EXTENSION IS A SINGLE INITIAL FOR THE PRIMARY ALGEBRA

by Louis H. Kauffman

I. INTRODUCTION

This note assumes the conventions of <u>Laws of Form</u> by G. Spencer-Brown [4]. I show that <u>extension</u>

$$a = \overline{\overline{a}\rceil\, b}\rceil\ \overline{\overline{a}\rceil\,\overline{b}\rceil}\rceil$$

can be used as a single initial for the primary algebra.

There are two sections to the paper. Section 2 contains the promised derivation. Section 3 discusses the historical context of this result, and its relation to the work of Huntington [2], Rodney Johnson, and Graham Ellsbury [1].

II. CONSEQUENCES OF EXTENSION

I shall assume extension, denoted E, and the conventions of substitution and replacement. Any variable can be replaced by an arbitrary algebraic expression — including the unmarked state (void substitution).

E. (Extension) $\quad a = \overline{\overline{a}\rceil\, b}\rceil\ \overline{\overline{a}\rceil\,\overline{b}\rceil}\rceil$

1°. $p\overline{p}\rceil = q\overline{q}\rceil$ for any p and q.

<u>Dem.</u>

$$p\overline{p}\rceil = \overline{\overline{p}\rceil\,\overline{q}\rceil}\rceil\ \overline{\overline{p}\rceil\,\overline{\overline{q}\rceil}\rceil}\rceil\ \overline{p}\rceil \qquad \text{(E.)}$$

$$= \overline{\overline{p}\rceil\,\overline{q}\rceil}\rceil\ \overline{\overline{p}\rceil\,\overline{\overline{q}\rceil}\rceil}\rceil\ \overline{\overline{\overline{p}\rceil}\rceil\,\overline{q}\rceil}\rceil\ \overline{\overline{\overline{p}\rceil}\rceil\,\overline{\overline{q}\rceil}\rceil}\rceil \qquad \text{(E.)}$$

$$= \overline{\overline{p}\rceil\,\overline{q}\rceil}\rceil\ \overline{\overline{\overline{p}\rceil}\rceil\,\overline{q}\rceil}\rceil\ \overline{\overline{p}\rceil\,\overline{\overline{q}\rceil}\rceil}\rceil\ \overline{\overline{\overline{p}\rceil}\rceil\,\overline{\overline{q}\rceil}\rceil}\rceil \qquad \text{(implicit commutativity)}$$

$$= q\ \overline{\overline{p}\rceil\,\overline{\overline{q}\rceil}\rceil}\rceil\ \overline{\overline{\overline{p}\rceil}\rceil\,\overline{\overline{q}\rceil}\rceil}\rceil \qquad \text{(E.)}$$

$$= q\overline{q}\rceil \qquad \text{(E.).}$$

2° (reflection) $\overline{\overline{a}|}| = a$

<u>Dem.</u> $\overline{\overline{a}|}| = \overline{\overline{\overline{\overline{a}|}|}|\;\overline{a}|}|\;\;\overline{\overline{\overline{a}|}|\;\overline{\overline{a}|}|}|$ (E.)

$= \overline{\overline{\overline{a}|}|\;\overline{a}|}|\;\;\overline{\overline{a}|\;\overline{\overline{a}|}|}|$ (1°)

$= a$ (E.).

3° (position) $\overline{\overline{p}|\,p}| =$

<u>Dem.</u> $\overline{p}|\,p = \overline{q}|\,q$ any p and q (1°)

Let q be unmarked.

$\overline{p}|\,p = \overline{\;\;}|$ (void subs.)

$\overline{\overline{p}|\,p}| = \overline{\overline{\;\;}|}|$ (subs.)

$\overline{\overline{p}|\,p}| =$ (2° with a void).

4° (iteration) $aa = a$

<u>Dem.</u> $aa = \overline{\overline{aa}|}|$ (2°)

$= \overline{\overline{a\,\overline{a}|}|\;\overline{aa}|}|$ (3.°)

$= \overline{\overline{\overline{\overline{a}|}|\;\overline{a}|}|\;\;\overline{\overline{\overline{a}|}|\;a}|}|$ (2.°)

$= \overline{\overline{a}|}|$ (E.)

$= a$ (2°).

5.° (occultation) $\overline{\overline{a}|\,b}|\,a = a$

<u>Dem.</u> $\overline{\overline{a}|\,b}|\,a = \overline{\overline{a}|\,b}|\,\overline{\overline{a}|\,b}|\,\overline{\overline{a}|\,\overline{b}|}|$ (E.)

$= \overline{\overline{a}|\,b}|\,\overline{\overline{a}|\,\overline{b}|}|$ (4.°)

$= a$ (E.).

6.° (generation) $\overline{a}|\,b = \overline{a\,b}|\,b$

<u>Dem.</u> $\overline{a}|\,b = \overline{\overline{\overline{a}|}|\,b}|\,\overline{\overline{\overline{a}|}|\,\overline{b}|}|\,b$ (E.)

$= \overline{\overline{\overline{a}|}|\,b}|\,b$ (5.°)

$= \overline{a\,b}|\,b$ (2.°).

7.° (transposition) $\overline{\overline{a}|\,\overline{b}|}|\,c = \overline{\overline{a\,c}|\,\overline{b\,c}|}|$

<u>Dem.</u> $\overline{\overline{a}|\,\overline{b}|}|\,c = \overline{\overline{a}|\,\overline{b}|\,c}|\,c$ (6.°)

$= \overline{\overline{a}|\,\overline{b}|\,c}|\,\overline{\overline{c}|}|$ (2.°)

$= \overline{\overline{a}|\,\overline{b}|\,c}|\,\overline{\overline{\overline{\overline{c}|}|\,a}|\,\overline{c}|}|$ (5.°)

$= \overline{\overline{a}|\,\overline{b}|\,c}|\,\overline{\overline{\overline{\overline{c}|}|\,a}|\,\overline{\overline{\overline{c}|}|\,b}|\,\overline{c}|}|$ (5.°)

$= \overline{\overline{a}|\,\overline{b}|\,c}|\,\overline{\overline{c\,a}|\,\overline{c\,b}|\,\overline{c}|}|$ (2.°)

$= \overline{\overline{a\,c}|\,\overline{b\,c}|\,c}|\,\overline{\overline{a\,c}|\,\overline{b\,c}|\,\overline{c}|}|$ (6.°)

$= \overline{\overline{\overline{\overline{a\,c}|\,\overline{b\,c}|}|}|\,c}|\,\overline{\overline{\overline{\overline{a\,c}|\,\overline{b\,c}|}|}|\,\overline{c}|}|$ (2.°)

$= \overline{\overline{a\,c}|\,\overline{b\,c}|}|$ (E.).

We have derived both <u>position</u> (3.) and <u>transposition</u> (6.) from extension. Position and transposition are the traditional initials for the primary algebra [4]. Therefore, <u>extension is a single initial for the primary algebra</u>.

III. HISTORY

This result has been in the folklore of Laws of Form for some years. It was mentioned to me by Spencer-Brown in 1980 – as a result obtained by Dr. Rodney Johnson, a participant in his seminar at the Naval Research Laboratory in Maryland, USA. According to Spencer-Brown, this demonstration was long and arduous. In any case, no manuscript appeared in the intervening years.

In April of 1988, I discovered the demonstration included herein. It is simple, short, and as natural as the demonstrations of the primary algebra in [4].

Upon relaying this result to Spencer-Brown I was informed that Mr. Graham Ellsbury (see [1]) had found an elegant demonstration that the cross-transposed form of extension

CE: (crosstransposed extension)

$$a = \overline{\overline{ab}\;\overline{a\,\overline{b}}}$$

is also a single initial for the primary algebra.

While these two initials (E and CE) are closely related, there is a particular mathematical difference between them. The initial of cross-transposed extension is more delicate in a technical sense. <u>If we do not allow void substitution (i.e. substituting the unmarked state for a variable) in the algebra then it is an unsolved problem* (see [3]) whether cross-transposed extension is a single initial for an algebra of primary type (equivalent to a boolean algebra).</u>

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* The Robbins Problem was solved by William McCune with a computer assistant in 1996. See "Robbins Algebras are Boolean" <www.cs.unm.edu/~mccune/papers/robbins/>

On the other hand , extension is the formal counterpart of an axiom used by Huntington [2] to give a three-axiom system for boolean algebra. Huntington's other two axioms were commutativity and associativity. Thus ,Huntington's work predates that of Rodney Johnson and myself. Of course Huntington was not working in the context of Laws of Form and he had to work harder in his domain because he did not allow void substitution. These issues are discussed at greater length in [3].

It has been the purpose of this paper to present a simple derivation of the primary algebra from extension, and to place this result in its historical context.

As this discussion indicates, the problem we now face is to characterize the structure of a single initial that can generate the entire primary algebra.

REFERENCES

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