal analogs of complex numbers – processes seen as timeless forms, then indicated (self-referentially) and re-entered into the discourse that engendered them. While the discussion in this paper is quite abstract, the ideas and models apply to a wide range of phenomena in mathematics, physics, linguistics, perception and thought.

1. Introduction

Our theme is best indicated by the following experiment, and by a corresponding passage from the beautiful work, *Cymatics* by Hans Jenny (1974):

> Sprinkle sand over the surface of a metal plate; draw a violin bow carefully along the plate boundary. The sand particles will toss about in a rapid dance, swarming and forming a characteristic pattern on the plate surface. This pattern is at once both form and process: individual grains of sand play continually in and out, while the general shape is maintained dynamically in response to the bowing vibration.

> "Since the various aspects of these phenomena are due to vibration, we are confronted with a spectrum which reveals patterned figurate formations at one pole and kinetic-dynamic processes at the other, the whole being generated and sustained by its essential periodicity. These aspects, however, are not separate entities but are derived from the vibrational phenomenon in which they appear in their unitariness . . . The three fields – the periodic as the fundamental field with the two poles of figure and dynamics inevitably appear as one. They are inconceivable without each other . . . nothing can be abstracted without the whole ceasing to exist. We cannot therefore label them one, two, three, but can only say that we have a morphology and a dynamics generated by vibrations, or more broadly by periodicity, but that all these exist together in true unitariness.
> . . . It is therefore warrantable to speak of a basic or primal phenomenon which exhibits this threefold mode of appearance."

Here, Jenny has allowed himself to speak generally about a wealth of experience and concrete experimentation with the effects of vibration on

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various media ranging from sand on a vibrating plate to fluids in three-dimensional space.

These are poetic ideas, metaphoric notions, and yet they have reflections in all fields from the wave/particle duality of quantum physics and the forms produced in concrete media by physical vibration, to the oscillations and distinctions that we make at every moment of our lives.

Our object is to find and explore a language that expresses these ideas and is sensitive to them. Such a language should be internally consistent and externally meaningful. The calculus of indications created by G. Spencer-Brown in his book *Laws of Form* is admirably suited for the project. The purpose of this paper is to *extend Brown's language to exhibit how a rich world of periodicities, waveforms and interference phenomena is inherent in the simple act of distinction.*

There is nothing new about the idea that an entire universe of forms comes into being with the making of one distinction. In mathematics this idea is reflected by the use of the binary system, and more centrally by the construction of the natural numbers from the empty set via the operation of forming a collection. We wish to focus on these same themes, but our aims are more fundamental. For example, we would give attention to the construction of the empty set itself. This set, $\phi = \{ \ \}$, is obtained by bracketing or *framing nothing.* Notationally, the frame is given by the brackets. These brackets *indicate* a distinction on the planar space upon which the brackets are written. The open space of the plane is construed as an indication of nothing (that is, the absence of set members).

Of course, these are remarks about the notation, about the specific choice of frame. In point of fact, we conceptualize the empty set by first framing nothing and then *throwing away the frame!* That is, we require that the mathematics be independent of the vagaries of notation. Nevertheless, it is significant that in the beginning of set theory or the beginning of Boolean algebra, the very process that the mathematics proposes to discuss is inevitably mirrored in its own language and notation. This mirroring quality of the language is essential for our understanding. We believe that recognition of this point leads to a number of fruitful avenues, and we hope to illustrate these avenues in the paper.

Our use of Spencer-Brown's notation also reflects these issues. His calculus of indications is based on the sign $\rceil$. Our comments about the empty set apply equally well to this mark $\rceil$ which indicates a division of the plane. This notation is actually very close to the notation of Boolean algebra. For example, one standard notation that corresponds to $\overline{a}\rceil$ is $\overline{a}$ (see Fig. 1).

Brownian	**Boolean**
$\overline{a}\rceil$	$\overline{a}$
ab	ab
$\overline{a}\rceil b$	$\overline{a}\, b$
$\overline{a}\rceil \overline{b}\rceil$	$\overline{a}\, \overline{b}$
$\overline{\overline{a}\rceil \overline{b}\rceil}\rceil\ c$	$\overline{(\overline{a}\, \overline{b})}\ c$

Fig. 1

These is nothing obscure or esoteric about our notational system. Typographically, it is very similar to standard notation. Furthermore, it allows parenthesis free expressions by generating its own divisions. Most importantly, it focuses on the fact that the algebra it supports can be interpreted as speaking about the distinctions engendered by its own typography.

It has often been pointed out that the formal structure of G. Spencer-Brown's work, and *a fortiori* of the present work, is equivalent to some form of Boolean algebras or switching automata. Although, in a strict formalist sense, this is correct, this line reasoning misses an essential point: A change of context and style (notation) may reveal the intuitive and conceptual underpinnings of a field, in a way that other, formally equivalent systems, do not. A simple example is Roman and Arabic numerals.

The relations between the formal structure of Boolean and indicational algebras are quite obvious, and were discussed by Spencer-Brown in his original work (1972, Appendix 2; see also Varela, 1975). The point is that one may see Boolean values (true or false) or switching algebras (on or off) as particular cases of a more fundamental ground which is the act of indication. A two-state abstraction is seen rooted in a specific cognitive *act*. From this starting point, an entirely new vision of traditional Boolean formalism can be obtained. Furthermore, the actual significance of some of the postulates in these algebras is made transparent in a way that was hitherto impossible. A good example of this occurs in what is here called a transposition algebra, to be discussed later on.

We wish to mention that the algebras here called brownian are formally isomorphic to De Morgan algebras. As there is a large literature on such non-standard Boolean algebras, we use Kauffman (1978*b*) as an entry point, rather than including an extensive list of references. Thus, if we make little mention in this paper to the vast literature on Boolean algebras and algebraic logic, it is not because we wish to ignore these sources, but rather because we wish to emphasize the change in context.

In order to accomplish this program, we first present an extension of Spencer-Brown's algebra that is capable of handling periodic indicational forms. We then discuss how oscillations relate naturally to the re-entry of forms, that is, to their self-referential quality. †

The following is the outline of the paper.

Section 2 recalls Spencer-Brown's calculus of indications and discusses re-entry and oscillations in a general way. Section 3 shows how to make an algebraic construction for elementary waveforms. (This construction has also been discussed in Kauffman (1978) where it is used to construct De Morgan algebras from Boolean algebras.) In sections 4 and 5 we develop an algebra analogue to Spencer-Brown's primary algebra. In Boolean terms it corresponds to dropping the law of the excluded middle. This allows wave-form models; we call such algebras *brownian* algebras. Sections 5 and 6 develop more models

†The present paper grew out of our interest in the work of Spencer-Brown on indicational mathematics, and its relevance for systems theory and mathematical foundations. In the cases of social and biological systems, the complementarity of pattern/dynamics is quite apparent. It points directly to an examination of recursive, self-referential dynamics generating coherent autonomous unities. For this background and for the motivations from systems theory the reader should consult the following papers: Goguen & Varela (1978, 1979), Kauffman (1978*a*, *b*), Varela & Goguen (1978), Varela (1975, 1978*a*, *b*), Wadsworth (1976). We need not explicitly retake this background and motivation in the present paper, as its theme is of independent interest.

for brownian algebras, discuss an algebra of periodic sequences of varying periods, and present a way in which waveforms can interfere with each other.

Sections 7 and 8 examine the relation between waveforms and recursions or re-entry of forms. Section 7 discusses some periodic properties of iteration for indicational operators. In section 8 we examine recursion in a broader sense, and establish some relations between waveforms and fixed points. (This construction for recursion and fixed points is more fully given elsewhere: Goguen & Varela, 1979; and Varela & Gguen, 1978.)

Section 9 discusses the relations between geometrical and indicational forms. Section 10 is a summary.

2. Recalling the calculus of indications

The calculus of indications (Spencer-Brown, 1969, 1972) is based on one symbol, ⌉, the mark. It can be viewed as an abbreviation of □ , and hence makes, by cleaving it, a distinction upon the plane in which it is written. In the context of the calculus of indications, the mark is taken to be the name of the outside part of a distinction in an arbitrary domain or indicational space *or* as an instruction to cross the boundary of this distinction.

This gives rise to two forms of equation:

A1. *Form of Condensation* ⌉ ⌉ = ⌉

A2. *Form of Cancellation* ⌉⌉ = .

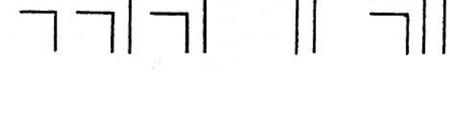
Here the blank indicates the un-marked state. These equations have various interpretations. Thus A1 is indeed the form of condensation, where two things are realized to be identical in form. The two marks in A1 may be seen differently by regarding the left mark as making a distinction in the plane, while the right mark is a token or name for the outside part of this distinction. When we see the mark that makes the distinction as *itself* indicating the outer space, then the two uses condense giving ⌉ ⌉ = ⌉.

Similarly, form A2 may be interpreted by the sentence: To cross from the marked state is to enter the unmarked state. Here the outer mark is interpreted as an instruction to cross from the state indicated within it. The outer mark is an operator in this interpretation. Operating on itself, it cancels itself.

Before proceeding into calculation, it should be remarked that the reader of this page is himself or herself, a mark distinguishing a space. Thus this calculus is self-referential in the broadest sense. In fact, one of the essential features of this approach is that the scribe and the forms of description are reflections of one another. We become individuals by making distinctions; The distinctions we make reveal (and sometimes conceal) who we really are.

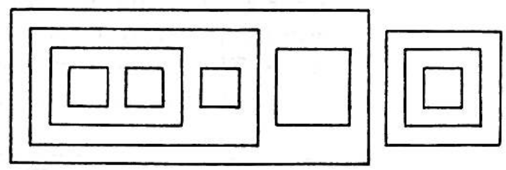
These initial forms of equation lead to a calculus of expressions that Spencer-Brown calls the *primary arithmetic*. In this system one considers arrangements of marks such as

e =

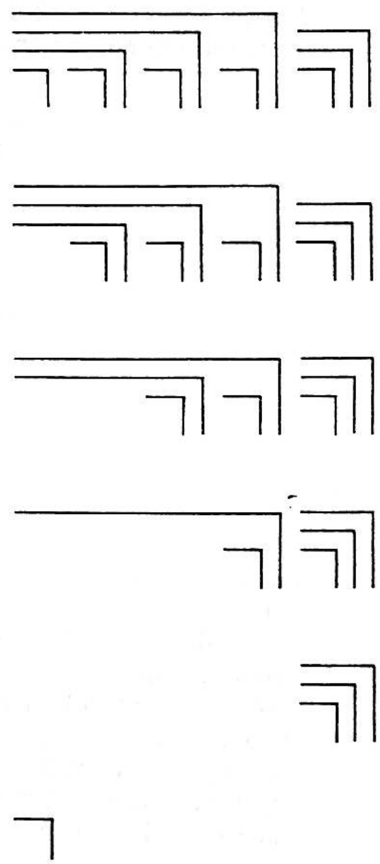


An expression like e is regarded as a clear distinction if for each mark in the expression there is no ambiguity about which marks it contains and which marks contain it. In this notation, this is made clear by examining the horizontal overhang of each mark. If □ were used instead of $\urcorner$, then a well-formed expression would simply be any finite disjoint collection of rectangles in the plane. Thus we would write

e =



For two expressions e and f, we write $e = f$ if there is a finite sequence of steps of type A1 (condensation) or A2 (cancellation) leading from one expression to the other. Thus



e =

= (A1)

= (A2)

= (A2)

= (A2)

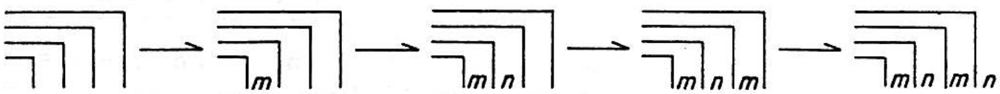
$\therefore$ e = $\urcorner$ (A2)

The following facts can be shown.

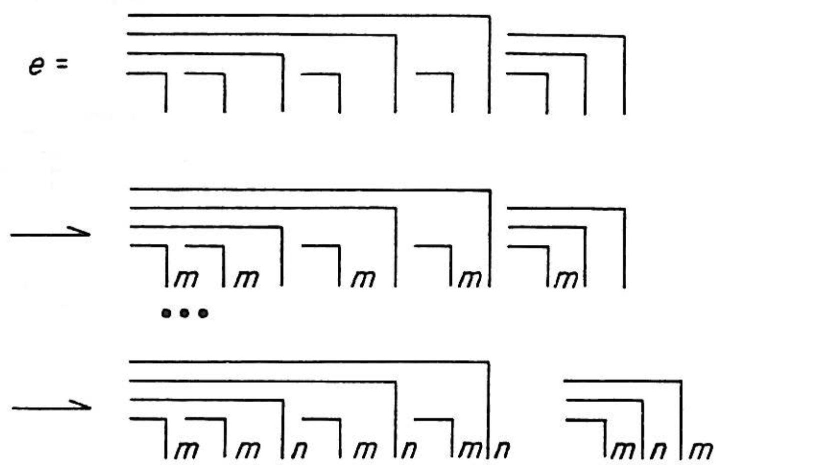
(a) If e is any well-formed expression, then e can be obtained by a sequence of steps from one of $\urcorner$ or .

(b) There is no sequence of steps leading from $\urcorner$ to .

Hence each expression is equivalent to either the marked or the unmarked state, and this will be referred to as its *value*. Two methods of evaluation are worth noting. The first method is in the form of calculation as indicated above: one looks into the deepest space of the expression, where there are marks that do not contain other marks. At such places condensation or cancellation may be applied. In the second method, one regards the deepest space as sending signals of value up through the expression to be combined into a global valuation. To do this, let m stand for the marked state, and n for the unmarked state. Thus $mm = m$, $mn = nm = m$, $nn = n$, and $\overline{m}\rceil = n$, $\overline{n}\rceil = m$. Now use these labels as *signals* as in the following example:

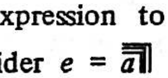


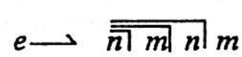
Here  has the value $n =$. This procedure starts from the deepest space and labels those values that are unambiguous until a value for the whole expression emerges. Here is one more example:



Hence $e = \urcorner$.

It should be noted that these methods are quite compatible. They reflect the dual nature of an expression as (self) operator or operand. Viewed as an operator, the expression filters its own inner signals, creating a pattern or waveform that culminates in its evaluation. Even at this level, the relation to the Cymatics metaphor of figure/vibration/dynamics (that is, to the descriptive pair figure/dynamics that may represent the polarities of vibration) is quite apparent: The geometric form of the expression represents the figure; calculational steps are an elementary dynamic; signals of value moving through the expression represent a kind of periodic vibration.

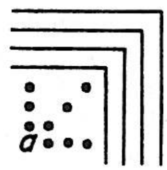
This relation of form and dynamics is even more apparent if we allow an expression to contain a variable that *vibrates* (i.e. changes in time). Consider 

If $a = n$ then 

If $a = m$ then $\qquad e \longrightarrow \overline{\overline{\overline{m|}\, n|}\, m|}\; n$.

If a varies from n to m periodically, then the signals n and m will form a moving pattern analogous to the moving light patterns on a sign in Times Square.

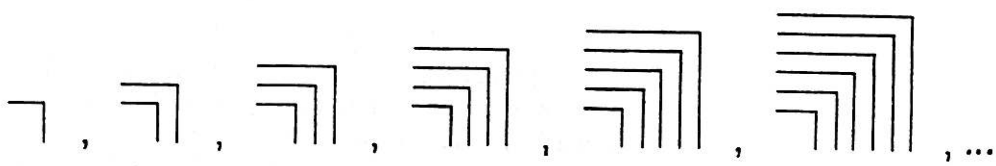
Taking another point of view, we may regard the signals as moving outward like ripples on a pond so that a time-like vibration by a yields a pattern of the form



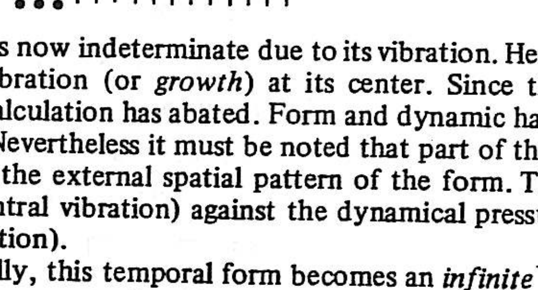
That is, we may suppose that each time $a = m$, a mark appears so that if time is represented as $t = 1, 2, 3, 4, \ldots$ then

$$a = \begin{cases} m & t \text{ odd} \\ n & t \text{ even} \end{cases}$$

and the outward expression grows in the pattern:



Viewed in space we would see something like



where the deepest space is now indeterminate due to its vibration. Here the form is maintained by the vibration (or *growth*) at its center. Since the deepest space is indeterminate, calculation has abated. Form and dynamic have become one with the vibration. Nevertheless it must be noted that part of the vibration has been remembered as the external spatial pattern of the form. This pattern is maintained (by the central vibration) against the dynamical pressure toward simplification (via calculation).

Viewed entirely spatially, this temporal form becomes an *infinite* expression consisting a descending sequence of marks. As such, its interior repeats itself.

The form becomes identical to part of itself. Thus, for f as given above, $f = \overline{f}| = \sqcup$ (see Varela, 1975). This *description*, $f = \overline{f}|$, where f re-enters itself, can thus be seen as a self-reference. This is the spatial context.

Temporally, we may view $f = \overline{f}|$ as a *prescription* for recursive action $f \longrightarrow \overline{f}|$. Thus $\overline{\ }| \longrightarrow \overline{\overline{\ }|}| \longrightarrow \overline{\overline{\overline{\ }|}|}| \longrightarrow \ldots$ and this regenerates the waveform. Thus vibration yields self-referential spatial form, while the associated recursive dynamic to the self-reference unfolds the vibration (once again) into a temporal oscillation. We shall deal with these relationships of recursion and re-entry of forms to oscillation in greater detail later in the paper.

It is now appropriate to consider Spencer-Brown's algebra for finite expressions: Various algebraic patterns are seen to be true in the calculus of indications. Thus $\overline{\overline{a}|}| = a$, $\overline{\overline{a}|\,\overline{b}|}|\,c = \overline{\overline{ac}|\,\overline{bc}|}|$ and $\overline{\overline{a}|\,a}| =$ for any finite forms a, b, c. This leads to an algebra with initial equations and a number of consequences. At the end of this section we have given a table consisting of these initials and consequences, forming the *primary algebra*. Formally, the primary algebra is an axiomatic form of Boolean algebra written in the notation of the calculus of indications. We feel that there is a significant conceptual gain in placing Boolean algebra within this wide indicational context.

It is also desirable to deal with infinite forms and/or waveforms algebraically. An apparent paradox seems to emerge: Suppose that the primary algebra is applicable to forms such as $f = \overline{f}|$. Then

$$f = \overline{f}| = \overline{f\,\overline{f}|}| = \overline{\overline{f}|\,f}| = \quad .$$

Hence $f = \overline{f}| \Rightarrow f = \quad$. How are we to interpret this? The simplest way out is to realize that in Spencer-Brown's calculus of indications we have the *marked state* which is purely spatial having *no temporal component*, and the unmarked state connecting *everything else*. Thus f, being vibratory, has been cast into the unmarked state. It has been so cast by the form of *position* $\overline{\overline{p}|\,p}| = \ $. If we wish to articulate temporal forms we can do so by *limiting the cancellation* given by the form of position. We shall show how to do this in the next section. The result is an algebra where there are many 'self-interference' terms of the form $\overline{\overline{p}|\,p}|$. Cancelling all such terms yields the primary algebra. Leaving them gives an algebra capable of caring for vibratory forms and self-reference.

Index for primary algebra

Initials

J1. $\overline{\overline{p}|\,p}| =$ (position)

J2. $\overline{\overline{pr}|\,\overline{qr}|}| = \overline{\overline{p}|\,\overline{q}|}|\,r$ (transposition)

Consequences

01. $\overline{\overline{a}|}| = a$ (reflection)

02. $\overline{a\,\overline{b}|}|\,b = \overline{a}|\,b$ (generation)

03. $\overline{\ }|\,a = \overline{\ }|$ (integration)

04. $\overline{\overline{a}|\,b}|\,a = a$ (occultation)

05. $a\,a = a$ (iteration)

06. $\overline{\overline{a|}\,\overline{b|}|}\;\overline{\overline{a|}\,b|} = a$ (extension)

07. $\overline{\overline{\overline{a|}\,b|}\,c|} = \overline{a\,c|}\;\overline{\overline{b|}\,c|}$ (echelon)

08. $\overline{\overline{a|}\,\overline{b\,r|}\,\overline{c\,r|}|} = \overline{\overline{a|}\,\overline{b|}\,\overline{c|}|}\;\overline{\overline{a|}\,r|}$ (modified transposition)

09. $\overline{\overline{\overline{b|}\,\overline{r|}|}\;\overline{\overline{a|}\,\overline{r|}|}\;\overline{\overline{x|}\,r|}\;\overline{\overline{y|}\,r|}|} = \overline{\overline{r|}\,a\,b|}\;\overline{r\,x\,y|}$. (crosstransposition)

The primary algebra is *complete* with respect to the primary arithmetic (calculus of indications). That is, $\alpha = \beta$ is a consequence of J1 and J2 if and only if $\alpha = \beta$ is a theorem about the arithmetic (see Spencer-Brown, 1972, p. 50).

3. A waveform arithmetic

Consider again $f = \overline{f|}$. By taking successive replacement of f in its equivalent form we have a sequence

$$f, \overline{f|}, \overline{\overline{f|}|}, \overline{\overline{\overline{f|}|}|}, \ldots$$

Now, if we let f take the two possible initial values $\overline{\ |}$ and $\overline{\overline{\ |}|}$, and apply condensation, we get two sequences

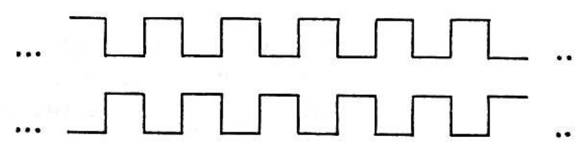
$$\overline{\ |},\ \overline{\overline{\ |}|},\ \overline{\ |},\ \overline{\overline{\ |}|},\ \overline{\ |},\ \overline{\overline{\ |}|},\ \overline{\ |},\ \ldots$$

and

$$\overline{\overline{\ |}|},\ \overline{\ |},\ \overline{\overline{\ |}|},\ \overline{\ |},\ \overline{\overline{\ |}|},\ \overline{\ |},\ \overline{\overline{\ |}|},\ \ldots$$

These two sequences can be looked at as the successive values of two basic *waveforms*

$i = \ldots$ $j = \ldots$



This *temporal interpretation* of re-entering expressions which can take no values in the primary arithmetic, was first proposed by Spencer-Brown himself in Chapter 11 of *Laws of Form.* He calls these temporal values *imaginary Boolean* values. A few years later, in the preface to the American edition of his book he says:

> What we do in Chapter 11, is to extend the concept (of imaginary numbers) to Boolean algebras . . . The implications of this, in the fields of logic, philosophy, mathematics, and even physics, are profound.
>
> What is fascinating about the imaginary Boolean values, once we admit them, is the light they apparently shed on our concepts of matter and time. It is, I guess, in the nature of us all to wonder why the universe appears just the way it does. Why, for example, does it not appear more symmetrical? (Spencer-Brown, 1972, p. xi).

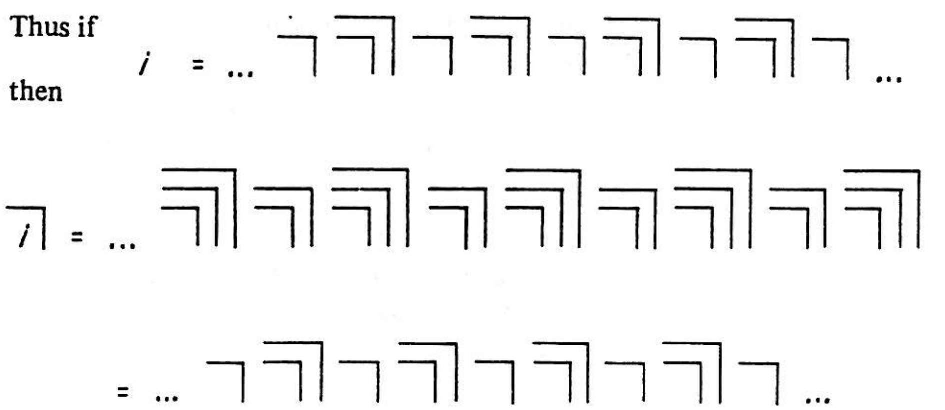
Actually, Spencer-Brown did not develop the formal side of these ideas extensively in Chapter II; he pointed into a rich direction, but did not exhibit specific constructions. On the other hand, his book is a courageous attempt to deal with these issues, beginning as it does in the simple, undefinable soil of distinctions and moving upward into formal realms. We now present a specific development of imaginary values. We hope that what is seen from our work may help to explain the importance of Spencer-Brown's ideas.

As a first step toward making temporal expressions well-defined objects, note that in the example provided above $\overline{i}|$ = i if we interpret ⌉ as *ordinary inversion plus a half-period shift.* That is, suppose we have a periodic pattern

x = . . . a b a b a b a b . . . then we *define*
$\overline{x}|$ = . . . $\overline{b}|$ $\overline{a}|$ $\overline{b}|$ $\overline{a}|$ $\overline{b}|$ $\overline{a}|$ $\overline{b}|$ $\overline{a}|$ . . .

Thus if

then



∴ $\overline{i}|$ = i.

Note also that ij = ⌉ since at any given time either i or j is marked.

Now the sequence . . . *ababab* . . . can be looked at in a different way, somewhat similar to the construction of the complex numbers from the real numbers. That is, we think of the ordered pair (a,b) as representing the essential features of the sequence. This shift in perspective permits a more detailed algebraic treatment of the notions described above.

3.1. *Definition.* Let B be any algebra (or arithmetic) satisfying the initials for the primary algebra. Let $\hat{B} = \{(a,b) \mid a,b \in B\}$
and define (i) $\overline{(a,b)}|$ = $(\overline{b}|, \overline{a}|)$ (inversion plus shift)
(ii) $(a,b)(c,d) = (ac,bd)$. $((a,b),(c,d) \in \hat{B})$

Let i = (⌉, $\overline{⌉}|$) and j = ($\overline{⌉}|$, ⌉). Make the identification $B \subset \hat{B}$ via $a = (a,a)$ for $a \in B$.

The smallest collection of such pairs representing waveforms is, in fact, one containing no variables, but only the constant elements ⌉, $\overline{⌉}|$, i, j. Call it the *waveform arithmetic* V. By the previous definition we see that:

3.2. *Proposition.* Let P = ⌉, $\overline{⌉}|$ denote the primary arithmetic. Then $V = \hat{P}$.

As required, the basic waveforms i, j arise out of the static forms of the primary arithmetic. They also return to it by the relation ⌉ = ij.

Next, note the following:

3.3. *Proposition.* In $\hat{B}$ occultation and transposition are valid, and $i = \overline{i}\rceil$, $j = \overline{j}\rceil$, $ij = \rceil$

Proof. We shall verify occultation and leave the rest of the proof for the reader. Let $X = (a,b)$, $Y = (c,d)$. We must show that $\overline{\overline{X}\rceil\;Y}\rceil\, X = X$ in $\hat{B}$:

$\overline{\overline{X}\rceil\;Y}\rceil\, X = \overline{(\overline{b}\rceil,\overline{a}\rceil)(c,d)}\rceil\;(a,b)$ (3.1 (i))

$= \overline{(\overline{b}\rceil\, c,\overline{a}\rceil\, d)}\rceil\;(a,b)$ (3.1 (ii))

$= (\overline{\overline{a}\rceil\, d}\rceil\,, \overline{\overline{b}\rceil\, c}\rceil\,)(a,b)$ (i)

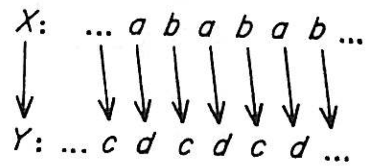
$= (\overline{\overline{a}\rceil\, d}\rceil\, a, \overline{\overline{b}\rceil\, c}\rceil\, b)$ (ii)

$= (a,b)$ (occultation in B)

$\therefore \overline{\overline{X}\rceil\;Y}\rceil\, X = X$.

This proposition suggests that in the waveform arithmetic and the related algebras $\hat{B}$ the relevant initials may be occultation and transposition. The next section develops this idea.

3.4. *Remark.* Note that (as in 3.3) if $X = (a,b)$, $Y = (c,d)$ then $\overline{X}\rceil\; Y = (\overline{b}\rceil\, c, \overline{a}\rceil\, d)$ in $\hat{B}$. It is interesting to consider this in the light of Spencer-Brown's interpretation of *implication (see Spencer-Brown*, 1972, p. 114). In interpreting the primary algebra for logic, Spencer-Brown lets $\rceil$ stand for T (true) and   stand for F (false). Then $a \rightarrow b$ becomes $a\ \ b$. Thus in $X\ Y$ resolves into two implications in B: $b \rightarrow c$ and $a \rightarrow b$:



Thus, if $X \rightarrow Y$ in $\hat{B}$ is taken to be $\overline{X}\rceil\; Y$ we see that *implication in $\hat{B}$ has a temporal component.* Each term of the Y-series is 'implied' by the previous term of the X-series. Further study of this remark promises to be very interesting.

4. Brownian algebras

Let $a,b, \ldots, p,q, \ldots$ be a collection of variables, and let expressions involving the cross, $\rceil$, and these variables be defined as in the primary algebra. That is, whenever A,B are expressions, then $\overline{A}\rceil$, $\overline{B}\rceil$, $\overline{A}\rceil\ B$, $A\overline{B}\rceil$, AB, are also expressions. The variables themselves are expressions, as are $\rceil$ and (blank). Take the following two initials as valid:

Initial 1. *Occultation*

I1 $\overline{\overline{p}\rceil\, q}\rceil\, p = p$ conceal $\rightleftharpoons$ reveal

Initial 2. *Transposition*

I2 $\overline{\overline{pr}\rceil\;\;\overline{qr}\rceil}\rceil = \overline{\overline{p}\rceil\,\overline{q}\rceil}\rceil\, r$ collect $\rightleftharpoons$ distribute

Call any algebra satisfying I1 and I2 a *brownian algebra.* As in the primary algebra, the cross operates on whole expressions and the juxtaposition operation (continence) $a,b \leftrightarrow ab$ also operates on any finite collection of expressions. Since it simply expresses the distinctness of these expressions viewed together as a whole, continence has no particular order properties. In more orthodox algebraic systems this would be expressed explicitly by introducing initials for commutativity and associativity. Here it is not so much that commutativity and associativity are tacitly assumed, but that we are articulating a level at which they do not yet exist! Service is paid to this point by ignoring commutativity and associativity in all subsequent demonstrations. Since the cat is already out of the bag, we must make do with tacit conventions to substitute for true simplicity. We now derive some forms of equations valid in these algebras.

Consequence 1. *Reflection*

C1. $\overline{\overline{a}\rceil}\rceil = a$ reflect $\rightleftharpoons$ reflect

Demonstration.

$$\overline{\overline{a}\rceil}\rceil = \overline{\overline{\overline{\overline{a}\rceil}\rceil\, a}\rceil\;\overline{a}\rceil}\rceil \qquad (I1)$$

$$= \overline{\overline{\overline{\overline{a}\rceil}\rceil\, a}\rceil\;\overline{\overline{\overline{a}\rceil}\rceil\, a}\rceil}\rceil \qquad (I1)$$

$$= \overline{\overline{\;}\rceil\,\overline{\;}\rceil}\rceil\;\overline{\overline{a}\rceil}\rceil\, a \qquad (I2)$$

$$= \overline{\overline{a}\rceil}\rceil\, a \qquad (I1)$$

$$= a \qquad (I1).$$

Consequence 2. *Iteration*

C2. $a\,a = a$ iterate $\rightleftharpoons$ reiterate

Demonstration. $a\,a = \overline{\overline{a}\rceil}\rceil\; a$ (C1)

$= a$ (I1).

Consequence 3. *Integration*

(C3) $\overline{\;}\rceil\, a = \overline{\;}\rceil$ reduce $\rightleftharpoons$ augment

∴

Demonstration $\overline{\;}\rceil\, a = \overline{\overline{\overline{\;}\rceil\, a}\rceil}\rceil$ (C1)

$= \overline{\qquad}\rceil$ (I1).

Consequence 4. *Echelon*

(C4) $\overline{\overline{\overline{a}\rceil\; b}\rceil\; c}\rceil = \overline{ac}\rceil\;\overline{\overline{b}\rceil\; c}\rceil$ break $\rightleftharpoons$ make

Demonstration

$$\overline{\overline{\overline{a}\rceil\ b}\rceil\ c}\rceil = \overline{\overline{\overline{a}\rceil\ \overline{\overline{b}\rceil}\rceil}\rceil\ c}\rceil \qquad \text{(C1)}$$

$$= \overline{\overline{\overline{ac}\rceil\ \overline{\overline{b}\rceil\ c}\rceil}\rceil}\rceil \qquad \text{(I2)}$$

$$= \overline{ac}\rceil\ \overline{\overline{b}\rceil\ c}\rceil \qquad \text{(C1)}$$

Consequence 5. *Combination*

(C5) $\overline{\overline{a}\rceil\ r}\rceil\ \overline{\overline{b}\rceil\ r}\rceil = \overline{\overline{ab}\rceil\ r}\rceil$ combine ⇌ split

Demonstration $\overline{\overline{a}\rceil\ r}\rceil\ \overline{\overline{b}\rceil\ r}\rceil = \overline{\overline{\overline{\overline{a}\rceil\ r}\rceil\ \overline{\overline{b}\rceil\ r}\rceil}\rceil}\rceil$ (C1)

$$= \overline{\overline{\overline{\overline{a}\rceil}\rceil\ \overline{\overline{b}\rceil}\rceil}\rceil\ r}\rceil \qquad \text{(I2)}$$

$$= \overline{\overline{ab}\rceil\ r}\rceil \qquad \text{(C1)}$$

The next two consequences are special cases of C5 and C4 respectively. We articulate them because they are related to corresponding forms in the primary algebra.

Consequence 6. *Catalysis*

(C6) $\overline{\overline{ab}\rceil\ b}\rceil = \overline{\overline{a}\rceil\ b}\rceil\ \overline{b\ \overline{b}\rceil}\rceil$ release ⇌ dissolve

Demonstration Apply C5.

Note that if $\overline{b\ \overline{b}\rceil}\rceil$ = (blank), as in the primary algebra, then C6 becomes a crossed form of *generation* (see the index for the primary algebra in section 2).

Consequence 7. *Tension*

(C7) $\overline{\overline{a}\rceil\ b}\rceil\ \overline{\overline{a}\rceil\ \overline{b}\rceil}\rceil = \overline{\overline{a}\rceil\ \overline{b\ \overline{b}\rceil}\rceil}\rceil$ atone ⇌ attend

Demonstration. Apply C4

If $\overline{b\ \overline{b}\rceil}\rceil$ = (blank), then tension becomes $\overline{\overline{a}\rceil\ b}\rceil\ \overline{\overline{a}\rceil\ \overline{b}\rceil}\rceil = \overline{\overline{a}\rceil}\rceil = a$, the form of *extension* in the primary algebra.

Consequence 8. *Modified transposition*

(C8) $\overline{\overline{a}\rceil\ \overline{br}\rceil\ \overline{cr}\rceil}\rceil = \overline{\overline{a}\rceil\ \overline{b}\rceil\ \overline{c}\rceil}\rceil\ \overline{\overline{a}\rceil\ \overline{r}\rceil}\rceil$ collect ⇌ distribute

Demonstration

$$\overline{\overline{a}\rceil\ \overline{br}\rceil\ \overline{cr}\rceil}\rceil = \overline{\overline{a}\rceil\ \overline{\overline{\overline{\overline{br}\rceil\ \overline{cr}\rceil}\rceil}\rceil}\rceil}\rceil \qquad \text{(C1)}$$

$$= \overline{\overline{a}\rceil\ \overline{\overline{\overline{\overline{b}\rceil\ \overline{c}\rceil}\rceil\ r}\rceil}\rceil}\rceil \qquad \text{(I2)}$$

$$= \overline{\overline{a}\rceil\ \overline{b}\rceil\ \overline{c}\rceil}\rceil\ \overline{\overline{a}\rceil\ \overline{r}\rceil}\rceil \qquad \text{(C4)}.$$

Consequence 9. *Distribution*

(C9) $\overline{\overline{p}\;\overline{q}}\;\overline{\overline{r}\;\overline{s}} = \overline{\overline{pr}\;\overline{ps}\;\overline{qr}\;\overline{qs}}$ shuffle ⇌ cut

Demonstration

$$\overline{\overline{p}\;\overline{q}}\;\overline{\overline{r}\;\overline{s}} = \overline{\overline{p\;\overline{\overline{r}\;\overline{s}}}\;\;\overline{q\;\overline{\overline{r}\;\overline{s}}}} \quad \text{(I2)}$$

$$= \overline{\overline{\overline{\overline{pr}\;\overline{ps}}}\;\;\overline{\overline{\overline{qr}\;\overline{qs}}}} \quad \text{(I2)}$$

$$= \overline{\overline{pr}\;\overline{ps}\;\overline{qr}\;\overline{qs}} \quad \text{(C1)}$$

Consequence 10. *Modal crosstransposition*

(C10) regulate ⇌ accommodate

$$\overline{\overline{a\,\overline{X}}\;\overline{\overline{a}\;\overline{Y}}\;\overline{a\;\overline{\overline{a}}\;\overline{Z}}} = \overline{a}\,\overline{X}\;\overline{\overline{a}}\;\overline{Y}\;\overline{\overline{a}}\;\overline{a}\;\overline{XYZ}$$

Demonstration

$$\overline{\overline{a\,\overline{X}}\;\overline{\overline{a}\;\overline{Y}}\;\overline{a\;\overline{\overline{a}}\;\overline{Z}}}$$

$$= \overline{\overline{\overline{X}}\;\overline{\overline{a}\;\overline{Y}}\;\overline{\overline{a}\;\overline{Z}}}\;\;\overline{\overline{a}\;\;\overline{\overline{a}\;\overline{Y}}} \quad \text{(C8)}$$

$$= \overline{\overline{\overline{X}}\;\;\overline{\overline{Y}}\;\;\overline{\overline{Z}}}\;\;\overline{\overline{a}\;\;\overline{\overline{X}}}\;\;\overline{\overline{a}\;\;\overline{\overline{a}\;\;\overline{\overline{Y}}}} \quad \text{(C8)}$$

$$= \overline{a\;\;\overline{X}}\;\;\overline{\overline{a}\;\;\overline{Y}}\;\;\overline{\overline{a}\;\;\overline{a}}\;\;\overline{XYZ} \quad \text{(C6, C1)}$$

(We thank Mark Kauderer for this demonstration.)

It is worth comparing the consequences in the brownian algebra with the corresponding results (see index at end of section 2) in the primary algebra. 01,02,03,04,05 and 08 are valid in both algebras. Catalysis is the brownian image of generation; tension corresponds to extension in the primary algebra; model crosstransposition corresponds to crosstransposition. In each of these cases cancellation of terms of the form $\overline{\overline{p}\;p}$ yields a corresponding result in the primary algebra.

By adopting the initials occultation and transposition we have constructed an algebra closely related to the primary algebra that automatically avoids the cancellation of $\overline{\overline{p}\;p}$. As we have seen, these self-interference terms $\overline{\overline{p}\;p}$ are important for handling waveforms. For these purposes they should be articulated rather than ignored.

By taking the initials (I1) and (I2) we have done no violence to Spencer-Brown's original grounds, but have simply adjusted our sights to perceive patterns already inherent in the form.

It is also worthwhile to compare this *change of language* (the primary algebra versus brownian algebra) to other more complex linguistic situations. For example, in Benjamin Wharf's studies of Hopi Indian languages (see Wharf, 1956) he finds language structures that allow a view of the world that is probably closer in spirit to the waveform algebra than to the primary algebra (which keeps the all or none quality of Aristotelian logic). To change viewpoint it is not enough to simply throw away a rule (say throw away $\overline{p\overline{p}\rceil}\rceil =$). This abandonment must be embedded in a positive framework (such as the brownian algebra) that consistently holds open the possibilities hoped for. Thus we have in the pair Primary algebra/brownian algebra a prototype for many issues involving change of perspective and shift of language.

5. Completeness and structure of brownian algebras

We have already seen that the arithmetic V generated by $\overline{\ }\rceil$, $\overline{\overline{\ }\rceil}\rceil$, i, j with $\overline{i}\rceil = i$, $\overline{j}\rceil = j$ and $ij = \overline{\ }\rceil$ satisfies I1 (occultation) and I2 (transposition). Hence V is a model for a brownian algebra. All of our consequences hold for expressions in V. In fact, the next theorem shows that brownian algebra is *complete* with respect to this arithmetic.

5.1. *Theorem.* Let α and β be two algebraic expressions. Then $\alpha = \beta$ is a consequence of I1 and I2 if and only if $\alpha = \beta$ is true in the arithmetic V.

The proof of this result requires some preliminary work as outlined below. The first result we need is an algebraic reduction of form (similar to Theorems 14 and 15 in *Laws of Form*).

5.2. *Proposition.* Let α be any expression in the brownian algebra. Then α can be reduced to an expression involving no more than four appearances of a given variable. More precisely, suppose that X is a variable in α. Then there are expressions A,B,C,D involving no appearance of X so that

$$\alpha = \overline{AX}\rceil\ \overline{B\overline{X}\rceil}\rceil\ \overline{C\,X\overline{X}\rceil}\rceil\, D\,.$$

Proof. First note that, by using echelon, any expression is equivalent to an expression no more than two crosses deep. Hence we find that

$$\alpha = \overline{\overline{Xa_1}\rceil\ b_1}\rceil \dots \overline{\overline{Xa_n}\rceil\ b_n}\rceil\ \overline{\overline{Xc_1}\rceil\ \overline{Xd_1}\rceil}\rceil \dots \overline{\overline{Xc_m}\rceil\ \overline{Xd_m}\rceil}\rceil\ \overline{Xe_1}\rceil \dots \overline{Xe_p}\rceil\, f$$

where $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_m, d_1, \dots, d_m, e_1, \dots, e_p, f$ are expressions in which X does not appear.

Note that $\overline{\overline{Xa}\rceil\ b}\rceil = \overline{\overline{X}\rceil\ b}\rceil\,\overline{\overline{a}\rceil\ b}\rceil$ by combination (C5). Similarly, $\overline{\overline{Xc}\rceil\ Xd}\rceil = \overline{X\overline{X}\rceil\ d}\rceil\ \overline{X\,\overline{c}\rceil\ d}\rceil$. Thus the proposition follows at once from these facts and repeated applications of C5.

The next result involves evaluating an expression at $\overline{\ }\rceil$, $\overline{\overline{\ }\rceil}\rceil$, i, j. That is, we shall have an algebraic expression $\alpha = \alpha(X)$ involving a variable X. The expression α may be viewed as a function on $V = \{\overline{\ }\rceil, \overline{\overline{\ }\rceil}\rceil, i, j\}$ that gives a value in V when all of its variables are replaced by elements of V. Similarly, $\alpha(\overline{\ }\rceil)$, $\alpha(\overline{\overline{\ }\rceil}\rceil)$, $\alpha(i)$ and $\alpha(j)$ are also functions on V. (Here X is replaced by $\overline{\ }\rceil$, $\overline{\overline{\ }\rceil}\rceil$, i and j respectively). In the next proposition the symbol = refers to equality of functions on V.

5.3. *Proposition.* Let $\alpha(X) = \overline{XA}\rceil\ \overline{\overline{X}\rceil\ B}\rceil\ \overline{X\,\overline{X}\rceil\ C}\rceil\ D$ where A,B,C,D are expressions involving no appearance of the variable X. The following equalities of functions on V are valid :

$$\alpha(\overline{\overline{\ }\rceil}\rceil) = \overline{A}\rceil\, D \quad , \quad \alpha(\overline{\ }\rceil) = \overline{B}\rceil\, D$$

$$\alpha(i)\,\alpha(j) = \overline{A}\rceil\,\overline{B}\rceil\,\overline{C}\rceil\, D$$

$$\overline{\overline{\alpha(i)}\rceil\ \overline{\alpha(j)}\rceil}\rceil = D\,.$$

Proof. The first two equations are obvious. For the last note, since $\overline{i}\rceil = i$, $\alpha(i) = \overline{i\overline{A}\rceil}\rceil\ \overline{i\overline{B}\rceil}\rceil\ \overline{i\overline{C}\rceil}\rceil\ D = \overline{i\overline{A}\rceil\overline{B}\rceil\overline{C}\rceil}\rceil D$ (using C5 twice). Similarly, $\alpha(j) = \overline{j\overline{A}\rceil\overline{B}\rceil\overline{C}\rceil}\rceil D$. Hence (letting $E = \overline{A}\rceil\overline{B}\rceil\overline{C}\rceil$) we have

$$\overline{\overline{\alpha(i)}\rceil\overline{\alpha(j)}\rceil}\rceil = \overline{\overline{\overline{iE}\rceil D}\rceil\overline{\overline{jE}\rceil D}\rceil}\rceil$$

$$= \overline{\overline{\overline{iE}\rceil}\rceil\overline{\overline{jE}\rceil}\rceil}\rceil D \qquad \text{(I2)}$$

$$= \overline{ijE}\rceil D \qquad \text{(C1, C2)}$$

$$= \overline{\rceil E}\rceil D \qquad (ij = \rceil)$$

$$\therefore\ \overline{\overline{\alpha(i)}\rceil\overline{\alpha(j)}\rceil}\rceil = D \qquad \text{(I1)}.$$

Finally,

$$\alpha(i)\ \alpha(j) = \overline{iE}\rceil D\ \overline{jE}\rceil D$$

$$= \overline{\overline{\overline{i}\rceil\ \overline{j}\rceil}\rceil E}\rceil D \qquad \text{(C2, C5)}$$

$$= \overline{\overline{ij}\rceil E}\rceil D \qquad (\overline{i}\rceil = j,\ \overline{j}\rceil = i)$$

$$= \overline{E}\rceil D \qquad (ij = \rceil)$$

$$\therefore\ \alpha(i)\,\alpha(j) = \overline{\overline{A}\rceil\overline{B}\rceil\overline{C}\rceil}\rceil D\ .$$

This completes the proof of the proposition.

Proof of Theorem 5.1. We are given two algebraic expressions α and β such that $\alpha = \beta$ can be proved as a theorem about the arithmetic V. This means that $\alpha = \beta$ as functions on V. We wish to show that under these conditions $\alpha = \beta$ is demonstrable from the initials I1 and I2. The proof will proceed by induction on the total number N of variables in the two expressions.

If $N = 0$, then $\alpha = D$ and $\beta = D'$ where D and D' are constants. By hypothesis, $D = D'$ and there is nothing to prove. Note that the only algebraic constants are $\rceil$ and (blank).

Thus we assume that $N > 0$ and that the theorem is true for a smaller N. Let X be a variable appearing in one or both of the expressions α, β. By Proposition 5.2 we can assume that $\alpha = \overline{X\overline{A}\rceil}\rceil\ \overline{\overline{X}\rceil B}\rceil\ \overline{X\overline{X}\rceil C}\rceil D$ and $\beta = \overline{XA'}\rceil\ \overline{\overline{X}\rceil B'}\rceil\ \overline{X\overline{X}\rceil C'}\rceil D'$ where $A, B, C, D, A', B', C', D'$ are expressions involving no appearance of X.

By the evaluations of Proposition 5.3 and the hypotheses of this theorem it then follows that the following formulas are demonstrable:

(i) $\overline{A}\rceil D = \overline{A'}\rceil D'$

(ii) $\overline{B}\rceil D = \overline{B'}\rceil D'$

(iii) $D = D'$

(iv) $\overline{A}\rceil\overline{B}\rceil\overline{C}\rceil D = \overline{A'}\rceil\overline{B'}\rceil\overline{C'}\rceil D'$.

We now apply modal crosstransposition (C10) to demonstrate $\alpha = \beta$:

$$\alpha = \overline{X\overline{A}\rceil}\rceil\ \overline{\overline{X}\rceil B}\rceil\ \overline{X\overline{X}\rceil C}\rceil D$$

$$= \overline{\overline{\overline{X\overline{A}\rceil}\rceil\overline{\overline{X}\rceil\overline{B}\rceil}\rceil\overline{X\overline{X}\rceil}\rceil\overline{\overline{A}\rceil\overline{B}\rceil\overline{C}\rceil}\rceil}\rceil}\rceil D \qquad \text{(C10, C1)}$$

$$= \overline{\overline{X\overline{A}\rceil D}\rceil\overline{\overline{X}\rceil\overline{B}\rceil D}\rceil\overline{X\overline{X}\rceil D}\rceil\overline{\overline{A}\rceil\overline{B}\rceil\overline{C}\rceil D}\rceil}\rceil \qquad \text{(I2)}\ .$$

Now substitute, using equations (i) – (iv), reverse steps, and conclude that $\alpha = \beta$. This completes the induction step and the proof of Theorem 1.

Some technical comments are in order here. We have actually proved that any *free* brownian algebra is complete with respect to V. That is, given a set S, one can form an algebra $B(S)$ by regarding the elements of S as variables with no special relations. We then form a set of expressions $E(S)$ by the following rules:

(1) s is an expression for each $s \in S$.

(2) $\urcorner$ and (blank) are expressions.

(3) If X and Y are expressions, then $\overline{X}\rceil$, $\overline{Y}\rceil$, and XY are expressions.

The initials I1 and I2 generate an equivalence relation $\sim$ on $E(S)$. We let $B(S) = E(S)/\sim$ and say $\alpha = \beta$ if $\alpha \sim \beta$ for $\alpha, \beta \in E(S)$. The collection of equivalence classes, $B(S)$, will be called the *free brownian algebra on the set S*.

Note that V itself is *not* a free algebra. The relations $\overline{i}\rceil = i$, $\overline{j}\rceil = j$ and $ij = \urcorner$ are not consequences of I1 and I2. If $S = \{i,j\}$, then V may be regarded as the result of placing extra relations on $B(S)$. We shall not formalize this notion now, but shall return to it in section 8 when algebraic structures for self-reference are discussed.

We can now examine not only the initials and consequences of a given brownian algebra, but also the relations between algebras. The structure preserving maps between the objects of a given class are at least as important as the objects themselves. Thus we now give the definition of homomorphisms between brownian algebras.

5.4. *Definition*. Let B, B' be brownian algebras. A homomorphism $h: B \to B'$ is a set-mapping such that

$$h(\ \ ) = $$

$$h(\urcorner) = \urcorner$$

$$\text{and } h(x\,y) = h(x)\,h\,(y)$$

$$h(\overline{x}\rceil) = \overline{h(x)}\rceil$$

for all elements $x, y \in B$.

It is well-known that in a free algebra $B(S)$ any homomorphism is determined by its values $h(s)$ for $s \in S$. In the case at hand, a homomorphism between a brownian algebra B and the waveform algebra V, amounts to assigning to each variable $s \in S$ a value in the waveform arithmetic. Using this language, Theorem 5.3 may be reformulated as follows:

5.5. *Theorem*. Let $B(S)$ be a free brownian algebra on the set S. Then for $\alpha, \beta \in B(S)$, $\alpha = \beta$ if and only if $h(\alpha) = h(\beta)$ for every homomorphism $h: B(S) \to V$.

This theorem, in turn, has another reformulation that places $B(S)$ *inside* a larger waveform algebra. We first need the notion of a cartesian product of algebras (not to be confused with the ^ construction of section 3).

5.6. *Definition*. Let B and B' be brownian algebras. Then the product algebra $B \times B'$ is defined by taking the cartesian product of the underlying sets and defining operations by

(i) $\overline{(a,b)}\rceil = (\overline{a}\rceil, \overline{b}\rceil)$ (inversion but *no shift*)

(ii) $(a,b)(c,d) = (ac,bd)$.

Similarly, if A is an indexing set and we have algebras B_a, $a \in A$ then we can form the product of all of these and denote it by $\prod_{a \in A} B_a$.

Remark. A wave interpretation for this product construction will be given more fully in the next section. However, suppose that $B = B' = \hat{R}$ where R is another brownian algebra. Then $a \in B$ means that $a = (x,y)$ where (x,y) connotes a periodic pattern $\ldots xyxy \ldots$. Thus

$$a = \ldots xyxyxy \ldots$$

$$b = \ldots zwzwzw \ldots$$

and

$$ab = \ldots (xz)(yw)(xz)(yw) \ldots \in B .$$

However, we may choose to view $\ldots xzywxzyw \ldots$ as a pattern of period 4. The element $(a,b) \in B \times B$ formalizes this notion. In general $B \times \ldots \times B$ (k factors) can be interpreted as an algebra of patterns of period 2^k.

5.7. *Theorem.* Let $B(S)$ be a free brownian algebra on the set S. Let $A = \{ h : B(S) \rightarrow V \}$ be the set of homomorphisms of $B(S)$ to the waveform arithmetic V. Let V_h denote (a copy of) V, corresponding to each homomorphism $h \in A$. Then there is an injective homomorphism

$$\Phi : B(S) \longrightarrow \prod_{h \in A} V_h .$$

Proof. Define Φ by $\Phi(x) = \prod_{h \in A} h(x)$ for each $x \in B(S)$. (Note that $h : B(S) \rightarrow V_h$.) Since $x = y$ in $B(S)$ if and only if $h(x) = h(y)$ for all $h \in A$, we have $x = y$ if and only if $\Phi(x) = \Phi(y)$. Hence Φ is injective.

In fact, Theorem 5.7 is true for arbitrary brownian algebras. The proof of this more general version follows from further reformulations and the use of deeper results about De Morgan algebras. For references, and a discussion of this point see Kauffman (1978*b*).

From our point of view, this result is quite significant, since it shows that *any* brownian algebra may be seen as a subalgebra of a wave-form algebra $\prod_{h \in A} V_h$. The latter is generated entirely by self-reflexive elements, that is, by solutions of $x = \overline{x}\rceil$.

Thus the wave forms associated with the simple re-entering form $x = \overline{x}\rceil = ⅂$ stand at the base of all our considerations. The 'real' logical or indicational values such as $\rceil$ are seen as combinations of synchronized waveforms ($ij = \rceil$). This principle remains true in the general context of all algebras satisfying occultation and transposition.

In this regard it is worth noting that the self-reflexive elements of a brownian algebra are *irreducible*. That is:

5.8. *Proposition.* Let B be a brownian algebra containing X such that $\overline{X}\rceil = X$. If $X = YZ$ with $\overline{Y}\rceil = Y$ and $\overline{Z}\rceil = Z$, then $X = Y = Z$.

Proof. Note that the proposition will follow if we show that $AB = AC \Rightarrow B = C$ when $\overline{A}\rceil = A, \overline{B}\rceil = B, \overline{C}\rceil = C$. ($X = YZ \Rightarrow YX = YZ \Rightarrow X = Z$ etc . . .) To prove this, we proceed as follows:

$$
\begin{aligned}
B &= \overline{\overline{B|}\;A|}\;\; B && (\mathrm{I1})\\
&= \overline{BA|}\;\; B && (B = \overline{B|})\\
&= \overline{CA|}\;\; B && (CA = AC = AB = BA)\\
&= \overline{\overline{C|}\,\overline{A|}|}\;\; B && (C = \overline{C|},\ A = \overline{A|})\\
&= \overline{\overline{CB|}\;\overline{AB|}|} && (\mathrm{I2})\\
&= \overline{\overline{CB|}\;\overline{AC|}|} && (AB = AC)\\
&= \overline{\overline{B|}\,\overline{A|}|}\;\; C && (\mathrm{I2})\\
&= \overline{BA\,|}\;\; C && (\overline{A|} = A,\ \overline{B|} = B)\\
&= \overline{CA\,|}\;\; C && (BA = CA)\\
&= \overline{\overline{C|}\;A|}\;\; C && (\overline{C|} = C)\\
\therefore\;\; B &= C && (\mathrm{I1})
\end{aligned}
$$

This completes the proof.

6. Varieties of waveforms and interference phenomena

In the algebra $\hat{B}$, associated to a brownian algebra B, we find period two sequences of elements from B. Thus for $a,b \in B$ we have the correspondence $(a,b) \leftrightarrow \ldots ababab \ldots$. So far, however, we have encountered only waveforms of period 2, namely those of the waveform arithmetic V. There is no reason to stick to patterns of period 2 (high frequency) in the context developed so far. Let us now discuss explicit constructions for waveforms of arbitrary period.

6.1 *Definition*. Let p be an even positive integer and let $k = p/2$ so that $p = 2k$ Let B be a given brownian algebra. Define $S_p(B)$ to be the set of *sequences* in B of period p. That is

$$S_p(B) = \{a = \{a_n\}\ ,\ n \in Z \ |a_n \in B \text{ and } a_{n+p} = a_n \text{ for all } n\}\ .$$

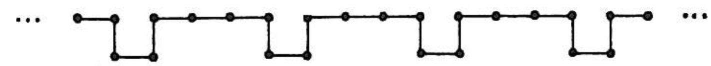
(Z denotes the set of integers.) This collection of sequences can be transformed into an algebra by extending the operations of continence and crossing in the following way:

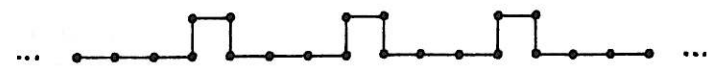
(i) $ab = \{(ab)_n\}\ ,\ (ab)_n = a_n b_n$ for $n \in Z$
and $a,b \in S_p(B)$.

(ii) $\overline{a|} = \{(\overline{a|})_n\}\ ,\ (\overline{a|})_n = a_{n-k}\ \ ,\ \ k = p/2$
for $n \in Z$ and $a \in S_p(B)$.

Thus crossing is accomplished by combining ordinary inversion for B with a half-period shift (whence p must be even). This extends the previous use of crossing for the Λ construction.

By way of illustration, consider a sequence of period 4, and the special case $B = P$ (the primary arithmetic); that is, consider $S_4(P)$. Take one such sequence a,

a = 

$\therefore\ \overline{a|}$ = 

$\therefore\ \overline{a|}\ a$ = 

Thus $\overline{a|}\,a = a$.

It is easy to verify that occultation and transposition hold in $S_p(B)$. Thus $S_p(B)$ is a brownian algebra. The next result gives a more precise idea of the structure of this algebra, by showing that it is indistinguishable from tuples of forms of period 2.

6.2. *Proposition.* Let B be a brownian algebra. Then we have the following isomorphism of algebras:

$$S_p(B) \cong \prod_{l=1}^{k} \hat{B}_l$$

where $\hat{B}_l$ ($l = 1, \dots, k$) denotes (a copy of) $\hat{B}$ corresponding to each of the integers 1 through k, and $k = p/2$.

Proof. Let $\overline{S}_p(B) = \{ (\alpha,\beta) \mid \alpha = (\alpha_1, \dots, \alpha_k), \beta = (\beta_1, \dots, \beta_k)$ and $\alpha_i, \beta_j \in B$ for $1 \le i, j \le k \}$. Define operations in $\overline{S}_p(B)$ as follows: $(\alpha,\beta)(\alpha',\beta') = ((\alpha_1\alpha'_1, \dots, \alpha_k\alpha'_k), (\beta_1\beta'_1, \dots, \beta_k\beta'_k))$, $\overline{(\alpha,\beta)|} = ((\overline{\beta_1|}, \dots, \overline{\beta_k|}), (\overline{\alpha_1|}, \dots, \overline{\alpha_k|}))$. Then $\overline{S}_p(B)$ is a brownian algebra, and we may map $S_p(B)$ to $\overline{S}_p(B)$ by the function h: $S_p(B) \longrightarrow \overline{S}_p(B)$ where $h(a) = ((a_1, \dots, a_k), (a_{k+1}, \dots, a_p))$. This is clearly an isomorphism. On the other hand, we have an isomorphism:

$$g: \overline{S}_p(B) \longrightarrow \prod_{l=1}^{k} \hat{B}_l$$

given by $g(\alpha,\beta) = ((\alpha_1,\beta_1), (\alpha_2,\beta_2), \dots, (\alpha_k,\beta_k))$. Hence the composition $g \circ h$ *is* the desired isomorphism.

When $p = 2$ this result shows that $S_2(B) \cong \hat{B}$, as expected. When $B = P$, the primary arithmetic, then,

$$S_p(P) \cong \prod_{l=1}^{k} V_l$$

(where V_l is a copy of V). For the example of $a \in S_4(P)$ discussed above, one obtains $S_4(P) \cong V \times V$ and $a \leftrightarrow ((\overline{\ \ |}, \overline{|}), (\overline{\overline{|}|}, \overline{|})) = (\overline{|}, j)$.

By combining two sequences in the same indicational space, we produce an *interference* between the waveforms they represent. This is what is involved in the extension of crossing in (ii) above. Interference follows quite naturally for waveforms of the same period (but arbitrary frequency). We are left with the question of interference of waveforms of widely different period, so that we may handle the resulting patterns in an adequate way. One approach, presented below, is to concentrate on the *least* possible period resulting from the interference.

Let *lcm* (p,q) denote the least common multiple of the integers p,q. For even integers p,q the least common multiple is also even, and we may define a mapping μ: $S_p(B) \times S_q(B) \longrightarrow S_r(B)$ where $r = lcm\ (p,q)$ and $\mu\ [(a,b)\ ]_n$ =

$a_n b_n$. Thus two sequences of different period interfere to form a new sequence whose least period divides the least common multiple of the initial periods. By choosing to look at interference in this way we are simply stressing the high frequency components of the interference pattern. We are also insisting on a rule for assigning period to the interference pattern that will work well algebraically. Least common multiple has this property, as we shall see.

What are the difficulties in constructing an algebra of sequences of varying periods? The first point can be seen in the waveform arithmetic V. There $ij = \urcorner$ where i and j have period 2. If we must assign a period to $\urcorner$, then $2 = lcm\ (2,2)$ seems to be the natural choice! We might say that the mark, $\urcorner$, in V resonates with period 2. This leads to no confusion since everything in V (or $\mathcal{B}$) has period 2. In the more general situation $\urcorner$ may have to be regarded differently depending upon the assigned period. Consequently integration in the form $\urcorner a = \urcorner$ will fail. It will hold only if we disregard the period of a. Since the marked state represents the observer, it is not surprising that we should find a spectrum of marked states, each corresponding to a different resonant condition. In order to capture this aspect, we define a *generalized brownian algebra* as follows.

6.3. *Definition*. A *generalized brownian algebra* is an algebra satisfying the initials

(i) $aa = a$ (iteration)

(ii) $\overline{\overline{a|}|} = a$ (reflection)

Note that such a generalized algebra has nothing more than exterior descriptions of calling ($\urcorner\urcorner = \urcorner$) and crossing ($\overline{\urcorner|}$ = (blank)).

6.4. *Definition*. Let B be a brownian algebra and let $S(B)$ denote the set of periodic sequences in B with even (assigned) period. If $a, b \in S(B)$, then we write $a = b$ when $a_n = b_n$ for all n, and $p(a) = p(b)$ ($p(a)$ = the period assigned to a). Operations are defined as follows:

(i) $(ab)_n = a_n b_n$, $p(ab) = lcm\ (p(a), p(b))$.

(ii) $(\overline{a|})_n = a_{n-k}$, $k = p(a)/2$

$p(\overline{a|}) = p(a)$.

Since $lcm(lcm\ (x,y),z) = lcm\ (x, lcm\ (y,z))$ for any integers x,y,z, the operation (i) is associative. Associativity becomes explicit at this point.

Another way of putting our observation about resonant marked states is to note that we may assign any period p to the *empty sequence* $\phi = \ldots, , , , , \ldots \in S(B)$. Then $\overline{\phi|} = \ldots \urcorner, \urcorner, \urcorner, \urcorner, \ldots$ also has period p. Thus we cannot write $\urcorner = \ldots \urcorner, \urcorner, \urcorner, \ldots$ without noting its period.

6.4. *Lemma*. $S(B)$ is a generalized brownian algebra.
The proof is immediate.

In order to understand the structure of $S(B)$ we now give conditions under which occultation and transposition hold.

6.5. *Theorem*. Let $a,b,c \in S(B)$ be members of the sequence algebra for a brownian algebra B. Then

(i) $\overline{\overline{a|}\ b|}\ a = a$ if and only if $p(b)$ divides $p(a)$;

(ii) $\overline{\overline{a|}\ \overline{b|}|}\ c = \overline{\overline{ac|}\ \overline{bc|}|}$ whenever the largest power of 2 dividing N is the same for $N = p(a)$, $N = p(b)$ and $N = p(c)$.

The following lemma will be used in the proof.

6.6. *Lemma*. Let $k(a) = \frac{1}{2}p(a)$ for any $a \in S(B)$. Then

(i) $p(b) \mid p(a)$ ($x \mid y$ means x divides y)
$\Rightarrow p(a) \mid (k(ab) + k(a))$.

(ii) If $p(a) = 2^{\alpha}N$, $p(b) = 2^{\alpha}M$ when N and M are odd,
then $p(a) \mid (k(ab) + k(a))$ and $p(b) \mid (k(ab) + k(b))$.

Proof of lemma. The proof is omitted.

Proof of theorem. If $\overline{\overline{a\rceil}\, b\rceil}\ a = a$ then $p(\overline{\overline{a\rceil}\, b\rceil}\ a) = p(a)$. But $p(\overline{\overline{a\rceil}\, b\rceil}\ a) = lcm\ (p(a), p(b))$ and $\therefore p(a) = lcm\ (p(a), p(b))$ and this implies that $p(b) \mid p(a)$.
Conversely, suppose that $p(b) \mid p(a)$. Then for any n, $a_{n-k(ab)-k(a)} = a_n$ since $k(ab) + k(a)$ is a multiple of $p(a)$.

Hence

$$
\begin{aligned}
(\overline{\overline{a\rceil}\, b\rceil}\ a)_n &= (\overline{\overline{a\rceil}\, b\rceil}\ )_n\, a_n \\
&= \overline{(\overline{a\rceil}\, b)_{n-k(ab)}\rceil}\ a_n \\
&= \overline{\overline{a_{n-k(ab)-k(a)}\rceil}\ b_{n-k(ab)}\rceil}\ a_n \\
&= \overline{\overline{a_n\rceil}\ b_{n-k(ab)}\rceil}\ a_n \\
&= a_n \quad \text{(occultation in } B).
\end{aligned}
$$

This proves (i).

To prove (ii) first note that since $lcm\ (lcm\ (x,y),z) = lcm\ (lcm\ (x,z), lcm\ (y,z))$ for any integers x,y,z, $p(\overline{\overline{a\rceil}\ \overline{b\rceil}\rceil}\ c) = p(\overline{\overline{ac\rceil}\ \overline{bc\rceil}\rceil}\ )$ for any $a,b,c \in S(B)$. Thus we must show that $\overline{\overline{a\rceil}\ \overline{b\rceil}\rceil}\ c$ and $\overline{\overline{ac\rceil}\ \overline{bc\rceil}\rceil}$ are identical term by term. We are given that $p(a) = 2^{\alpha}N$, $p(b) = 2^{\alpha}M$, $p(c) = 2^{\alpha}R$ where N,M and R are odd.

$$
\begin{aligned}
\text{Then } (\overline{\overline{a\rceil}\ \overline{b\rceil}\rceil}\ c)_n &= \overline{(\overline{a\rceil}\ \overline{b\rceil}\ )_{n-k(ab)}\rceil}\ c_n \\
&= \overline{\overline{a_{n-k(ab)-k(a)}\rceil}\ \overline{b_{n-k(ab)-k(b)}\rceil}\rceil}\ c_n \\
&= \overline{\overline{a_n\rceil}\ \overline{b_n\rceil}\rceil}\ c_n
\end{aligned}
$$

Since $p(a) \mid k(ab) + k(a)$ and $p(b) \mid k(ab) + k(b)$ (by Lemme 6.6.) .

On the other hand,

$$
\begin{aligned}
(\overline{\overline{ac\rceil}\ \overline{bc\rceil}\rceil}\ )_n &= \overline{(\overline{ac\rceil}\ \overline{bc\rceil}\ )_{n-k(abc)}\rceil} \\
&= \overline{\overline{(ac)_{n-k(abc)-k(ac)}\rceil}\ \overline{(bc)_{n-k(abc)-k(bc)}\rceil}\rceil} \\
&= \overline{\overline{(ac)_n\rceil}\ \overline{(bc)_n\rceil}\rceil}
\end{aligned}
$$

Since $p(ac) \mid k(abc) + k(ac)$ and $p(bc) \mid k(abc) + k(bc)$ (by Lemma 6,6) .

$$
\begin{aligned}
&= \overline{\overline{a_n\, c_n\rceil}\ \overline{b_n\, c_n\rceil}\rceil} \\
&= \overline{\overline{a_n\rceil}\ \overline{b_n\rceil}\rceil}\ c_n \\
&= (\overline{\overline{a\rceil}\ \overline{b\rceil}\rceil}\ c)_n
\end{aligned}
$$

This completes the proof of the theorem.

Remark. If we let $S^{\alpha}(B)$ denote the set of elements $a \in S(B)$ such that $p(a) = 2^{\alpha}M$ for M odd, then transposition is satisfied in $S^{\alpha}(B)$. Call an algebra satisfying iteration, reflection and transposition a *transposition algebra*. We see that for each a, $S^a(B)$ is a transposition algebra. By (i) of 6.5 it is not a brownian since $\overline{\overline{a}\rceil\;\; b}\rceil\;\; a \;=\; a$ if and only if $p(b) \mid p(a)$. It is easy to verify that echelon, combination, catalysis, tension, modified transposition and modal cross-transposition are valid in any transposition algebra. Occultation is not a consequence, as the models $S^a(B)$ show.

Remark. It is worth noting exactly how close a transposition algebra comes to being a brownian algebra. If we include the initial $\overline{\rceil}\rceil a = \rceil$, then we can get occultation as a consequence:

$$\overline{\overline{a}\rceil\, b}\rceil\; a = \overline{\overline{a}\rceil\;\overline{\overline{b}\rceil}\rceil}\rceil\; a = \overline{\overline{aa}\rceil\;\;\overline{\overline{b}\rceil\, a}\rceil}\rceil = \overline{\overline{a}\rceil\;\;\overline{\overline{b}\rceil\, a}\rceil}\rceil = \overline{\rceil\;\;\overline{\overline{b}\rceil}\rceil}\rceil\; a = \overline{\rceil}\rceil a = a.$$

Thus $\rceil$ *becomes relativized to the period of the sequence that it interacts with in a generalized brownian algebra.* We can assert that $\overline{\rceil}\rceil b = \rceil$ for any b only by allowing the frequency of $\rceil$ to change with b. Nevertheless, the rules $\rceil\rceil = \rceil$ and $\overline{\rceil}\rceil$ = (blank) still hold. Even if $\rceil$ is seen to be resonating at a given frequency, it still calls and cancels itself.

7. Constructing waveforms

In dealing with waveforms, we have so far assumed that there are sequences of elements from an algebra B. The relationship between the sequences and the underlying algebra has remained mysterious. We now show that the operations of the algebra B itself are capable of generating oscillations, by the simple expedient of recursion. That is, given an algebra B and an algebraic operation $T:B \to B$, we consider the iterates $T^0 = 1$, $T^1 = T$, $T^2 = T \circ T, \ldots, T^{n+1} = T^n \circ T$.

If there is an integer p such that $T^{n+p} = T^n$ for all n, then T can be used to produce sequences of period p.

For example, let $T(x) = \overline{x}\rceil$. Then $T^2(x) = x$ and $T^{n+2} = T^n$ for all n. T produces the sequence: $x, \overline{x}\rceil, x, \overline{x}\rceil, x, \overline{x}\rceil, \ldots$ In this case we have an algebraic version of the sequence. That is, if $x \in B$, then $\alpha = (x, \overline{x}\rceil)$ and $\beta = (\overline{x}\rceil, x)$ belong to $\hat{B}$ and represent two phase-shifted versions:

$\alpha : \ldots x\, \overline{x}\rceil\, x\, \overline{x}\rceil\, x\, \overline{x}\rceil \ldots$

$\beta : \ldots \overline{x}\rceil\, x\, \overline{x}\rceil\, x\, \overline{x}\rceil\, x \ldots$

We are given $T:B \to B$ and obtain the corresponding mapping $\hat{T}: \hat{B} \to \hat{B}$ defined by the same formula. Note that

$$\hat{T}(\alpha) = \overline{\alpha}\rceil = \overline{(x, \overline{x}\rceil)}\rceil = (\overline{x}\rceil, \overline{\overline{x}\rceil}\rceil) = (x, \overline{x}\rceil) = \alpha$$

and $\hat{T}(\beta) = \beta$. Thus the sequences generated by T become the fixed points of $\hat{T}$. This correspondence holds more generally, as is shown in the next two results.

7.1. *Proposition.* Let B be an algebra (or arithmetic) satisfying all the initials for the primary algebra. We shall say that B is *primary*. Then an algebraic mapping $T:B \to B$ will generate sequences of period at most 2.

Proof. Since T is an operation in the primary algebra, it has a canonical structure with respect to the variables that it operates on; in fact $T(x) = \overline{\overline{ax}\rceil\;\; b\; \overline{x}\rceil}\rceil\;\; c$ for some $a,b,c \in B$ not containing x. A simple calculation shows that $T^2(x) = T(T(x)) = \overline{ab\, x}\rceil\;\; \overline{\overline{a}\rceil\; \overline{b}\rceil\;\; \overline{x}\rceil}\rceil\; c$ and that $T^3(x) = T(x)$. Hence by induction on n, $T^{n+2}(x) = T^n(x)$ for all n.

7.2. *Theorem*. Let B be primary, and $T{:}B \to B$ an algebraic mapping. Let $\hat{T}$ be the corresponding mapping on $\hat{B}$. Then there exists a $Z \in \hat{B}$ such that $\hat{T}(Z) = Z$. In fact, we may take $Z = (T(x), T^2(x))$ or $Z = (T^2(x), T(x))$ for any $x \in B$.

Proof. To see that $Z = (T(x), T^2(x))$ is a fixed point for $\hat{T}$, first note that for any $Z = (\alpha, \beta) \in \hat{B}$,

$$
\begin{aligned}
\hat{T}(Z) &= \overline{aZ}\;\overline{b\overline{Z}}\;c \\
&= \overline{(a\alpha, a\beta)}\;\overline{b(\overline{\beta}, \overline{\alpha})}\;c \\
&= (\overline{a\beta}, \overline{a\alpha})\,(\overline{b\overline{\alpha}}, \overline{b\overline{\beta}})\,c \\
&= (\overline{a\beta}\;\overline{b\overline{\alpha}}\;c,\ \overline{a\alpha}\;\overline{b\overline{\beta}}\;c)\,.
\end{aligned}
$$

Thus it suffices to show that

$$T(x) = \overline{aT^2(x)}\;\overline{b\,\overline{T(x)}}\;c$$

and

$$T^2(x) = \overline{aT(x)}\;\overline{b\overline{T^2(x)}}\;c$$

when

$$T(x) = \overline{ax}\;\overline{b\overline{x}}\;c.$$

This is a straightforward computation.

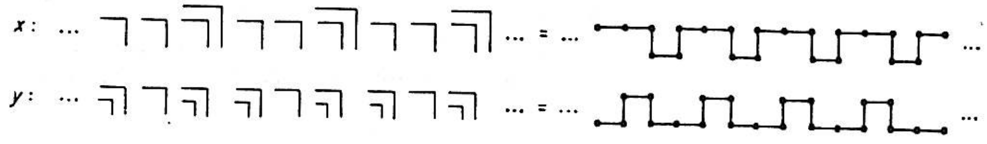
Thus the algebraic structure of $\hat{B}$ reflects the properties of periodic sequences that are generated from B. *Sequences from B become algebraic fixed points in* $\hat{B}$.

What we see emerging here is a beautiful harmony among oscillations, re-entering forms, algebraic operations and their fixed points. So far we have seen that the re-entry of a form to its own indicational space, as in $x = \overline{x}$ = ⊐, gives rise to a fundamentally new arithmetic V, where we have the waveforms i and j. These are fixed points for the cross ⏋ in the new algebra V. We can attempt to generalize this situation by showing how every re-entering form will give rise to an oscillation: the fixed points of the operator represent the spatial view of the oscillation, while its associated sequences represent the temporal context. Now in order to do this, we have to be able to construct an algebra where infinite expressions are *defined*, so that we are assured that every algebraic expression actually has a fixed point in the *same* algebra, rather than in a larger one (as in the waveform arithmetic where i is in V but not in P). We provide such a construction in the next section, and in so doing we will see that the correspondence between pattern (re-entry) and oscillation (sequences) will be partially lost. Although every re-entry form will oscillate, a given waveform can be generated through many alternative operators when they are allowed to re-enter.

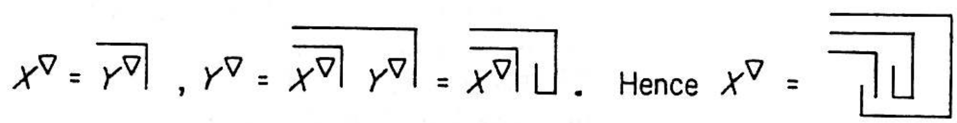
The remainder of this section will be devoted to procedures for generating sequences of arbitrary period. As we have seen, sequences of period 2 may be generated by an operator with a single variable. In order to generate sequences of period other than 2, it is necessary to use recursion on more than one variable. For example, let $T{:}P \times P \to P \times P$ (P denotes the primary arithmetic) be defined by $T(x,y) = (\overline{y}, \overline{\overline{x}\,y})$. Then

T (⌉ , ⌉⌉) = (⌉ , ⌉)
T (⌉ , ⌉) = (⌉⌉ , ⌉⌉)
T (⌉⌉ , ⌉⌉) = (⌉ , ⌉⌉)

and hence T produces two entrained oscillations of period 3:



We say that T produces an oscillation of period 3 and *dimension* 2. Notice that we may also represent this two-dimensional waveform by the spatial pattern of re-entry of the operation which generates it: That is, $T(X^{\nabla}, Y^{\nabla}) = (X^{\nabla}, Y^{\nabla})$ represents the spatial fixed point and in this case,



We now show that it is a simple matter to determine operators T that produce a given wave-train.

7.3. *Definition*. Let P denote the primary arithmetic, and $P^n = P \times P \times \ldots \times P$ the n-fold cartesian product of P with itself. An algebraic operator $T: P^n \rightarrow P^n$ is a function

$$T(x_1, \ldots, x_n) = (T_1(x_1, \ldots, x_n), \ldots, T_n(x_1, \ldots, x_n))$$

where each $T_k(x_1, \ldots, x_n)$ is an expression in the primary algebra involving the variables $x_1, \ldots, x_n$. We say that T is *periodic* if there exists an integer p such that $T^{p+n} = T^n$ for all integers $n > N$ (p and n are non-negative integers) where N is some specified integer.

Example. Let $T: P^2 \rightarrow P^2$ be defined by $T(x,y)$ = (⌉x⌉ y⌉⌉ , x y⌉). Now P^2 = a,b,c,d where a = (⌉, ⌉), b = (⌉, **⌉**), c = (**⌉**, **⌉**) and d = (**⌉**, ⌉). It is easy to verify that $T(a) = b$, $T(b) = c$, $T(c) = d$ and $T(d) = c$. Thus we say that T has period 2 since $T^{n+2} = T^n$ for $n > 1$. Note that $T^3 \neq T$ since $T(a) = b$ while $T^3(a) = d$. With this definition we immediately obtain the following result:

7.4. *Theorem*. Every algebraic operator is periodic.

Proof. P^n has cardinality 2^n (with respect to arithmetic values). Hence for any fixed $x = (x_1, \ldots, x_n) \in P^n$, the set $\{T^n(x) \mid n = 1,2, \ldots\}$ is finite. Thus the sequence $T(x), T^2(x), \ldots$ must be eventually periodic for each x. Since there are a finite number of such x, the least common multiple of the corresponding periods is necessarily a period for T.

7.5. *Theorem*. Let $\pi: P^n \rightarrow P$ be the projection to the first co-ordinate, $\pi(\alpha_1, \ldots, \alpha_n) = \alpha_1$. Let $\{a = \{a_n\} \mid n = 1,2, \ldots\}$ be any periodic sequence of values from P. Let p be the least period of a, and choose n so that $2^{n-2} < p \leq 2^{n-1}$. Then there exists an operator T: $P^n \to P^n$ of least period p, and a starting vector of values $x \in P^n$ so that $a_n = \pi(T^n(x))$ for $n = 1,2,\ldots$.

Thus the sequence a can be seen as the first component of an n-dimensional entrained oscillation.

Proof. We shall give an algorithm for producing the requisite operator. The following notation is convenient. Let $b = (b_1, b_2, \ldots, b_n) \in P^n$, and let $\Delta(b) = \Delta_1\Delta_2 \ldots \Delta$ where $\Delta_i = 1$ if $b_i = \boldsymbol{\urcorner}$ and $\Delta_i = 0$ if $b_i = \urcorner$. Regard $\Delta(b)$ as an integer expressed in the binary system. Let $\Omega(b)$ be the corresponding decimal integer. Let $\vartheta(b)$ be the following operator:

$$\vartheta(b)(x_1, \ldots, x_n) = \overline{b_1(x_1) b_2(x_2) \ldots b_n(x_n)}\rceil$$

where $b_i(x) = \overline{x}\rceil$ if $b_i = \urcorner$

x if $b_i = \boldsymbol{\urcorner}$.

Note that $\vartheta(b): P^n \to P$ and $\vartheta(b)(x) = \urcorner \Leftrightarrow x = b$. For computations it is often useful to use $\Omega(b)$ as the name of $\vartheta(b)$.

Now choose $b_1, b_2, \ldots, b_p \in P^n$ so that

(i) $b_i \neq b_j$ if $i \neq j$.

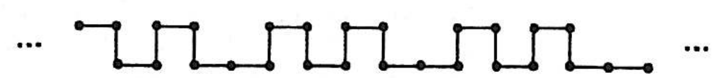
(ii) $\pi(b_k) = a_k$, $k = 1,2,\ldots,p$.

This can be done since $2^{n-2} < p \leq 2^{n-1}$.

Let $T_k(x_1, \ldots, x_n) = \vartheta(b\alpha_1)\,\vartheta(b\alpha_2) \ldots \vartheta(b\alpha_e)$ where $\alpha_1, \ldots, \alpha_e$ is the set of indices α such that kth co-ordinate of $b(\alpha+1)$ is marked (we view α modulo p so that $b_{p+1} = b_1$). Finally, let $T(x_1, \ldots, x_n) = (T_1(x), \ldots, T_n(x))$.

It is easy to verify that $T(b_k) = b_{k+1}$ and $T(b_p) = b_1$. Thus T produces the desired periodic sequence. This completes the proof of the theorem.

In order to illustrate the foregoing theorem, suppose that we wish to produce the period 5 oscillation:



That is, $a_1 = \urcorner, a_2 = \boldsymbol{\urcorner}, a_3 = \urcorner, a_4 = \boldsymbol{\urcorner}, a_5 = \boldsymbol{\urcorner}$. Then we take $n = 3$ and we may choose $b_1, \ldots, b_5$ as in the chart Fig. 2).

Hence

$$\Omega(b_1) = 4, \Omega(b_2) = 0, \Omega(b_3) = 5, \Omega(b_4) = 1, \Omega(b_5) = 2.$$

Hence

$$T_1 = 2\,0 = \overline{x\,\overline{y}\rceil\,z}\rceil\ \overline{xyz}\rceil = \overline{xz}\rceil$$

$$T_2 = 1 = \overline{xy\,\overline{z}\rceil}\rceil$$

$$T_3 = 0\,5 = \overline{xyz}\rceil\ \overline{\overline{x}\rceil\,y\,\overline{z}\rceil}\rceil$$

Thus $T(x,y,z) = (\overline{xz}\rceil, \overline{xy\,\overline{z}\rceil}\rceil, \overline{xyz}\rceil\ \overline{\overline{x}\rceil\,y\,\overline{z}\rceil}\rceil)$.

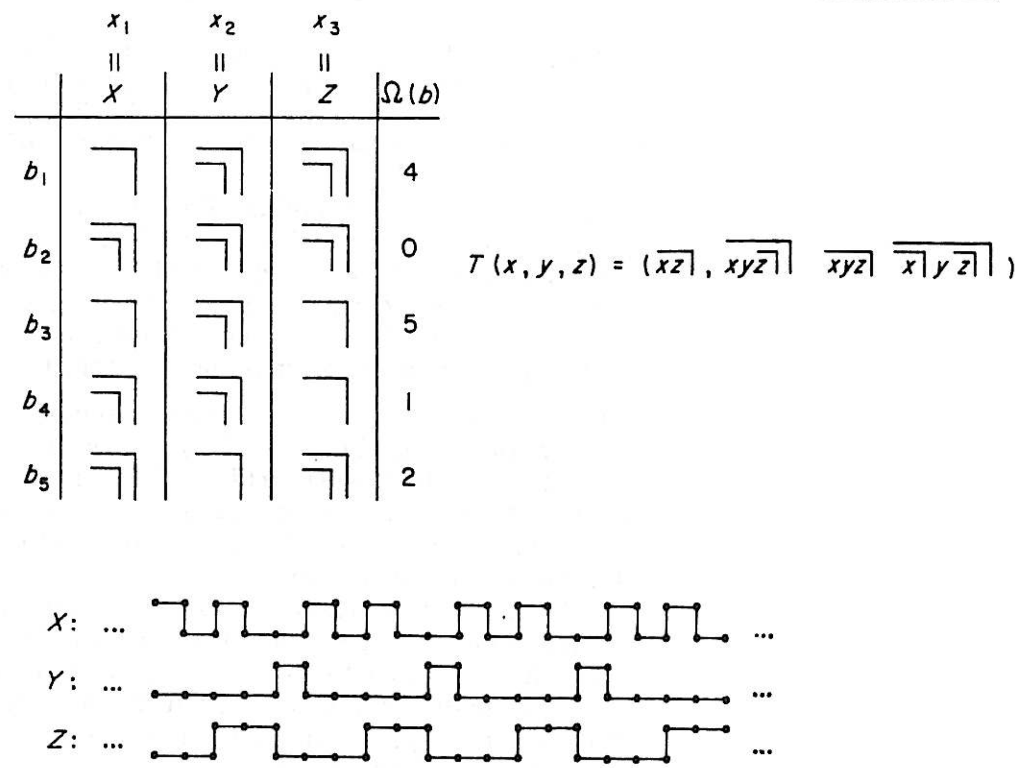


Fig. 2

In general there will be many operators T such that $a_n = \pi T^n(x)$ for a given sequence $a = \{ a_n \}$. We can easily generalize this lemma to embed any entrained oscillation (of arbitrary dimension) in a higher-dimensional oscillation without repetitions (in the sense that $b_i \neq b_j$ for $i \neq j$). Applying the same algorithm, we thereby obtain an operator T whose projected iterates give rise to the original entrainment.

In the next section we look more closely at the formal structure of this category of operators.

8. Re-entering forms and infinite expressions

In the primary arithmetic there is no separation of operator and operand. The mark, ⌉, takes both roles as context and viewpoint demand. We have constructed certain 'complex' arithmetical forms (such as i and j of the waveform arithmetic) but these have not been constructed as operators. Rather, they are seen as *fixed points* of an operator. Thus $\overline{i}| = i$ and $\overline{j}| = j$. In order to return to a viewpoint wherein such fixed points are seen also as operators, we need to re-consider the self-referential context.

What is the meaning of $f =$ ⊐ ? We have suggested in section 2 that ⊐ may be viewed as an infinite form $f =$ ⊐ $= \overline{\overline{\cdots}|}|$ so that f is literally contained as a proper part of its own indicational space ($f = \overline{f}\rceil$). When we write $f = \overline{\overline{\overline{\;}\rceil}\rceil}\rceil = \overline{f}\rceil$ we are not saying that f may be calculated from $\overline{f}\rceil$, but rather that f and $\overline{f}\rceil$ are *identical* as infinite forms. Seen in this light, f operates *on itself* by way of the re-entry. This operation is the very lifeblood of f = ⊐ giving it the *stability* expressed by $\overline{⊐}\rceil$ = ⊐ .

In this sense the very notion of a self-contained form is closely tied to self-reference. The extent to which a form is seen to be autonomous/self-contained is directly proportional to how we find its stabilities. We push at it here, pull at it there. It seems to react, stabilize, retain shape and intelligibility. How can this come about? Even to a pure solipsist, the experience of relatively stable, seemingly external forms must present something of a puzzle. And yet the simple form of ⊐ $= \overline{\overline{\overline{\;}\rceil}\rceil}\rceil$ presents a shift of perspective. Let be given only an operator T (say $T(x) = \overline{x}\rceil$). Allow the operator to operate on itself infinitely as in $A = T(T(T(T(T(\ldots)))))$. Then $T(A) = A$ is a formal identity. The infinite concatenation of the operator on itself gives rise to a stability with respect to this operation. It becomes possible to say of A: it has form. It is an object with a life of its own.

It is no denial of reality to suggest that what we call objects are in fact nothing more than operations taken to such a limit. In fact we do suggest this and invite the reader to consider the idea in all of its remifications. When we investigate natural forms we continually come upon circularities of structure that lend stability. Are not these precisely the same as our formal images of self-containment and self-reference?

In order to understand the main idea for allowing infinite forms, consider the following example (due to Spencer-Brown). Let $T(x) = \overline{\overline{x\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil$. Then iterating T we have, as expected,

$$T^2(x) = \overline{\overline{\overline{\overline{x\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil = \overline{\overline{x\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil = T(x).$$

Thus

$$\overline{\overline{x\overline{a}\rceil}\,\overline{b}\rceil}\rceil = \overline{\overline{\overline{\overline{x\overline{a}\rceil}\,\overline{b}\rceil}\rceil\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil = \overline{\overline{\overline{\overline{\overline{\overline{x\overline{a}\rceil}\,\overline{b}\rceil}\rceil\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil = \overline{\overline{\overline{\overline{\overline{\overline{\overline{\overline{x\overline{a}\rceil}\,\overline{b}\rceil}\rceil\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil$$

and if we allow this process to proceed indefinitely, we are led to contemplate the infinite expression,

$$x^{\nabla} = \ldots \overline{\overline{\overline{\overline{a}\rceil\,\overline{b}\rceil}\rceil\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil \; .$$

This form contains a copy of itself, and thus re-enters its own indicational space

$$x^{\nabla} = \overline{\overline{x^{\nabla}\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil \; .$$

By going to an infinite expression, we have eliminated x as a variable and obtained a form, or spatial pattern, which embodies the operation.

In other language, x^{∇} is the fixed point of T, for

$$T(x^{\nabla}) = \overline{\overline{x^{\nabla}\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil = \ldots \overline{\overline{\overline{\overline{a}\rceil\,\overline{b}\rceil}\rceil\,\overline{a}\rceil}\,\overline{b}\rceil}\rceil = x^{\nabla} \; .$$

The equation $T(x^{\nabla}) = x^{\nabla}$ is an expression of the direct identity of these expressions; it is not a statement that one can be calculated from the other. In general, by going into a suitable structure, where infinite expressions are allowed, we are assured that every operation will have a fixed point solution, for we can form

$$x^{\nabla} = T(T(T(\ldots))) = \lim_{n \to \infty} T^n,$$

the infinite concatenation of this operator. In this universe of infinite forms we are free to express the re-entry of forms. What needs to be examined is the relation between spatial re-entry and its temporal quality: given a pattern, how does it vibrate?

In order to explore this question we first have to construct a universe of infinite expressions and see how re-entry is expressed in them more precisely. We have two basic clues from the previous discussion. First, in order to form an infinite expression we may allow it to grow step by step, and never command the process to come to a halt. This involves introducing the idea of an *order* in the class of expressions, so that at each successive step the new expression is a better approximation to the infinite one. In the limit, the sequence of approximations defines the infinite expression. This is a process somewhat reminiscent of the idea of order, approximation and limit in calculus. (The order being introduced here, however, is quite different from numerical approximation). Secondly, in order to have re-entry, it is enough to consider *equations* among these infinite expressions, or, in the other words, fixed points for operators in this extended domain.

As an example, with the operation $T(x) = \overline{\overline{x\ a}|\ b}|$ we may consider a sequence of approximate expressions thus :

$$\perp \sqsubseteq \overline{\overline{\perp\ a}|\ b}| \sqsubseteq \overline{\overline{\overline{\overline{\perp\ a}|\ b}|\ a}|\ b}| \sqsubseteq \ldots$$

where we start with an undefined expression $\perp$, and successively apply T. Here $\sqsubseteq$ denotes the order relation and $x^{\nabla} = \lim_{n \to \infty} T^n(\perp)$. The reader will have to forgive a full exposition of the details of order and approximation for infinite expressions. We hope that we have given the flavor of it; the full technical description may be found in Goguen & Varela (1979) and Varela & Goguen (1978).

Thus we now assume that we have before us a well defined class B_∞ of indicational expressions finite *or* infinite. Elements of B_∞ look just like ordinary indicational forms, except that they might grow indefinitely! Call B_∞ the class of *continuous forms.*

Here is a rapid sketch of some algebraic ideas that can be handled in B_∞:

8.1. *Definition*. Let $B_\infty(X^n)$ be the class of continuous forms on $X^n = \{x_1, x_2, \ldots, x_n\}$. That is, the variables $x_1, \ldots, x_n$ may appear in the expressions of $B_\infty(X^n)$. A *system of equations* in $B_\infty(X^n)$ is a function $E: X^n \to B^n_\infty(X^n)$ where $B^n_\infty(X^n)$ denotes the n-fold cartesian product of $B_\infty(X^n)$ with itself. We shall write $x_i = E_i(x)$ as the ith equation of E $(x = (x_1, x_2, \ldots, x_n))$. The following can be proved.

8.2. *Proposition*. let E be a system of equations in $B_\infty(X^n)$. Then there is $A_E \in B^n_\infty(X^n)$ which is a *minimum fixed point* for E, $E(A_E) = A_E$. A_E is called the solution of E over $B_\infty(X^n)$. In fact $A_E = \lim_{n \to \infty} E^n(\perp, \perp, \ldots, \perp)$.

This proposition assures us that this new universe B_∞ is large enough to handle all kinds of re-entry. We have no idea how complex the equation E could be; perhaps it is infinite. Thus, for the present purposes we will narrow our scope, and consider only those infinite expressions in B_∞ that correspond to *finite* re-entry (e.g. where the re-entry can be indicated on a sheet of paper.)

8.3. *Definition.* A *finite* system of equations over $B_\infty(X^n)$ is a system of equations E such that $E: X^n \to B^n(X^n)$. Here $B^n(X^n)$ designates the set of n-tuples of primary algebraic expressions in the variables $x_1, x_2, \ldots, x_n$.

In this way we focus on the forms in B_∞ that arise from (finite) operations in B.

8.4. *Definition.* The set R_B of rational expressions of dimension n is the subset of B_∞ satisfying: $R_B = \{ A \in B_\infty \mid \exists A_2, \ldots, A_n \in B_\infty$ and a finite system of equations E so that $E(A) = A$ where $A = (A_1, A_2, A_3, \ldots, A_n)\}$. That is, the rational expressions are components of fixed points for finite operators. They are single components of n-dimensional re-entering forms.

R_B is an enchanted land for self-referential forms – every operation immediately gets an associated value (fixed point), a re-entering form computed through the fixed point construction. Conversely, each rational form has a corresponding operator. While this correspondence between operators and operands is not one-to-one, we are certainly justified in asserting that it is a matter of point of view whether a rational expression is regarded as a value, or as a transformation. In this sense, operands (i.e. elements of R_B) and operators (i.e. certain maps $R_B \to R_B$) are interchangeable.†

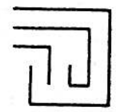
Let us now examine the relation of R_B to the temporal context. Given $A \in R_B$ We know that there exists $A = (A_1, A_2, \ldots, A_n) \in R_B^n$ so that $A = A_1$ and a finite algebraic operator T of dimension n so that $T(A) = A$. We know that $A = \lim_{n \to \infty} T^n (\bot, \bot, \ldots, \bot)$. Thus A has an associated sequence $A(n)$ $(n = 0,1,2, \ldots)$ where $A(n) = T^n(\bot, \bot, \ldots, \bot) = T^n(\mathbf{I})$. At every *finite* n, each term of this sequence is finite, and thus we can reduce them algebraically by choosing an *initial* value (vector) for the indeterminate terms in $V = \mathbf{I}$. We immediately see, by Theorem 7.4, that each sequence will be (eventually) periodic. If we change the initial value we either generate the same sequence with a phase shift, or we find ourselves in an entirely new periodic sequence. These associated periodic sequences may be thought of as (temporal) states of the expression A. Thus every rational expression *oscillates*. To each pair (A, T) such that $T(A) = A$ there is a corresponding oscillation and characteristic frequency.

Thus we may say that (A, T) has a dual nature of *particle* $(T(A) = A)$ and *wave* $(V, T(V), T^2(V), \ldots)$. Which viewpoint comes to the fore depends on our bias. If we insist on invariance of form, then the particle nature is apparent. If we allow the calculational dynamic and assign values, then the wave-nature is seen. Both viewpoints rest ultimately in the essential periodicity of the process

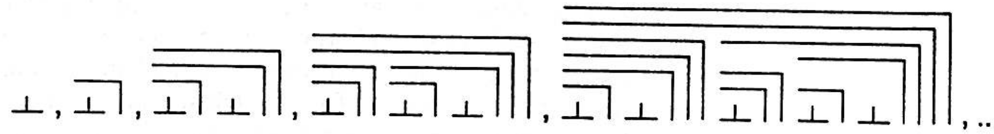
†This is meant to point to the idea that formal domains *can* be reflexive, that is, type-free. The full extent of this idea has been proposed and explored in combinatorial logic and topology by Dana Scott (1971, 1972; see also Wadsworth, 1978). For further discussion on the notion of rational elements of continuous algebras see Goguen *et al.* (1977) and Wright (1976). Obviously what we say here is very informal and expository, and the interested reader is encouraged to look at the aforementioned papers for a detailed discussion.

we have considered.

For example, consider the re-entering form presented in section 7:

X^{∇} = 

Since (being infinite) it must belong to R_B, we know that it must be part of the fixed point of a system of equations. In fact $X^{\nabla} = \overline{\overline{x^{\nabla}}\rceil\lrcorner}\rceil$, and this reveals that it is, in fact, the first component of a system $(x^{\nabla}, y^{\nabla}) = (\overline{y^{\nabla}}\rceil, \overline{\overline{x^{\nabla}}\rceil\, y^{\nabla}}\rceil)$. Thus the corresponding transformation is given by $T(x,y) = (\overline{y}\rceil, \overline{\overline{x}\rceil\ y}\rceil)$. To see the waveform that corresponds to x^{∇} we compute the sequence of approximations to x^{∇} (these are the first co-ordinates of $T^n(\perp, \perp)$ for $n = 1,1,2,\ldots$):



Thus if $\perp$ = **⏋**. then the wave is

⏋, ⏋, ⏋**⏋**, ⏋, ⏋, ...

and if $\perp$ = ⏋ then the wave becomes

⏋, **⏋**, ⏋, ⏋, **⏋**, ⏋, ...

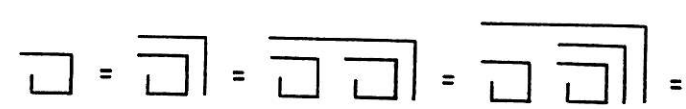
In each case we have a wave train of period 3. The change in choice of initial value produces a phase shift.

The inverse process is more complex. For a given sequence, there are many operators that will generate it, and therefore several elements of R_B can be associated to it. For example, $i \in V$ can be produced by $T(x) = \overline{x}\rceil$, but it also appears entrained with other oscillations as in $T(x,y) = (\overline{y}\rceil, \overline{x}\rceil)$. Notice also that the re-entrant form ⊟ will correspond to both i and j depending on how $\perp$ is evaluated. As it should be in the static world of forms, phases are irrelevant. The waveforms i and j *condense* to the spatial form ⧠.

Much remains to be explored about re-entry forms. For example, we have not discussed the matter of algebraic structures. We feel that more work is needed in this area, but shall end this section with an example.

Consider $x = \overline{x x}\rceil$. Certainly, this equation is satisfied by the infinite form ⊔⊔. If we allow iteration ($aa = a$) as an algebraic rule about infinite forms, then ⧠ will also satisfy the equation. The infinite forms ⧠ and ⊔⊔ should be regarded as distinct, and yet algebraically they are both solutions to the same equation. In general, we can impose algebraic initials on the class of infinite forms *after* these forms have been fully constructed. In this fashion the universe of infinite forms becomes a brownian algebra.

Note that it is too much to ask the infinite forms to be a primary algebra since, if so then



The form of position sends the entire structure into the void!

Just as a self-referential expression has many possible stable states and patterns so have we produced many viewpoints clustering around the general self-referential form. Each view has its own coherence and different views are related in remarkable ways. These remarks, unfolding the self-referential form, only skim the surface. We see that each self-referential form has a natural operator / operand (or wave/particle) duality. and furthermore, each self-referential expression has an associated brownian algebra. We have emphasized individual self-containing expressions because this context places us closest to our experience. Each multiplicity of forms is seen within a wider form to which it condenses.

9. On geometrical form

At this stage it is easy to see the beginnings of the connections between our notions of form and the classical views of geometry and topology. The geometer considers forms as topological spaces and asks for properties invariant under various groups of geometrical transformations. In the abstract, there is a direct analogy with our invariances $T(A) = A$. It would be an incredibly complex task to translate most standard geometric notions directly into this context. Nevertheless, this could in principle be done, since the structure of an axiom system for geometry can express all of its operations as analogous formal operations. The 'geometric' objects can then be constructed as formal concatenations of these operators. This may give insight into the processes of geometry.

This general view of invariance is attractive in that it embraces forms of all kinds from the abstract geometric forms to the shapes of living things that are so obviously continually reformed by recursive process such as we have discussed.

A well-known example of a geometrical re-entry form is given by the division of a golden rectangle: (Fig. 3)

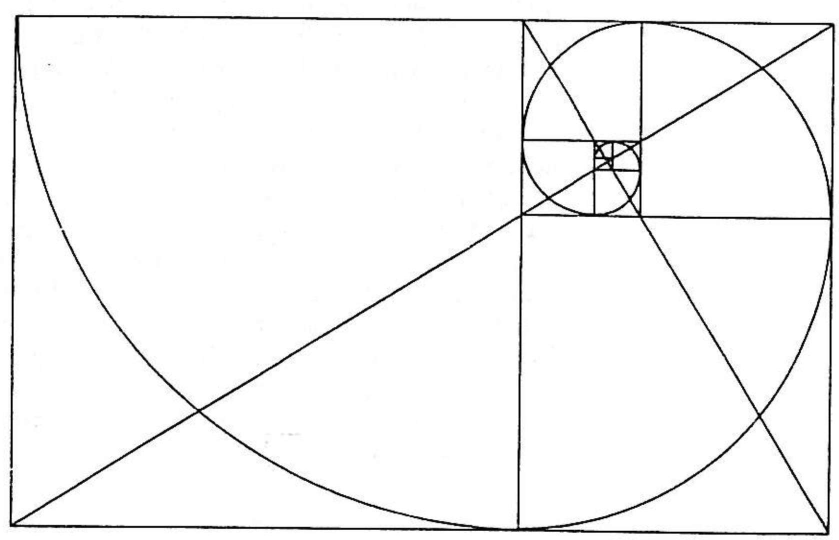


Fig. 3

Here we are given a rectangle with sides in the ratio $1:\frac{1}{2}(1+\sqrt{5})$. Successive cuttings of squares leads to a spiraling sequence of similar rectangles. The given rectangle is of the same form as the new rectangle obtained by adding a square to its longer side. This essential identity of part and whole is seen again and again in natural forms. The whole can be seen in any part, and at any scale at which the form is examined. Forms of this type (self-similar geometric forms) have been given much attention recently under the name of fractals (see Mandelbrot, 1977).

The line between the recursion and the perceived form varies just as in our abstract wave/particle duality. When the particles are at the fore, their global inter-relationships lead to exterior forms of description best suited to algebra and geometry. Nevertheless, the recursion and waveform sounds continually below and throughout these structures.

As a final example, contemplate the complex numbers. Here is indeed the mathematical expression of an enchanted realm where waveform and geometry live joyfully together. All mathematicians know this, and yet our point of view gives even this place a subtle shift.

Reconsider $x^2 + 1 = 0$. View its solution as the happy wave-form solution to the 'paradox' $x = -x^{-1}$. That is, let $T(x) = -x^{-1}$. Whence the infinite eigenform E so that $T(E) = E$ is given by $E = T(T(T(\ldots)))) = -(-(-(-\ldots)^{-1})^{-1})^{-1})^{-1})$ and the associated wave train is $\ldots, -1, +1, -1, +1, -1, +1, -1, +1 \ldots$.

Let R denote all real numbers, and let $\hat{R} = R \times R$ with elements $[a,b]$ with $a,b \in R$. Regard $[a,b]$ as representative of the waveform . . . *abababab* . . . . Make definitions:

(1) $[a,b] + [c,d] = [a+c, b+d]$

(2) $\overline{[a,b]} = [b,a]$ (conjugation is a phase-shift)

(3) $\sqrt{-1} = [+1,-1]$, $1 = [1,1]$

(4) $[a,b] * [c,d] = [ac,bd]$

(5) $a[b,c] = [a,a] * [b,c] = [ab,ac]$

Now demand a multiplication $\alpha,\beta \mapsto \alpha\beta$ so that $(\sqrt{-1})^2 = -1$, that is commutative, associative and distributes over addition. Show that *necessarily:*

$$\alpha\beta = \tfrac{1}{2}(\alpha * \beta + \bar{\alpha} * \beta + \alpha * \bar{\beta} - \bar{\alpha} * \bar{\beta}).$$

Thus the complex numbers are easily viewed as waveforms. Multiplication becomes a special operation involving combinations of phase shifts.

$$a + b\sqrt{-1} = a1 + b\sqrt{-1} = a[1,1] + b[1,-1] = [a+b, a-b].$$

Thus $a+b\sqrt{-1}$ oscillates between $a+b$ and $a-b$.

Re-introduce the well-known geometry so that $a+b\sqrt{-1}$ is represented as (a,b) in the cartesian plane. View the oscillation as a circular orbiting of a point at a distance $|b|$ from a on R (see Fig. 4).

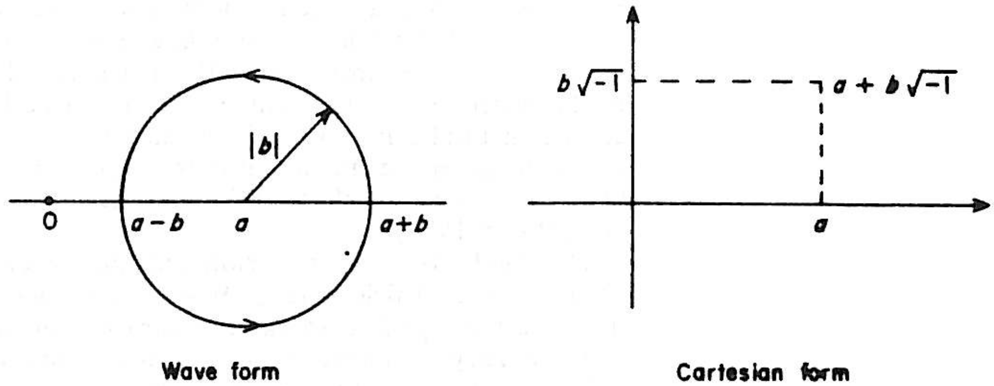


Fig. 4

Combine the two (Fig. 5) by associating the orbit to each complex point on the unit circle in the complex plane. In the diagram each circular orbit corresponds to two complex numbers $a \pm b\sqrt{-1}$ on the cartesian circle, +1 and −1 are degenerate circles, $\pm\sqrt{-1}$ is the large unit circle.

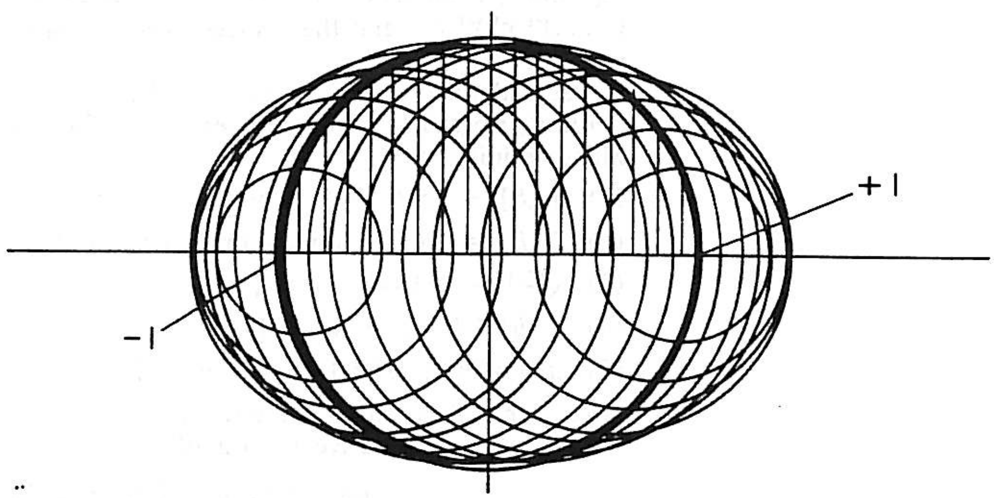


Fig. 5

This view of the complex numbers expands the real line not to the plane but to a dancing buzzing line with an infinity of synchronized circular orbits associated to each point.

The two points of view are in mutual support. While the temporality has in the usual case been restricted to the one circular orbit about the origin (and its projections as sine and cosine), nevertheless it pervades the complexes from whatever view we take.

Thus we are only stating the obvious, but we must repeat: the complex numbers are to the real numbers as the brownian wave-algebra is to primary algebra. Resolution of paradox at the point of self-reference leads to the emergence of new forms, temporal and spatial.

10. Coda

We intended, in this paper, a discussion of form that began as simply as possible. What is simplicity? The word simple derives from the Latin simplex, a combination of *semel* (once) and *plex* (fold). Thus to be simple is to be *of one fold*. This look into the roots of simplicity propels us at once into an entire complex of ideas surrounding the notion of form.

Let that which is folded be some fabric of unspecified qualities. The fold provides a distinction within the space of the fabric. And yet the fabric itself remains essentially whole and undivided. It is our perception of the fold (our making of it) that divides the fabric for us. The distinction is mutable – a pull on the fabric or a change in viewpoint will restore the wholeness at once. The act of distinction, of seeing the fold as a fold in the fabric, becomes the form as seen. In this sense there can be no separation between distinctions and acts of distinguishing. Herein lies the point of utmost simplicity.

Even to distinguish ourselves from the fabric is to move into a complexity of cleavages that is far from being of one fold. In simplicity we can not distinguish ourselves from the fabric. The fold is our fold. The act *is* the object acted upon. The distinction is made and dissolved alternately and simultaneously. For there is no sequential time in simplicity – only in the inevitable movements into and away from complexity does the notion of time occur.

Yet in perceiving a form we are aware of time's passage, of the succession of that form that is seen to be the same form. To see the form again and yet to see it as unchanged is a periodicity, a repetition of form. An underlying periodic vibration, if you will.

The periodicity may be purely temporal as in music, or almost entirely spatial as in the ornate frieze patterns used to decorate walls and boundaries. These seemingly complementary views of periodicity merge when we realize that the viewing or making of any boundary involves a periodic oscillation or dance across it. There may in fact be no boundary other than this musical dance. The boundary becomes vibration at a distance. Here space and time move into one another as our perceptions move toward the one fold.

We must conclude at this point. This paper has been a combination of two things: on the one hand we have presented a formalization for those intuitions which are clear to us; on the other hand we have hinted at many other possible routes in an informal and metaphorical way. This seems to be necessary at this stage of study of form dynamics.

Our approach can be summarized as follows: from the basic notion of indication and the primary algebra of indications of G. Spencer-Brown, we have moved into two essentially complementary directions in order to bring out the dynamic component immanent in a form. First we developed the notion of a brownian algebra, where waveforms can be represented through sequences. Second, we expanded indicational forms to infinitary indicational algebras where re-entry can be expressed properly. The relations between oscillations and pattern can then be established through an analysis of algebraic operators which emobody the dynamic quality of spatial forms.

Even at the level of scrutiny we have achieved so far, there is great beauty in the harmony of shapes and the vibrational quality of indicational expressions. So far, the music of these spheres seemed to have escaped our notice, except in some more special forms (such as the wave/particle duality in quantum mechanics). What we see in the present context is that all of these periodic phenomena in matter, nature, and art seem to be fundamentally the same. They stem from the basic *act* of distinction, of creating a duality of this and that. This primordial act is pregnant with time, space, pattern and their dance.

Acknowledgements

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